# TESSELLATING THE HYPERBOLIC PLANE

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Abstract. The main goal of this paper will be to determine which hyperbolic polygons can be used to tessellate the hyperbolic plane. Sections 1-4 will be devoted to providing the context of the hyperbolic plane and developing the basic tools needed to prove the key theorems of this paper. In these sections I will cover two common models of the plane and the isometries of these spaces. Section 4 will be a brief digression into the nature of triangles which will provide a good background for more general theorems about tessellation. Section 5 will cover the basics of tessellation. Poincaré's theorem for compact polygons which is the central element of the argument of this paper will be introduced at the end of section 5. Section 6 will be devoted to the mechanical tools needed to prove this theorem. In the final two sections,Poincaré's theorem is proved and the implications of the proof are discussed. Significant information supplementary to the highlighted theorem is discovered in the course of its proof.

Contents



A note on the Philosophy of Geometry:

When describing geometric surfaces, there seem to be two approaches, you can either begin with axioms from which all properties of the surface are discovered organically or you can begin by defining a priori points, lines, distances, and angles on a set and later end up 'realizing' some convenient properties such as the fact that 'lines' are the shortest paths between two points. My approach to constructing the Hyperbolic plane will be a combination of the two, more heavily weighted toward the latter approach. It is important to remember that there is an axiomatic basis to the world we are about to describe however, actually building it up from this basis seems tedious. Instead I will start by defining the hyperbolic plane as a set equipped with an arc-length function.

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## 1. Some historical notes and a brief explanation of motivations for the hyperbolic plane

Hyperbolic geometry was first discovered by geometers who were trying to prove that the omission Euclid's fifth postulate, which is sometimes referred to as the parallel postulate, would lead to a contradiction with the other Euclidean axioms. The famous axiom in question can be stated as:

> *Given a line l and a point p not on that line, there exists a unique line through the point p which does not intersect l.*

The hyperbolic plane can be established by replacing this with:

> *Given a line l and a point p not on that line, there is more than one non-identical line through the point p and not intersecting l.*

Any two spaces fulfilling this axiom together with the other axioms of Euclidean space are isomorphic.

This space was first discovered by Bolyai and Lobachevsky separately. Gauss also seems to have known of it around the same time (late 19th century) although he didn't publish his findings, possibly because he was unable to find a complete embedding of hyperbolic space in $\mathbb{R}^3$. Hilbert proved in 1901 that a complete (meaning that all lines can be infinitely extended) embedding of hyperbolic space in $\mathbb{R}^3$ does not exist but useful models were constructed within $\mathbb{R}^2$ by Poincaré and Beltrami. This paper will be working within the Poincaré half plane model and the Poincaré disc model.

Another important motivating concept behind hyperbolic geometry is that of Gaussian curvature which I wont discuss here except to say that the hyperbolic plane was motivated by a search for a surface of constant negative curvature analagous to the sphere which has constant positve curvature. The Euclidean plane has zero curvature. An idea of curvature can be provided by giving the definition for a two dimensional surface embedded in $\mathbb{R}^3$ which is not applicable to the hyperbolic plane because the hyperbolic plane cannot be embedded in $\mathbb{R}^3$ but this definition is helpful in visualizing the concept.

**Definition 1.1.** The *curvature* $\kappa$ of a path $K$ at a point $p$ on a manifold $S$ embedded in $\mathbb{R}^3$ is $1/\rho$ where $\rho$ is the radius of the circle in $\mathbb{R}^3$ which most closely approximates K at p.

The curvature of $S$ at $p$ is $K(p) = \kappa_{min}\kappa_{max}$ where $\kappa_{min}$ and $\kappa_{max}$ are respectively the largest and smallest curvatures of paths through $p$ on $S$.

Gauss proved that the parameter of curvature is independent of the choice of coordinate system and can be calculated directly from the first fundamental form. One of the consequences of this theorem is that the concept of curvature can be applied to manifolds without reference to an embedding. Gauss was so impressed by this theorem he named it "Theorema Egregium" or "Magnificent Theorem."

## 2. The Poincaré Half Plane model

Because the hyperbolic plane cannot be embedded in $\mathbb{R}^3$ we will begin by looking at the upper half of the complex plane

$$\mathbb{H}^2 = \{z \in \mathbb{C} | Im(z) > 0\}$$

with the length function

$$L(\gamma, 0, 1) = \int_0^1 \sqrt{(1/y^2)(dx^2 + dy^2)}$$

Here $x = Im(z)$ and $y = Re(z)$.

Hyperbolic lines will be defined as vetical lines in the complex plane and. These lines are not arbitrarily chosen but are in fact geodesics, or locally shortest paths with respect to the length function. Although this is not true on all surfaces, in the hyperbolic plane geodesics are also unversally shortest paths, meaning there is a unique geodesic between any two points and it is the shortest path between them. This is notably not true on the sphere. Hyperbolic angles will be defined as the Euclidean angles between the Euclidean line tangent to the two intersecting hyperbolic lines at the point of intersection.

