TWIST & SHOUT: EXPLORING ISOMETRIES OF HYPERBOLIC SURFACES

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Let Γ be a cocompact discrete subgroup of the isometry group of $\mathbb{H}^2$, also known as a cocompact Fuchsian group. Recall that $\mathbb{H}^2$ is the upper half complex plane $\{z \in \mathbb{C} \mid \text{Im}z > 0\}$ equipped with the metric

$$< v, w >_z^{Hyp} = \frac{< v, w >^{Euclidean}}{\text{Im}z}$$

at each point $z \in \mathbb{H}^2$, making it a model of the hyperbolic plane.

Consider the closed compact surface $\Sigma = \mathbb{H}^2/\Gamma$. We aim to show that its isometry group is finite, and furthermore that we can achieve a sharp upper bound directly proportional to the genus of the surface. Consider, by comparison, the isometries of $\mathbb{R}^2/\mathbb{Z}^2$, the Euclidean torus T^2. Any Euclidean translation of $\mathbb{R}^2$ descends to an isometry of T^2, yielding an uncountablely large isometry group. Recall that a homeomorphism, diffeomorphism, or isometry Φ descends to a surface from its covering space when there exists an equivalent morphism ϕ on Σ such that, if π is the covering map, $\Phi \circ \pi = \pi \circ \phi$. We equivalently refer to Φ as a lift of ϕ.

We will show that the isometry group is finite ultimately by considering its action on special points whose property is preserved under isometry. Given an element of the unit tangent bundle, there exists a unique complete geodesic realized by parallel transport (this is equivalent to saying that two geodesics passing through the same point on a surface must be the same or else be proceeding in different directions). We record this as **Fact 0**. We will restrict our attention to elements of the unit tangent bundle on Σ whose associated geodesic is closed, of length L, and whose assosciated geodesic has a double point at its assosciated point of our surface; recall that a double point occurs where a geodesic crosses itself transversely.

1. Finitude of the isometry group

First, a lemma showing that we may comfortably proceed concerning ourselves only with orientation preserving isometries of our surface.

Lemma 1.1. *The group of orientation preserving isometries* $Isom_+(\Sigma)$ *is either index 1 or 2 in* $Isom(\Sigma)$*.*

Proof. Since our surface is orientable, any element of $\mathrm{Isom}(\Sigma)$ will have a uniformly positive or negative Jacobian regardless of what point it is evaluated at. Furthermore, since the Jacobian of the product is the product of the Jacobians, so long as there are orientation reversing elements of $\mathrm{Isom}(\Sigma)$ we have a well defined group epimorphism from $\mathrm{Isom}(\Sigma)$ to $\mathbb{Z}/2\mathbb{Z}$ sending an element of $\mathrm{Isom}(\Sigma)$ to 1 if its Jacobian is positive and -1 if it is negative. If there are no such elements sent to -1, then Isom_+ is clearly index 1. If there are, $\mathrm{Isom}_+(\Sigma)$ forms a normal subgroup of $\mathrm{Isom}(\Sigma)$ similarly, since conjugation by two negative Jacobian isometries produces again a positive determinant isometry. Thus we have the following exact sequence:

$$1 \to \mathrm{Isom}_+(\Sigma) \to \mathrm{Isom}(\Sigma) \to \mathbb{Z}/2\mathbb{Z} \to 1$$

and the orientation preserving isometries are index 2. □

We seek to show that the number of double points whose closed geodesics are of a given length is finite. We proceed by showing that the number of closed geodesics of a certain length on our closed compact surface is finite, trivially proving that those with double points are finite as well.

Lemma 1.2. *The set of closed geodesics of a given length* L *on* Σ *is finite.*

Proof. Choose a fundamental domain K in $\mathbb{H}^2$ for Σ. Consider the set of all closed circles of radius L with center points in our fundamental domain. Since K is compact (equivalently since Γ is cocompact) and thus bounded, the union of all our closed circles is bounded. Consequently, by virtue of being closed and bounded we can conclude that their union forms a compact set.

