TWO COORDINATIZATION THEOREMS FOR PROJECTIVE PLANES

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A projective plane Π consists of a set of points Π_p, a set of lines Π_L, and a relation between them, denoted $\in$ and read "on", satisfying the following three requirements:

(1) Given any two distinct points P and Q, there is a unique line on which they both lie, denoted $[P \cdot Q]$.
(2) Given any two distinct lines l and m, there is a unique point which lies on both, denoted $l \cap m$.
(3) There are four distinct points, no three of which have a line on which they all lie.

Since the points of a line uniquely determine the line, it is safe to talk about lines as sets of points, but for convenience we will often regard them separately. We will talk about projective planes using the usual language of points and lines - "lies on", "passes through", "collinear", "concurrent", etc.

The third requirement implies its dual, that there are four lines, no three of which are concurrent (consider all the lines determined by the four points), and vice versa, so for any plane we can form the dual plane by swapping the points and lines. Thus for any statement true for all projective planes, the dual statement is also true.

A system satisfying the first two requirements but not the third is called a degenerate projective plane. We will not prove this here, but by case analysis it can be seen that all degenerate projective planes are as follows[1]:

(1) No points, no lines
(2) 1 point, no lines
(3) 1 line, no points
(4) n points, m lines. There is a point P and a line l such that every line passes through P, every point lies on l, while every non-P point lies only on l and every non-l line passes only through P.
(5) n points, n lines. There is a point P and a line l such that every line except l passes through P, every point except P lies on l, while every non-P point Q lies only on l and $[P \cdot Q]$, and every non-l line m passes only through P and $l \cap m$.

A non-degenerate projective plane must have the same number of points $n + 1$ on every line, and furthermore $n+1$ lines through every point; n is called the order of the plane. We can set up a bijection between the points of l and the points of m by picking a point P not on either and mapping $Q \in l$ to $[Q \cdot P] \cap m$; similarly for the lines between any two points. We can set up a bijection between the points of l and the lines through P by mapping $Q \in l$ to $Q \cdot P$, as long as $P \notin l$; otherwise, we can simply note that P has the same number of lines through it as any other point, at least one of which is not on l. In addition, given a point P, there are $n+1$ lines through it, each of which contains n non-P points, all distinct; every non-P point lies on one, so there are n^2+n+1 points total; by the dual argument, there are n^2+n+1 lines total. Note that a projective plane must have at least 4 points, hence order at least 2 (and hence at least 7 points).

A function $f : \Pi \to \Sigma$ will refer to a function $f_p : \Pi_p \to \Sigma_p$ along with a function $f_L : \Pi_L \to \Sigma_L$. We say a bijection $f : \Pi \to \Sigma$ is a collineation if $f(P) \in f(l) \Leftrightarrow P \in l$ for all $P \in \Pi_p, l \in \Pi_L$.

If we have a collineation f from a plane Π to itself, we can consider the set of fixed points and lines and the induced incidence relation on them; note that these fit the first and second requirements for a projective plane (if P and Q fixed, then $f[P \cdot Q] = [f(P) \cdot f(Q)] = [P \cdot Q]$, and dually for the lines), and so either form a subplane or a degenerate subplane.

We say a collineation $f : \Pi \to \Pi$ is (P, l)-central for a point P and a line l if every line passing through P is fixed and every point on l is fixed. P is called the center of f and l is called the axis.

Theorem 0.1. *Let $f : \Pi \to \Pi$ be a central collinneation. Then if f is not the identity, the center and axis are unique.*

Proof. Let f be both (P, l)-central and (Q, m)-central, $P \neq Q$. Then for any point $R \notin [P \cdot Q]$, $f(R) = f([R \cdot P] \cap [R \cdot Q]) = f[R \cdot P] \cap f[R \cdot Q] = [R \cdot P] \cap [R \cdot Q] = R$ as P and Q are both centers. So any line containing at least two points not on $[P \cdot Q]$ is fixed, but this includes every line not equal to $[P \cdot Q]$, while $[P \cdot Q]$ is fixed as it is the line between two fixed points. So every line is fixed, and intersecting these, every point is fixed, and so f is the identity. If instead $l \neq m$, then f is the identity by the dual argument, and so if f is not the identity, the center and axis are unique. □