The next piece of information we need to really give shape to this universe is an idea of the isometries, or distance preserving maps, on this space. Because it is not the main focus of this paper I am going to abbreviate my discussion of the isometries of the hyperbolic plane, although a thorough understanding of them will be essential to later arguments. A really good reference for this is [1]. At the end of the following section there is a table listing isometries and their geometric meaning.

It is important to note: Although many hyperbolic isometries take the same form as Euclidean isometries they usually have different geometric meaning. Euclidean dilations translate hyperbolic lines along the y axis, leaving the y-axis invariant, Euclidean translations are not Hyperbolic translations and have no analogue in Euclidean space, although it may be helpful to think of them as rotations about the infinite limit of the imaginary axis.

**Definition 2.1.** An *isometry* of the hyperbolic plane is a function

$$T : \mathbb{H}^2 \to \mathbb{H}^2$$

such that $L(T \circ \gamma) = L(\gamma)$ for all paths on $\mathbb{H}^2$

**Definition 2.2.** An *inversion* in the circle $C_{d,r}$ with center d and radius r is a function of the form

$$T(x) = \frac{d\bar{z} + r^2 - d\bar{d}}{\bar{z} - \bar{d}}$$

This function translates a point $w$ along the line through itself and $d$, so that, if the euclidean distance of $w$ from $d$ is given by $\rho$ then the Euclidean distance of $T(w)$ form $d$ is $r^2/\rho$. $T$ sends everything outside the circle to the inside of the circle and vice versa. By taking limits of $T$ and $T^{-1}$ as $w \to d$ we see that the center of the circle and the "point at infinity" switch places. It is as if the entire complex plane is turned inside out leaving the boundary of the circle invariant.

Calculations of the type used below the table show that inversions through circles with centers on the x-axis as well as the first two isometries listed in the table are in fact isometries. Inversions are hyperbolic reflections while the first two isometries from the table are respectively, limit rotations and translations. Unlike these three, hyperbolic rotations are difficult to see in $\mathbb{H}^2$ so it is time to introduce another model.

## 3. The Poincaré Conformal Disc Model

We would like to create this new model, $\mathbb{D}^2$, in such a way that the knowledge we already have about $\mathbb{H}^2$ is still usefull. Specifically we need a function $J(z) : \mathbb{H}^2 \to \mathbb{D}^2$ such that if $I(z)$ is an isometry in $\mathbb{H}^2$ then $J(I(z))$ is an isometry in $\mathbb{D}^2$. This means that we will require the disc to be equipped with the metric it inherits from $\mathbb{H}^2$. In other words a curve $\gamma$ on $\mathbb{D}^2$ will have the $\mathbb{H}^2$ length of $J^{-1}(\gamma)$. Because of this requirement, $J$ will take geodesics in $\mathbb{H}^2$ to geodesics in $\mathbb{D}^2$ and if two $\mathbb{H}^2$ lines, $l_1$ and $l_2$ intersect in a $\mathbb{H}^2$ angle $\alpha$ thend $J(l_1)$ and $J(l_2)$ will intersect in a $\mathbb{D}^2$ angle $\alpha$. The specification of $\mathbb{H}^2$ angles versus $\mathbb{D}^2$ angles is really unecessary because they will both be defined in the same way. Most of the tessellating in this paper will be done on the $\mathbb{D}^2$ model

To map the upper half plane onto the unit-disc we first invert the complex plane in the circle $C_{-i,\sqrt{2}}$, which exchanges the origin and the point $i$ because the origin is at a distance of 1 from the center and so is sent to a point at a distance (2/1) which is the point i, while i, being at distance 2 is sent to the point (2/2) = 1 away from the center. We now have an image of $\mathbb{H}^2$ in the unit circle centered at the origin, but it is upside down because the image of the origin is at the top while image of the infinite limit of the plane is at $-i$. To correct this we then preform a reflection of the entire plane about the x axis, interchanging the image of O and the point $-i$ at the bottom of the circle. This pair of transformations is given by:

$$J(z) = (\bar{z}) \circ (\frac{-i\bar{z}+1}{\bar{z}-i}) = \overline{(\frac{-i\bar{z}+1}{\bar{z}-i})} = \frac{iz+1}{z+1}$$

$$J^{-1}(z) = \frac{-iw+1}{w-i}$$

$\mathbb{H}^2$ -lines are mapped by this transformation to circles which intersect the boundary at right angles, and $\mathbb{H}^2$ angles are preserved. It will be useful to calculate the arc length function on $\mathbb{D}^2$ because, if a function $T$ preserves $\mathbb{D}^2$ arc length then $J^{-1} \circ T \circ J$ necessarily preserves $\mathbb{H}^2$ arc length.