We assume for contradiction that we have an infinite number of $\gamma_i \in \Gamma$ such that for each γ_i there exists an $x_i \in K$ such that the distance between x_i and $\gamma_i(x_i)$ is less than L. Choose a representative for each γ_i which we shall for the sake of convenience refer to as x_i. By Bolzano-Weierstrass, there exists a sequence $\{x_i\}$ which converges in K by its compactness. Call its limit x.

$\{\gamma_i(x_i)\}$ also has a convergent subsequence, again by Bolzano-Weierstrass (but this time due to the compactness of the L-neighborhood of K). Call y_i the image $\gamma_i(x_i)$, and so by extension $\{y_i\}_k$ our convergent subsequence. Say it converges to y. Then $\gamma_i(x) \to y$ since every γ_i is an isometry. This clearly violates our discreteness criterion for Γ, and so we reach a contradiction.

We have a finite number of γ_i which satisfy our condition, so by extension we also have only a finite number of conjugacy classes of our γ_i. Since every closed geodesic corresponds to a conjugacy class of covering transformations (e.g. there is a $\gamma \in \Gamma$ which maps the endpoints of our geodesic to the same point on our surface, as does any other conjugate covering transformation under a suitably different lift), if we choose to lift our geodesics into K there are only a finite number of conjugacy classes whose associated covering transformation closes lifted geodesics of length L. □

One more quick lemma is needed:

Lemma 1.3. *Let $I \subset \Sigma$ be a geodesic segment. If an orientation preserving isometry ϕ fixes I, then ϕ is the identity.*

Proof. Let Φ be a lift of ϕ with base point in the fundamental domain, and $\widetilde{I}$ be a lift of I. Since Φ fixes the geodesic segment $\widetilde{I}$ in $\mathbb{H}^2$ it must the the identity; since the identity is the only element of $\mathrm{PSL}_2(\mathbb{R})$ which fixes a segment. The identity descends to the identity on Σ since any element of Γ descends to the identity. □

Theorem 1.4. *$Isom_+(\Sigma)$ is finite.*

Proof. Elements of Isom_+ act by diffeomorphisms on $T\Sigma$ preserving norms, and so we get an action on the unit tangent bundle $T_1\Sigma$. Let $\mathcal{S}_L$ be the set of points in $T_1\Sigma$ whose induced geodesic is closed, of length L, and has a double point at the assosciated point of our surface. We proved above that $\mathcal{S}_L$ is finite. If L is less than the injectivity radius of $\mathrm{inj}(\Sigma)$ then $\mathcal{S}_L$ is empty, where $\mathrm{inj}(\Sigma)$ is the supremum of the set of numbers r such that for all $z \in \mathbb{H}^2$ and $\gamma \in \Gamma$ we have $d_{\mathbb{H}^2}(z, \gamma(z) \geq 2r$. For that reason, we choose L to be larger than $\mathrm{inj}(\Sigma)$.

Furthermore, we can choose our L so that demonstrating $\mathcal{S}_L$ non-empty is trivial. Consider the group presentation for $\pi_1(\Sigma, x_0)$ given by $< a_1, b_1, a_2, b_2, \ldots, a_g, b_g >$, g the genus of Σ. The curve $\xi = b_1 * b_2^{-1}$ has algebraic self-intersection number of 1 (written $\mathrm{i}(\xi) = 1$). Suppose ξ is homotopic to some ξ'. It is a theorem of algebraic intersection theory (which unfortunately we do not have either the machinery or space to prove) that $\mathrm{i}(\xi) = \mathrm{i}(\xi') \mod (2)$. Thus, since any simple closed curve has algebraic self-intersection of 0, ξ is not homotopic to any simple closed curve.

Take a lift of ξ to $\mathbb{H}^2$. Since ξ closes, $\widetilde{\xi}$'s endpoints must differ by an element $\gamma \in \Gamma$. γ is hyperbolic, and so has a unique invariant geodesic to which ξ is homotopic. The projection of this geodesic is itself a geodesic, still with the same end points in Σ. Furthermore, this projected geodesic is homotopic to ξ by homotopy of lifts. By the above argument, our induced geodesic also must have algebraic self-intersection equal to 1 mod 2. Since any self-intersecting geodesic intersects itself transversely (by Fact 0 given in the introduction), we have exhibited a nonsimple closed geodesic and thus a double point in the unit tangent space on Σ. Let L be the length of our induced geodesic.