Theorem 0.2. *Let $f : \Pi \to \Pi$ be a central collineation, not the identity. Then every fixed point is P or lies on l, and every fixed line is l or passes through P.*

Proof. Let Q a fixed point not on l, then for any line m through Q, $m \cap l$ is fixed, and as Q is fixed, m is fixed; hence Q is the center and equals P. By the dual argument, if m is a fixed line not through P, m is the axis and equals l. □

Theorem 0.3. *Let $f : \Pi \to \Pi$ be a collineation of order 2. Then every point lies on a fixed line, and every line contains a fixed point.*

Proof. Again, we will only prove the first, as the second follows by duality. If P is not fixed, then $f[P \cdot f(P)] = [f(P) \cdot P]$, and P lies on a fixed line. If P is fixed, and any other point Q is also fixed, then $[P \cdot Q]$ is fixed. Finally, if P is fixed but no other point is fixed, take another point Q and take another point $R \notin [Q \cdot f(Q)]$. Then $[R \cdot f(R)] \neq [Q \cdot f(Q)]$, so their intersection is fixed; as no other points are fixed, this intersection must be P, lying on both these fixed lines. So every point lies on a fixed line, and dually, every line contains a fixed point. □

While not really relevant to the rest of the paper, this useful property allows us to say a bit more about involutive collineations, if they don't fix a proper subplane.

Theorem 0.4. *Let $f : \Pi \to \Pi$ be a collineation of order 2; then if f does not fix a proper subplane, f is central.*

Proof. Consider the system Σ of fixed points and fixed lines; it is not all of Π nor a proper subplane and so must be a degenerate projective plane. Furthermore it must contain at least one point and at least one line as f an involution, so there is a point P and a line l of Σ such that all points of Σ except possibly P lie on l, and all lines of Σ except possibly l pass through P. We will show f is (P, l)-central.

Take any line m passing through P. If $m = l$, m is fixed. Otherwise, take a point Q of m not equal to P and not on l. As f is an involution, Q must lie on some fixed line. However, every fixed line either is l or passes through P, and Q does not lie on l, so the line $m = [Q \cdot P]$ is fixed. Thus any line passing through P is fixed. By duality, any point lying on l is fixed. Thus f is (P, l)-central. □

As a side note, which we will not prove, if Π is a projective plane, and f is an involutive collineation, with the subplane of fixed points and lines being of order m, then Π must have order m^2. [1]

We can introduce coordinates to a projective plane as follows: Let n be the order of a projective plane Π. Pick four distinct points, no three of which are collinear, called X, Y, O, and I. These will be called the "coordinatizing quadrangle". Take a set R of n symbols, two of which will be called 0 and 1 (0 and 1 are not equal.) Each point will be assigned a name of the form (x, y) for $x, y \in R$, (m) for $m \in R$, or $()$. Each line will be assigned a name of the form $[m, k]$ for $m \in R$, $[x]$ for $x \in R$, or $[]$. We do this as follows: $[X \cdot Y]$ is the line $[]$. Arbitrarily assign names of the form (x, x) to the points of $[O \cdot I]$ other than $[O \cdot I] \cap []$, requiring $O = (0, 0)$ and $I = (1, 1)$. Assign names to every other point not on $[]$ by $(x, y) = [Y \cdot (x, x)] \cap [X \cdot (y, y)]$. Y gets the name $()$; assign names to the other points of $[]$ by $(m) = [(0, 0) \cdot (1, m)] \cap []$. Then let $[m, k] = (0, k) \cdot (m), [x] = [(x, 0) \cdot ()]$. Note that $[0, k]$ consists of the points of the form (x, k) along with 0; $[x]$ consists of the points of the form (x, y) along with $()$; and $[1, 0] = [O \cdot I]$ and consists of points of the form (x, x) along with (1).