Recall: $L_H(\gamma) = \int \frac{|dz|}{Im(z)}$

$$L_D(\gamma) = L(_H J^{-1} \circ \gamma) = \int \frac{|d(J^{-1}(z))|}{Im(J^{-1}(z))} =$$

$$\int \frac{|\frac{-2dw}{(w-i)^2}|}{Im(\frac{(1-iw)(\bar{w}+i)}{|w-i|^2})} = \int \frac{|2dw|}{1-|w|^2}$$

**Lemma 3.1.** *$T(w) = e^{i\theta}w$ is a $\mathbb{D}^2$ isometry.*

Geometrically, this isometry represents a rotation of the disc around the origin. Previous calculations imply that $J^{-1}(T)$ is an $\mathbb{H}^2$ isometry which should leave the same number of points invariant as T does. In fact, T turns out to be an $\mathbb{H}^2$ rotation about i.

*Proof.* T leaves the expression $\frac{|2dw|}{1-|w|^2}$ unchanged □

The reader should verify for herself that all of the transformations are in fact isometries of their respective models. I will only demonstrate this for the second one in $\mathbb{H}^2$ (because it is the simplest).

| $\mathbb{H}^2 \overset{J(z)=\frac{iz+1}{z+1}}{\longrightarrow} \mathbb{D}^2$ | | Geometric Meaning |
|---|---|---|
| $t(z) = \alpha + z$ | $s(w) = \frac{i(\alpha+w)+1}{w+\alpha+i}$ | Limit Rotation: fixes no points but both $\lim_{w\to i}$, $w \in \mathbb{D}^2$ and $\lim_{z\to\infty}$, $z \in \mathbb{H}^2$ are fixed. |
| $t(z) = \rho z$ | $s(w) = \frac{\rho i w+1}{\rho w+i}$ | Translation: along arcs which are Euclidean lines through O in $\mathbb{H}^2$(only the y-axis as an $\mathbb{H}^2$ line) and along Euclidean circles through $\pm i$ in $\mathbb{D}^2$ (again only the y-axis is a $\mathbb{D}^2$ line). Translation preserves these lines and fixes their endpoints: 0 and $\infty$ in $\mathbb{H}^2$, $\pm i$ in $\mathbb{D}^2$. |
| $t(z) = \frac{-ie^{i\theta}w+1}{e^{i\theta}w-i}$ | $s(w) = e^{i\theta}w$ | Rotation: about O in $\mathbb{D}^2$, fixes only one point: O in $\mathbb{D}^2$ and $i$ in $\mathbb{H}^2$, leaves invariant circles with center O in $\mathbb{D}^2$ |
| $t(z) = \frac{1}{\bar{z}}$ | $s(w) = \bar{w}$ | Reflection: across the real axis in $\mathbb{D}^2$ and across the unit circle in $\mathbb{H}^2$(inversion). |

TABLE 1. Dictionary of isometries in the Half Plane and the Conformal Disc

In the case of Euclidean dilations $T(z) = \rho z$. Let a path $\gamma$ be given by

$$\gamma(t) = \gamma_x(t) + i\gamma_y(t)$$

and $dx = \frac{\partial(\gamma_x(t))}{\partial t}dt, dy = \frac{\partial(\gamma_y(t))}{\partial t}dt$ then

$$L(T \circ \gamma) = \int \frac{\rho\sqrt{dx^2+dy^2}}{\rho\gamma_y(t)} = \int \frac{\sqrt{dx^2+dy^2}}{\gamma_y(t)} = L(\gamma)$$

**Lemma 3.2.** *Any point in $\mathbb{D}^2$ is uniquely determined by its $\mathbb{D}^2$-distance from three non-collinear points on the disc.*

*Proof.* Using the fact that in hyperbolic space, geodesics are universally shortest paths, if the sides of a triangle are length A, B, C, the sum of any two lengths is greater than or equal to the other length because otherwise $A+B < C$ and $C$ is not a geodesic. Suppose that there are two distinct points equidistant from three non-colinear points $a, b$, and $c$. We will show that the set of points equidistant from both $p$ and $p'$ is an $\mathbb{H}^2$ line and so this is impossible. In figure 1 it is clear that the two triangles $\triangle apb$ $\triangle bp'a$ are congruent because $|\bar{ap}| = |\bar{ap'}|$ and $|\bar{bp}| = |\bar{bp'}|$. We can also see from the figure that reflection about the line through $a$ and $b$ interchanges these triangles and threfore also interchanges the points $p$ and $p'$. This shows that all points on that line are equidistant from $p$ and $p'$. It now only remains to show that any point not on this line cannot be equidistant from both $p$ and $p'$.