Let $k = |\mathcal{S}_L|$. Isom_+ permutes elements of $\mathcal{S}_L$. Let H be a subgroup of $\text{Isom}_+(\Sigma)$ fixing a geodesic segment; that is, fixing a point in $T_1\Sigma$. By the previous lemma, $H = 1$. H is clearly normal in $\text{Isom}_+(\Sigma)$, and so by combining the last two facts we get the following exact sequence:

$$1 \to H \to \text{Isom}_+(\Sigma) \to S_k$$

implying that H is of finite index in $\text{Isom}_+(\Sigma)$. Since H is finite and of finite index, $\text{Isom}_+(\Sigma)$ is finite. □

2. The Riemann-Hurwitz Theorem

Now that we know that it's finite, we can go about trying to give a bound. We will presume a formula known as the Riemann-Hurwitz formula which relates the Euler characteristic of a surface to the Euler characteristic of its quotient by its finite order orientation-preserving isometry group. To use it though we first need to understand why the relationship is not just simply a matter of dividing by the order of the isometry group which we are quotienting out by. After all, since it is finite it is clearly properly discontinous, and if it acted freely then Σ would just form a d sheeted cover of Σ/Isom_+, where d is the order of the group of orientation preserving isometries. However, it is not necessarily free, and it is precisely its deviation from freeness which we need to quantify.

Since our surface is compact, our failures can only be cone points and not cusps; that is, where our quotient fails it fails by wrapping a portion of our surface over itself some number of times everywhere in the neighborhood of a point except for that point itself. Call v_x the order of $\text{Stab}(x)$ in $\text{Isom}_+(\Sigma)$. Since $\text{Isom}_+(\Sigma)$ permutes the elements of $\text{Stab}(x)$, the same group elements will fix elements of the quotient, and so $v_{[x]}$, the order of the stabilizer of the projection of x, is equal to v_x.

To see that the number of x for which $v_x \geq 2$ is finite, we need the following lemma.

Lemma 2.1. *If $\phi \in Isom_+$ fixes two points x, y then $d_\Sigma(x, y) \geq inj(\Sigma)$, where $inj(\Sigma)$ is the supremum of the set of numbers r such that for all $z \in \mathbb{H}^2$ and $\gamma \in \Gamma$ we have $d_{\mathbb{H}^2}(z, \gamma(z) \geq 2r))$.*

Proof. Consider $\widetilde{x}$ a lift of x and Φ a lift of ϕ fixing $\widetilde{x}$. Let $\widetilde{y}$ similarly be a lift of y with $d_{\mathbb{H}^2}(\widetilde{x}, \widetilde{y}) = d_\Sigma(x, y)$. $\Phi(\widetilde{y})$ must also be a lift of y such that $d_{\mathbb{H}^2}(\widetilde{x}, \Phi\widetilde{y}) = d_\Sigma(x, y)$. Then

$$2\text{inj}(\Sigma) \leq 2d_{\mathbb{H}^2}(\widetilde{y}, \Phi\widetilde{y}) \leq d_{\mathbb{H}^2}(\widetilde{y}, \widetilde{x}) + d_{\mathbb{H}^2}(\widetilde{x}, \Phi\widetilde{y}) = 2d_\Sigma(x, y)$$

□

Thus by compactness an isometry of Σ fixes at most a finite number of points. Call the set of $x \in \Sigma$ with $v_x \geq 2$ by $\mathcal{F}$. Since $\mathcal{F}$ is finite, it is closed. $\mathcal{F}$ is preserved by the action of $\text{Isom}_+(\Sigma)$. $\Sigma - \mathcal{F}$ is then a compact surface upon which $\text{Isom}_+(\Sigma)$ acts freely and properly discontinuously; thus, $(\Sigma - \mathcal{F}/\text{Isom}_+(\Sigma))$ is a compact surface with a d-sheeted cover of Σ. Adding back in points in $\mathcal{F}$ then should change the Euler characteristic in a consistent way. Thus, it is logical (although we do not prove it) that we should get

Theorem 2.2. *(Riemann-Hurwitz) The quotient $\Sigma/Isom_+(\Sigma)$ is an orientable surface with Euler characteristic*

$$\chi(\Sigma/Isom_+(\Sigma)) = \frac{1}{d}\chi(\Sigma) + \sum_{[x]\in\Sigma/Isom_+(\Sigma)} \left(1 - \frac{1}{v[x]}\right)$$

With this formula firmly in hand, we are able to compute a sharp upper bound on the number of isometries of a hyperbolic surface.