We define a ternary operation F on R by $F(x, m, k) = y \Leftrightarrow (x, y) \in [m, k]$. This has the following properties:

(1) $F(a, 0, b) = F(0, a, b) = b$ for all $a, b \in R$.
(2) $F(a, 1, 0) = F(1, a, 0) = a$ for all $a \in$.
(3) Given $a, b, c \in R$, there is a unique $x \in R$ such that $F(a, b, x) = c$.
(4) Given $a, b, c, d \in R$ with $a \neq c$, there is a unique $x \in R$ such that $F(x, a, b) = F(x, c, d)$
(5) Give $a, b, c, d \in R$ with $a \neq c$, there are unique $x, y \in R$ such that $F(a, x, y) = b, F(c, x, y) = d$

Any such system (R, F) is called a planar ternary ring (abbreviated "PTR"); we can also turn a planar ternary ring into a projective plane by taking points (x, y), (m), and $()$, lines $[m, k]$, $[x]$, $[]$, and defining the incidence relation as follows[1]:

(1) $(x, y) \in [m, k] \Leftrightarrow y = F(x, m, k)$
(2) $(x, y) \in [a] \Leftrightarrow x = a$
(3) $(x, y) \notin []$
(4) $(a) \in [m, k] \Leftrightarrow a = m$
(5) $(m) \notin [x]$
(6) $(m) \in []$
(7) $() \notin [m, k]$
(8) $() \in [x]$
(9) $() \in []$

Note that the PTR we get from a given plane can depend on the coordinatizing quadrangle and not just on the PTR itself. With this, we can relate geometric properties of the plane to algebraic properties of the PTR.

We define addition and multiplication on a PTR by $a + b = F(a, 1, b)$ and $ab = (a, b, 0)$. This gives us that 0 is an additive identity, 1 is a multiplicative identity, $a0 = 0a = 0$, and $a + x = b, x + a = b, xa = b, ax = b$ all have unique solutions x (the latter two only if $a \neq 0$). However, addition and multiplication are not necessarily commutative, associative, or related in any way. Things are slightly easier if the PTR is linear, meaning it satisfies $F(a, b, c) = ab + c$.

Finally, the main theorems.

Definition 0.5. A projective plane Π is (P, l)-transitive for a point P and a line l if for every point Q of Π not equal to P or lying on l, and any point $R \in [Q \cdot P]$ other than P and not lying on l, there is a (P, l)-central collineation taking Q to P.

We will need the following theorem:

Theorem 0.6. *Let Π be a projective plane, P be a point of Π and l be a line of Π. If there is a line m through P and a point Q on m, such that for any $R \in m$, not equal to P and not on l, there is a (P, l)-central collineation taking Q to R, then Π is (P, l)-transitive.*

Proof. Let Q' be a point of Π not equal to P and not on l, and let R' be another point on $m' = [Q' \cdot P]$, not equal to P and not on l. We must find a (P, l)-central collineation taking Q' to R'. There are two cases. If Q' and R' are on m, take $f \circ g^{-1}$, where g is a (P, l)-central collineation taking Q to Q', and f is a (P, l)-central collineation taking Q to R'.

Otherwise, let $S = [Q' \cdot Q] \cap l$, and R=$[S \cdot R'] \cap m$. R is on m. R is not on l, as then R' would lie on $[S \cdot R] = l$. R is not P, as then S would lie on $[P \cdot R'] = m$, and thus would equal $m \cap [S \cdot Q'] = Q$, meaning $Q \in l$. So there is a (P, l)-central collineation taking Q to R. As it is (P, l)-central, S is fixed and m' is fixed, so $Q' = m' \cap [S \cdot Q]$ goes to $R' = m' \cap [S \cdot R]$. □

Theorem 0.7. *Let Π be a coordinatized projective plane with PTR R. Then R is linear with associative multiplication if and only if Π is $((0), [0])$-transitive.*

Proof. First, let R be linear with associative multiplication. (Note then that $(R - \{0\}, \cdot)$ forms a group.) Then for $a \neq 0$ in R, define $G : \Pi \to \Pi$ by $G(x, y) = (xa, y), G(m) = (a^{-1}m), G() = (), G[m, k] = [a^-1m, k], G[k] = [ka], G[] = []$.

That this is a collineation can be seen as if $(x, y) \in [m, k], y = xm + k, y = (xa)(a^{-1}m) + k, (xa, y) \in [a^{-1}m, k]$; the other cases are obvious. This collineation is $((0), [0])$-central as lines through (0) are of the form $[0, k]$ or $[]$ and so are fixed, while points on $[0]$ are of the form $(0, y)$ or $()$ and so are fixed. It takes $(1, 0)$ to $(a, 0)$, that is, to any point of $[0, 0] = [(1, 0) \cdot (0)]$ other than (0) or $(0, 0) = [0, 0] \cap [0]$. So Π is $((0), [0])$-transitive.