Since this point, $c$, is not on the line $\bar{ab}$, it must be on one side or the other of it and we can assume, without loss of generality, that it is on the same side as the point $p$. Again, referring to figure 1, we know from above that the lengths $B_2$

and $|\bar{dp}|$ are equivalent. We also have the inequality $B_1 + |\bar{dp}| \geq A$ with equality holding only if the point $c$ is co-linear with $p$ and $p'$. The only point co-linear with $p$ and $p'$ and equidistnt from both is the point at the intersecton of $\bar{pp'}$ and $\bar{ab}$ and, by assumption, $c$ is not co-linear with $a$ and $b$. Therefore, we have shown that $B$ is greater than $A$ contradicting our assumption that they are equal. □

**Theorem 3.3.** *Three Reflections Theorem for* $\mathbb{H}^2$
*All hyperbolic reflections are the product of one, two or three hyperbolic reflections.*

*Proof.* This proof is identical to the proof of the analogous theorem for Euclidean Space which is done in [1] among other places. □

**Theorem 3.4.** *Poincaré [1882]*
*The complex functions* $T(z) = \frac{az+b}{cz+d}$ *or* $T(z) = \frac{-a\bar{z}+b}{-c\bar{z}+d}$ *with* $ad - bc = 1$ *are all of the* $\mathbb{H}^2$ *isometries.*

*Proof.* This theorem follows directly from the three reflections theorem and the proven isometries from the table: All reflections in $\mathbb{H}^2$ are either inversions through circles with center on the x axis or reflections across lines of the form $x = \epsilon$.

Inversions in the circle with center $\alpha$ and radius 1 are represented by the linear transformation $I(z) = \frac{\alpha\bar{z}+1^2-\alpha^2}{\bar{z}-\alpha}$. Taking $a = -\alpha, b = 1 - \alpha^2, c = -1, d = -\alpha$, this isometry is of the second kind and $ad - bc = 1$. Reflections through lines $x = \epsilon$ are of the form $-\bar{z} + 2\epsilon$. This is a transformation also of the second type with $a = 1, b = 2\epsilon, c = 0, d = 1$ so that $ad - bc = 1$.

Products of two reflections therefore have the form $\frac{az+b}{cz+d}$ with $ad - bc = 1$ . Products of three reflections have the form $\frac{a'\bar{z}+b'}{c'\bar{z}+d'}$ with determinant -1 which is the same as $\frac{-az+b}{-cz+d}$ with determinant 1.

On the other hand, a few calculations show that any function of the form

$$t(z) = \frac{az+b}{cz+d} = (\frac{a}{c}) - \frac{1}{c(cz+d)}$$

. If $d > 0$ this is the product of three $\mathbb{H}^2$ reflections. If $d < 0$ the calculations may be repeated for $\frac{-a\bar{z}+b}{-c\bar{z}+d}$. □

**Corollary 3.5.** $\mathbb{D}^2$ *isometries are of the form* $s(w) = \frac{az+b}{\bar{b}z+\bar{a}}$ *and* $s(w) = \frac{a\bar{z}+b}{\bar{b}\bar{z}+\bar{a}}$ *where, in both cases* $a, b \in$ *and* $|a|^2 + |b|^2 = 1$.

In the rest of this paper, I will refer to the group of isometries of $\mathbb{H}^2$ as $\Gamma_H$ and the group of isometries of $\mathbb{D}^2$ as $\Gamma_D$.

## 4. Hyperbolic Triangles

As in Euclidean space, triangles are a fundamental polygon to understand. The proof that all polygons can be triangulated carries over directly from the proof in the Euclidean case so, while it is a simple topic, triangles can be the key to many complex geometric topics. I will not prove the results in this very short section but they can be found in [1].

The first thing to notice is that while the angles of Euclidean triangles are constrained to summing to precisely $\pi$, there is no such constraint in the hyperbolic plane, in fact the angles of hyperbolic triangles all sum to less than $\pi$. The area of a hyperbolic triangle can be found by integration. The result of such calculations is that the area depends only on its angle measurements. This means that there are no similar triangles - triangles with the same angles have the same area and are congruent! These facts carry over to the case of larger polygons by triangulation - a non-convex hyperbolic n-gon has angles summing to less than $\pi(n-2)$. The following formula will be useful:

The area of a triangle in $\mathbb{D}^2$ is given by $\pi - (\alpha+\beta+\gamma)$ if $\alpha, \beta$, and $\gamma$ are the angles of the triangle. This formula expresses the fact that the smaller a neighborhood in hyperbolic space, the more closely triangles in this space will look like Euclidean triangles. The converse of the idea that very large triangles have angles which sum to very small amounts.

In hyperbolic space there are two types of parallel lines: asymptotically parallel and ultraparallel. Both terms refer to pairs of lines which do not intersect, however, asymptotically parallel lines share a common limit point on the boundary of the disc, in the $^2$ model, while ultraparallel lines do not. The angle between two asymptotically parallel lines is defined to be zero.

## 5. Groups and Tessellations

In this section we will only be discussing tessellations of The Euclidean plane and tessellations of the two models for the hyperbolic plane given in the previous sections, so, for the purposes of simplicity, all definitions and theorems will be restricted to these two spaces even though many of them generalize to more Riemannian surfaces.

**Definition 5.1.** A polygon D is a closed region which is the union of an open set $D'$ and $\partial D'$ the boundary of $D'$. $\partial D'$ is a finite union of line segments, called edges, appropriate to the space which D inhabits. $\partial D'$ satisfies the following:

(1) Two edges are either disjoint or intersect in exactly one point which is an end point of both edges.
(2) A point of intersection of two edges is called a vertex and is an element of exactly two edges.
(3) Each edge intersects exactly two other edges.