3. Hurwitz's theorem

The result which captures this is known as Hurwitz's theorem. It states that the number of isometries of a hyperbolic surface of genus g is at most $168(g-1)$, including both the orientation preserving and orientation reversing varieties. Since we know the index of $\text{Isom}_+(\Sigma)$ is at most two, to prove Hurwitz's theorem it suffices to show that the order of $\text{Isom}_+(\Sigma)$ is at most $84(g-1)$, or equivalently $-42\chi(\Sigma)$ (since $\chi(\Sigma) = 2 - 2g$). By simple algebraic manipulation we get that

$$d = \frac{\chi(\Sigma)}{\chi(\Sigma/\text{Isom}_+(\Sigma) - \sum_{x\in\Sigma}\left(1 - \frac{1}{v_x}\right)}$$

$\chi(\Sigma)$ is always negative, since by the Gauss-Bonnet theorem $2\pi\chi(\Sigma) = \int_\Sigma \kappa d\sigma$, where κ is the curvature of the surface and $d\sigma$ is the area form and as a hyperbolic surface κ is always -1. That is to say, $\chi(\Sigma) = \frac{-v_\Sigma}{2\pi}$

where v_Σ is the area of Σ. Since $\chi(\Sigma)$ is always negative, we need only concern ourselves with determining how high we can get the denominator of the above equation to be while still being negative. If it were positive, it would imply that our isometry group had negative order, which is about as absurd as a moon made of cheese. An empty isometry group would be similarly ridiculous; at the very least, the identity is always an isometry.

Our first possibility is that the Euler characteristic of our quotient by the orientation preserving isometry group is -1 or less. Call it $-n$. Then

$$\begin{aligned} d &= \frac{2-2g}{-n-\sum_{x\in\Sigma}\left(1-\frac{1}{v_x}\right)} \\ &\leq \frac{2-2g}{-1-\sum_{x\in\Sigma}\left(1-\frac{1}{v_x}\right)} \\ &\leq (2-2g)(-1) = 2(g-1) \leq 84(g-1) \end{aligned}$$

since our genus is always 2 or higher.

The next possibility is that our quotient is genus 0, topologically a torus. In this case, our equation reduces to

$$d = \frac{2-2g}{-\sum_{x\in\Sigma}\left(1-\frac{1}{v_x}\right)}$$

which is clearly maximized when there is precisely one cone point of order two. A greater number of cone points would result in a denominator which was at least -1, a number clearly less than $-\frac{1}{2}$. A greater order cone point would similarly give us a denominator which was greater than $-\frac{1}{2}$. No cone points would imply that we should divide by zero, a horrible possibility indeed.

Since Σ is orientable and $\mathrm{Isom}_+(\Sigma)$ is orientation preserving our quotient must be orientable. Therefore by the classification of surfaces we can now consider our only remaining possibility, a topological sphere of Euler characteristic 2. By exhaustive calculation, we find that having three cone points, of order 2, 3, and 7 respectively, provide us with our maximum. We find that the sum of the $(1-\frac{1}{v_x})$ is precisely $\frac{85}{42}$; thus, subtracting it from 2 we get exactly $\frac{84}{42}-\frac{85}{42}=\frac{-1}{42}$. Plugging this back into our equation we get that

$$\begin{aligned} d &= \frac{2-2g}{\frac{-1}{42}} \\ &\leq 84(g-1) \end{aligned}$$

And so, a hyperbolic surface whose quotient by its orientation preserving isometry group is precisely a sphere with three cone points (one of 2, one of 3, and one of 7) will have the maximum number of isometries. Does any such exist? Indeed they do, and many exhibit remarkable properties. The first is Klein's quartic, the unique such surface achieving the bound for genus 3. In fact, there are an infinite number of genera for which such surfaces exist, and, perhaps even cooler, an infinite number for which they do not.