Now let Π be $((0), [0])$-transitive, and let G be a $((0), [0])$-central collineation taking $(1, 0)$ to $(a, 0)$. Then $[]$ passes through (0) and so is fixed, while $()$ lies on $[0]$ and so is fixed. Thus points of the form (m) go to other points of $[]$, but not to $()$, and we can define $g_0 : R \to R$ by $G(m) = (g_0(m))$. Similarly lines of the form $[x]$ go to other lines of the same form, and we can define $g_1 : R \to R$ by $G[x] = [g_1(x)]$.

It then follows that $G(x,y) = (g_1(x), y)$ as $(x,y) \in [x]$ so $G(x,y) \in G[x]$, as well as $(x,y) \in [0,y]$, which is fixed, passing through (0). (Note therefore $g_1(1) = a$.) Similarly, $G[m,k] = [g_0(m), k]$, as $G(m) = (g_0(m))$ and $G(0,k) = (0,k)$.

Now $F(x,m,k) = y \Leftrightarrow F(g_1(x), g_0(m), k) = y$, that is, $F(g_1(x), g_0(m), k) = F(x,m,k)$. Letting $k = 0$, $xm = g_1(x)g0_(m)$; letting $x = 1$, we get $m = ag_0(m)$ for any $m \in R$. Letting $m = a$, $a = ag_0(a)$, so cancelling, $1 = g_0(a)$. Consider again $xm = g_1(x)g_0(m)$, and let $m = a$, then we get $xa = g_1(x)$ for any $x \in R$. So for any $x, m \in R$, $xm = (xa)g_0(m)$. Letting $m = ay$, $x(ay) = (xa)g_0(ay) = (xa)y$ for any $x, a, y \in R$ with $a \neq 0$; of course, if $a = 0$, associativity follows immediately as both sides equal 0. Thus associativity holds. For linearity, note that we now have $F(x,m,k) = (xa, a^{-1}m, k)$; letting $m = a$, $F(x,m,k) = F(xm, 1, k) = xm + k$. $\square$

Theorem 0.8. *Let Π be a coordinatized projective plane with PTR R. Then Π is $((), [0,0])$-transitive and $((0), [0])$-transitive if and only if R is linear, has associative multiplication, and is right-distributive.*

Proof. If R meets the conditions above, Π is $((0), [0])$-transitive by the above theorem. For $a \neq 0$ define a collineation G of Π by $G(x,y) = (x, ya), G(m) = (ma), G() = (), G[m,k] = [ma, ka], G[x] = [x], G[] = []$.

That this is a collineation can be seen as if $(x,y) \in [m,k], y = xm + k, ya = (xm + k)a = (xm)a + ka = x(ma) + ka, (x, ya) \in [ma, ka]$; the other cases are obvious. This collineation is $((), [0,0])$-central as lines through $()$ are of the form $[x]$ or $[]$ and are fixed, while points on $[0,0]$ are of the form $(0,y)$ or (0) and are fixed (as $a0 = 0$ for any $a \in R$). Furthermore, $G(0,1) = (0,a)$, so $(0,1)$ can go to any point of $[0]$ other than $()$ (the center) or $(0,0)$ (the intersection with the axis), and so Π is $((), [0,0])$-transitive.

For the converse, let Π be $((), [0,0])$-transitive and $((0), [0])$-transitive. Thus R is linear and multiplication is associative. For any $a \in R$, $a \neq 0$, there is a $((), [0,0])$-central collineation G such that $G(0,1) = (0,a)$. Lines through $()$ are fixed and so $G[] = []$, $G[x] = [x]$ for all $x \in R$; also $G() = ()$. As $[]$ is a fixed line, and $()$ is a fixed point, $G(m) = (g_0(m))$ for some function $g_0 : R \to R$. As $[0]$ is a fixed line and $()$ is a fixed point, we must have $G(0,y) = (0, g_1(y))$ for some $g_1 : R \to R$. As $[x]$ is fixed, we must have $G(x,y) \in [x]$, and $G(x,y) \in [0, g_1(y)]$, so $G(x,y) = (x, g_1(y))$. Also, $G(m) = (g_0(m)) \in G[m,k]$, and $G(0,k) = (0, g_1(k)) \in G[m,k]$, so $G[m,k] = [g_0(m), g_1(k)]$.