**Definition 5.2.** A *tessellation* of a plane S which is either the Euclidean plane or of the Hyperbolic plane is a subdivision of the plane into polygonal tiles $t_i$ Such that the tiles have the following properties:

(1) (the tiles are all "the same shape")given any two tiles $t_i$ and $t_j$ $\exists$ an isometry $T_{i,j} \in \Gamma_{Euc}, \Gamma_H$, or $\Gamma_D$ as appropriate such that $T_{i,j}(t_i) = t_j$
(2) (No overlapping tiles) If $t_i$ and $t_j$ are not identical then only one of the following is true:
    - $t_i \cap t_j = \emptyset$
    - $t_i \cap t_j = p$ where p is a single point in S
    - $t_i \cap t_j = e$ where e is a line segment in S
(3) (No gaps between tiles)given any point p $\in$ S, there is at least one tile $t_i$ such that p $\in t_1$

The most well known tessellation is probably the chess board tessellation pictured in figure 2 with the unit square in which each square has lower left coordinate $(n, m)$ where $n$ and $m$ are integers.

Unfortunately, this definition also admits some awkward and inconvenient tessellations which we would like to exclude, like the two examples in figure 3. The tessellation on the left in figure 3 shows a standard chess board tessellation which has been modified. In this tessellation, a square which had its lower left corner at coordinate $(n, m)$ in figure 2, has been moved up by $1/n$ so that that corner is now at $(n, m + (1/n))$. The tessellation on the right in figure 3 shows a subdivision of the chessboard tessellation into right triangles obtained by arbitrarily choosing one of the two diagonals in every square.

**Definition 5.3.** a tessellation in *standard form* is a tessellation with the following properties:

(1) as in definition 5.2.
(2) If $t_i$ and $t_j$ are not identical then only one of the following is true:
   - $t_i \cap t_j = \emptyset$
   - $t_i \cap t_j = p$ where p is a single point in S **and p is a vertex of both $t_i$ and $T_j$**
   - $t_i \cap t_j = e$ where e is a line segment in S **and e is an entire edge of both $t_i$ and $T_j$**

(3) Given any point p ∈ S, there is at least one tile $t_i$ such that p ∈ $t_1$. **If P is in exactly one tile then p is in the interior of a tile. If p is in exactly two tiles then p is in an edge and if P is in more than two tiles then p is a vertex**

Notice that the brick tessellation in figure 4 can still be considered to be in standard form if it is viewed as a hexagonal tessellation rather than a rectangular one but that the tessellation on the left of figure 3 cannot because polygons are restricted to having a finite number of sides.

Because of the first requirement of a tessellation, any tessellation can be paired with a characteristic group H where H is the set of all the symmetries of the tessellation or the subset of $\Gamma_S$ of isometries which take all tiles to other tiles. If the polygonal tiles of the tessellation themselves have symmetries, these will be included in H.

**Example 5.4.** The group $H_{squares}$ of symmetries of the chess board tessellation is the group generated by:

g = reflection across one side of the square
h = rotation by 90 deg around the center of one of the squares

The group $H_{triangles}$ of symmetries of the tessellation on the right of figure 3 is generated by:

g = translation by one unit to the left
h = translation by one unit up

These are the only isometries which preserve the structure of the tessellation because of its random nature. Observe that $H_{triangles}$ does not contain enough isometries to move any tile to any other tile in the tessellation.

**Definition 5.5.** A tessellation, T is *symmetric* if the group $H_T$ contains at least one member from each set $T_{i,j} = \{t_{i,j} | t_{i,j}(t_i) = t_j\}$ where the $t_{i,j}$ are isometries.

The tessellation on the right of figure 3 is not symmetric. From this point the tessellations considered in this paper will be limited to symmetric tessellations in standard form. You can imagine that while a tessellation, T determines the associated group H of symmetries, H seems to suggest the shape of T. Given only the group $H_{squares}$ from example 5.4 and asked to guess what the associated tessellation looked like, you could very well guess correctly. It would, however, be equally reasonable for you to suggest a tessellation by the iscocolese right triangles which form quarters of the square.

The following definitions will help to use groups of isometries to produce tessellations of the Hyperbolic plane, a far more efficient way to discover them than guessing shapes and attempting to make them tessellate. In these definitions $S$ will refer to a space which could be either the Euclidean Plane or the Hyperbolic Plane.

**Definition 5.6.** If $x \in S$ is a point and $H$ is a group of isometries of $S$, the *orbit of $x$ under $H$* is

$$\text{orbit}_H(x) = \{hx | h \in H\}$$

**Definition 5.7.** $x$ and $x'$ are *images* under H if they are in the same orbit.

**Definition 5.8.** A point $p$ in $S$ is a *limit point* of $x$ under $H$ if $\forall U$ open neighborhoods of $y$, $U$ contains an infinite number of images of $x$. This definition implies that if x is a fixed point for $h \in H$ and $h$ has infinite order then $x$ is a limit point.

**Definition 5.9.** A group $H$ acts *discontinuously* in $S$ if $\forall x \in S$ there are no limit points of $x$ in $S$.

**Definition 5.10.** A region $D$ in $S$ is a fundamental region of $S$ under $H$ if $D = D' \cup \partial D'$ where $\partial D'$ is the boundary of $D'$ and the following are true:

(1) Given $x \in D'$ there are no images of $x$ in $D'$.
(2) $\forall x \in S \exists x'$ an image of $x$ in $D$.
(3) $\partial D'$ contains no limit points.

**Lemma 5.11.** *If $T$ is a tessellation and $H$ the associated group of symmetries, $H$ acts discontinuously on $S$.*

*Proof.* Suppose that $H$ does not act discontinuously. Then there is a limit point, $p$ of $x$ in $S$. Let $U$ be a small open neighborhood of $p$. Let $m(U)$ be the number of tiles in T which share points with $U$ and choose $U$ so that $m$ is minimized, that is, if $p$ is in the interior of a tile then $m(U) = 1$, if $p$ is on an edge then $m(U) = 2$ and if p is a vertex then $m(u) = n$ for $n$ a finite integer greater than 2. Because $p$ is a limit point, $U$ contains an infinite number of images of $x$, and because $U$ shares points with only a finite number of tiles, there is at least one tile, $t$, which contains an infinite number of images of $x$. $H$ may now be restricted only to those isometries which are symmetries of t:

$H'_t = \{h \in H | h(t) = t\forall\}$

There are many proofs that $H'_t$ has finite order and clearly any two points $x, x' \in t$ which are images under H are also images under $H'_t$. This generates a contradiction since there cannot be infinitely many images produced by a group of finite order. □

**Theorem 5.12.** *If a polygon $\Pi$ is a fundamental region for a group $\Gamma$ then there is a tessellation of $S$ with $\Pi$.*

This is a very important theorem. Once this fact is established the fundamental question of this paper changes from "Which polygons tessellate the hyperbolic plane?" to "Which polygons can be fundamental regions for subgroups of the isometries of the hyperbolic plane?" In order to prove this theorem we will first prove it for polygons which are fundamental regions for orientation preserving groups and then for all polygons. In order to do this we will use the elements of the orientation preserving group to identify edges of polygons in systematic way to create a hyperbolic surface. Before this will be useful however, we need to show that the hyperbolic plane is a covering space for all hyperbolic surfaces and that the tessellation can be lifted from the surface to the plane.

## 6. Hyperbolic Surfaces, Covering Spaces, and Lifts

All of the proofs and definitions in this section are given for hyperbolic space but most of them can be made (nearly word for word) to apply to Euclidean space as well.

**Definition 6.1.** $S$, a set equipped with a distance function, is a *hyperbolic surface* if for each point $x \in S$, there is an $\epsilon$ such that the disc of radius $\epsilon$ with respect to the distance function on $S$ around $x$ is isometric to a disc in the hyperbolic plane.

**Definition 6.2.** $E$ is a *covering space* of $B$ if there is a map $p : E \to B$ such that $p$ has the following properties:

(1) Every point of $B$ has a open neighborhood $U$ such that $p^{-1}(U)$ can be written as a disjoint union of open sets $U_\alpha$ in $E$. These open sets are called *partitions.*
(2) The restriction of $p$ to one of the open neighborhoods $U_\alpha$ is a homeomorphism. I won't give a formal definition of homeomorphism in this paper.

The goal of this section will be to prove that any hyperbolic surface can be covered by the hyperbolic plane. We will use the $\mathbb{D}^2$ model to show this but it can be done equally well using any other model. This was proven by Hopf in 1925 for Euclidean surfaces and the Euclidean plane and the proof I give here is draws heavily from the presentation of that proof in [1].

**Definition 6.3.** Pencil Map:

The idea behind a pencil map is that because locally the surface is isometric to a region of the hyperbolic plane, if the surface is complete, lines can be extended from one of these isometries. The pencil map will be the natural extension of one of the local maps from $\mathbb{D}^2$ to S. This definition relies on the fact given in lemma 3.2, that a point in $\mathbb{D}^2$ can be given by $\mathbb{D}^2$ coordinates. Because $\mathbb{D}^2$ looks the same everywhere,without loss of generality, we can say that all points on S have neighborhoods isometric to a neighborhood around the origin

If S is a hyperbolic surface the pencil map $p : \mathbb{D}^2 \to S$ is defined as follows. Select a point $O_s \in S$ and an isometry $p : U_\epsilon(O) \to U_\epsilon(O_s)$. For any point z in $\mathbb{D}^2$ p(z) may be found by taking the line segment on S $p(\overline{OP} \cap U_\epsilon)$ and extending it to the distance $|\overline{OP}|$

**Theorem 6.4.** *A pencil map has the following properties which make is a covering map:*

*(1) Each point in $\mathbb{D}^2$ has a neighborhood on which p is an isometry.*
*(2) p is surjective*