Now, $F(x,m,k) = y \Leftrightarrow F(x, g_0(m), g_1(k)) = g_1(y)$; by linearity, $xm + k = y \Leftrightarrow xg_0(m) + g_1(k) = g_1(y)$. Letting $k = 0$ and noting $g_1(0) = 0$ as $(0,0)$, lying on the axis, is fixed, we get $xm = y \Leftrightarrow xg_0(m) = g_1(y)$; letting $x = 1$, we see $g_0 = g_1$. Finally, $g_1(1) = a$ by definition, so letting $m = 1, k = 0$ in the equations above, we get $x = y \Leftrightarrow xa = g_0(y)$, or $g_0(x) = xa$ for all $x \in R$. So if $xm + k = y, x(ma) + ka = ya = (xm + k)a$; letting $m = 1$, $xa + ka = (x + k)a$. $\square$

Left distributivity is similar, but we will include the proof anyway.

Theorem 0.9. *Let Π be a coordinatized projective plane with PTR R. Then Π is $((0,0), [])$-transitive and $((0), [0])$-transitive if and only if R is linear, has associative multiplication, and is left-distributive.*

Proof. If R meets the conditions above, Π is $((0), [0])$-transitive by the above theorem. For $a \neq 0$ define a collineation G of Π by $G(x,y) = (ax, ay), G(m) = (m), G() = (), G[m,k] = [m, ak], G[ax] = [ax], G[] = []$.

That this is a collineation can be seen as if $(x, y) \in [m, k], y = xm + k, ay = a(xm + k) = a(xm) + ak = (ax)m + k, (ax, ay) \in [m, ak]$; the other cases are obvious. This collineation is $((0, 0), [])$-central as points on $[]$ are of the form (m) or $()$ and are fixed, while lines through $(0, 0)$ are of the form $[0, k]$ or $[0]$ and are fixed (as $a0 = 0$ for any $a \in R$). Furthermore, $G(1, 0) = (a, 0)$, so $(1, 0)$ can go to any point of $[0, 0]$ other than (0) (on the axis) or $(0, 0)$ (the center), and so Π is $((), [0, 0])$-transitive.

For the converse, let Π be $((0, 0), [])$-transitive and $((0), [0])$-transitive. Thus R is linear and multiplication is associative. For any $a \in R$, $a \neq 0$, there is a $((0, 0), [])$-central collineation G such that $G(1, 0) = (a, 0)$. Points on $[]$ are fixed and so $G() = ()$, $G(m) = (m)$ for all $m \in R$; also $G[] = []$. As $()$ is a fixed point, and $[]$ is a fixed line, $G[x] = [g_0(x)]$ for some function $g_0 : R \to R$. As (0) is a fixed point and $[]$ is a fixed line, we must have $G[0, k] = [0, g_1(y)]$ for some $g_1 : R \to R$. As (m) is fixed, we must have $(m) \in G[m, k]$, and $(0, g_1(k)) \in G[m, k]$, so $G[m, k] = [m, g_1(k)]$. Also, $G(x, y) \in G[x] = [g_0(x)]$, and $G(x, y) \in G[0, y] = [0, g_1(y)]$, so $G(x, y) = [g_0(x), g_1(y)]$.

Now, $F(x, m, k) = y \Leftrightarrow F(g_0(x), m, g_1(k)) = g_1(y)$; by linearity, $xm + k = y \Leftrightarrow g_0(x)m + g_1(k) = g_1(y)$. Letting $k = 0$ and noting $g_1(0) = 0$ as $[0, 0]$, passing through the center, is fixed, we get $xm = y \Leftrightarrow g_0(x)m = g_1(y)$; letting $m = 1$, we see $g_0 = g_1$. Finally, $g_0(1) = a$ as $G(1, 0) = (a, 0)$, so letting $x = 1, k = 0$ in the equations above, we get $x = y \Leftrightarrow ax = g_0(y)$, or $g_0(x) = ax$ for all $x \in R$. So if $xm + k = y, (ax)m) + ak = ay = a(xm + k)$; letting $m = 1$, $ax + ak = a(x + k)$. □

References

[1] A. A. Albert, R. Sandler. An Introduction to Finite Projective Planes. Rinehart and Winston. 1968.