These conditions cause $p$ to be a covering map because, since $p$ is onto, each point $s \in S$ has a preimage $p^{-1}(s)$ which is a non empty set of points in $\mathbb{D}^2$. By condition (1) each of these points, $s_i$ has a small neighborhood $U_{s_i}$ on which it is isometric to a small neighborhood $p(U_{s_i})$ of s. This forces all of the points in $p^{-1}(s)$ to be disjoint. Because an isometry is a special case of a homeomorphism, the intersection of the $p(U_{s_i})$'s will be a neighborhood fulfilling the conditions demanded by definition 5.2. It is a bit tricky and technical to verify that the intersection above can always be chosen so that it is not only the point $s$ but this can be done. If $S$ were not complete these conditions would not be enough to cause $p$ to be a covering map but in our case they are.

*Proof.* This proof is long and can be found in [1]. □

**Definition 6.5.** If $p$ is a covering map $p : \mathbb{D}^2 \to S$, $f$ is a map from a space $X$ to $S$ then a *lift* of $f$ is a map $\tilde{f} : X \to \mathbb{D}^2$ such that $p \circ \tilde{f} = f$.

**Lemma 6.6.** *Path lifting lemma*

*If $\gamma$ is a path on a complete hyperbolic surface $S$, $\gamma : [0, 1] \to S$ with $\gamma(0) = s_0$ and $p$ is a covering map $p : \mathbb{D}^2 \to S$, $p(e_0) = s_0$ then there is a unique lift, $\tilde{\gamma}$, of $\gamma$ onto $\mathbb{D}^2$ with $\tilde{\gamma}(0) = e_0$.*

*Proof.* $\tilde{\gamma}$ will be constructed explicitly in steps. Begin by choosing a covering of S by open sets such that each open set fulfills both conditions of definition 6.2 and choose a subdivision $0 = s_0 \leq s_1 \leq \ldots \leq s_n = 1$ such that for each $i$ $f[s_i, s_{i+1}]$ is contained entirely in one of the open sets. Define $\tilde{\gamma}$ as follows: $\tilde{\gamma}(0) = e_0$. Supposing $\tilde{\gamma}(s)$ is defined for $0 \leq s \leq s_i$ then $\tilde{\gamma}(s_i)$ lies in one of the partitions of the open set to which $f[s_i, s_{i+1}]$ belongs, say $U_0$. $\tilde{\gamma}[s_i, s_{i+1}]$ will be defined as $p^{-1}|_{U_0}(f[s_i, s_{i+1}])$.

The reader can verify that $\tilde{\gamma}$ is continuous. The fact that $p \circ \tilde{\gamma} = \gamma$ is clear from construction. It is unique because, if $\tilde{\tilde{\gamma}}$ is another lift of $\gamma$ with $\tilde{\tilde{\gamma}}(0) = e_0$, not identical to $\tilde{\gamma}$ then the two must diverge at some point $t$ such that $s_i < t \leq s_{i+1}$ for some $i$. If $U_0$ is the partition of the open set to which $f[s_i, s_{i+1}]$ belongs which contains $\tilde{\tilde{\gamma}}(s_i)$ , $\tilde{\tilde{\gamma}}[t, s_{i+1}]$ necessarily belongs to a different partition and is therefore not continuous. □

## 7. Polygons Which are Fundamental Regions for Subgroups of $\Gamma_D$

**Definition 7.1.** An *orientation preserving* isometry is one with determinant greater than 0. An isometry which is not orientation preserving is *orientation reversing.*

Specifically, orientation preserving isometries of $\mathbb{H}^2$ are of the form $T(z) = \frac{az+b}{cz+d}$ with $ad - bc = 1$, and the orientation preserving isometries of $\mathbb{D}^2$ are of the form $S(w) = \frac{aw+b}{\bar{b}w+\bar{a}}$ with $|a|^2 - |b|^2 = 1$.

The reader can verify that, as a consequence of Theorem 3.3, orientation preserving isometries are the products of even numbers of reflections while orientation reversing isometries are the odd products of reflections. The composition of two orientation preserving isometries is another orientation preserving isometry while the composition of two orientation reversing isometries is an orientation preserving isometry. I will not verify these facts here because it is easy enough to do and is done in many books for the Euclidean Plane.

It now seems logical to ask which polygons are fundamental regions for only orientation preserving isometries.

**Proposition 7.2.** *If $\Pi$ is a fundamental region for a group $G^+$ of orientation preserving isometries than the following conditions hold:*

(1) *For each side $s_i$ of $\Pi$ there is exactly one other side $s_j, i \neq j$ and one isometry $g \in G^+$ such that $g(s_j) = s_i$. Clearly if this is true then $g^{-1}(s_i) = s_j$ and the sides are matched in pairs. Also, $g(\Pi) \neq \Pi$ and the vertices which are endpoints of $s_i$ and $s_j$ are identified, also in cycles.*
(2) *The sum of the angles in each set of identified vertices is $2\pi/p$ where $p$ is a positive integer*

*Proof.* (1) Let $g_1\Pi \dots g_k\Pi$ be the tiles which share edges or parts of edges with $\Pi$. Supposing $s$ is a shared line segment of $\pi$ and $g_i\Pi$. If $s$ is not an entire edge of $\Pi$ then, by placing vertices at the endpoints of $s$, one edge of $\Pi$ may be subdivided into two or three edges, one of which is $s$. then $g_i^{-1}(s)$ is a shared side $s'$ of $g_i^{-1}\Pi$ and $g_i^{-1}g_i\Pi = \Pi$ thus also a side of $\Pi$. $s'$ is the only side paired with $s$ because, were another side, $s''$ also paired with $s$ via $g_j$, then there are points near $s''$ and $s'$ in the interior of $\Pi$ which are images of each other and $\Pi$ is not a fundamental region. Therefore, each side is paired with another side unless $g_i^{-1}(s)$ is $s$. In this case, because $g_i^{-1}$ is orientation preserving it must be rotation through the center of $s$ by $\pi$ and $s$ may be split by a vertex in the center into $s_1$ and $s_2$ so that $g_i(s_1) = s_2$.
(2) Suppose the vertices of $\Pi$ are $v_1 \dots v_k$. Suppose, without loss of generality, that $v_1$ is one of the end points of $s$ as above and that $g^{-1}v_1$ is one of the endpoints of $g^{-1}(s)$. The vertices of $\Pi$ can be devided into discrete sets of vertices which are identified by the side pairing operations specified in 1 and clearly there will be $k$ or fewer of these sets of one or more vertex. If $(v_1 \dots v_l)$ is the vertex cycle at $(v_1)$ then it is also the vertex cycle at $g^{-1}(v_1)$ and $g^{-1}(v_1)$ is one of the vertices in the cycle. In fact, the vertex cycle induced by the side pairing transformations is the same at each vertex in the cycle. Clearly the sum of these angles evenly divides $2\pi$

A consequence of this is that the group $G^+$ for which $\Pi$ is fundamental region is generated precisely by the isometries associated with side pairing operations. Filling of the plane by $\Pi$ under $G^+$ is insured by the angle conditions which cause the polygons to"fit together" nicely. □

**Proposition 7.3.** *If $\Omega$ is a fundamental region for a group $G$ of isometries which are not all orientation preserving and which acts discontinously, then it is possible to find a polygon $\Pi$ which is a fundamental region of $G^+ \subseteq G$ such that $\Pi$ can be subdivided evenly into copies of $\Omega$.*

This proposition insures that if Theorem 5.12 holds for polygons which are fundamental regions for orientation preserving isometries then it holds for all polygons which are fundamental regions.

*Proof.* Assume that $G$ is generated by isometries $g_1 \dots g_n$ and that only $g_1$ is orientation reversing. It may be assumed without loss of generality that $g_i(\Omega), 0 \leq i \leq n$ shares a line segment with $\Omega$. The first step is to add necessary vertices to insure that each $g_i(\Omega)$ shares a full side with $\Omega$, although because not all the isometries are orientation preserving we cannot ensure that no side is paired with itself. Define $\Pi$ as the union $\Omega \cap g_1\Omega$. $\Pi$ is then a fundamental region for the group $G^+$ generated by $(g_1^2), g_2 \dots g_n$: The group $G$ is the disjoint union of $G^+$ and $g_1G^+ = \{g_1h | h \in G^+\}$ Therefore every image of $\Omega$ in the plane is of the form $g\Omega$ where $g$ is either in $G^+$ or $g_1G^+$.

All the conditions of fundamental region are therefore fulfilled unless $\Omega \cap g_1\Omega$ is not a polygon. Of the conditions in definition 5.1 only the third could be violated since both $\Omega$ and $g_1\Omega$ are polygons the other two conditions are immediate. A violation of the third criterion implies that the boundary of $\Pi$ is not a simple closed curve. The polygonal Jordan curve theorem says that if $\partial\Pi$ is not a simple closed curve, then $\Pi$ separates a polygonal region $K$ from the rest of the plane. In this case, there must by an image of $\Pi$ contained completely within this region which by the same logic, separates a subregion of $K$ from the rest of $K$. This can be continued ad infinatum, implying the existence of a limit point in $K$. This contradicts the fact that $\Omega$ is a fundamental region which by lemma 5.11 means that $G$ acts discontinuously on the hyperbolic plane. □

It now remains only to show that Theorem 5.12 holds for polygons with the properties outlined in proposition 7.2. We will do this by stitching together the polygons generated by the group $G^+$ of orientation preserving isometries to build a hyperbolic surface. Using the lifting lemma, this tessellation can be lifted to the hyperbolic plane. The two delicate parts of this proof are i) verifying that we have created a hyperbolic surface and ii) verifying that the lift of the tessellation to $\mathbb{D}^2$ (or $\mathbb{H}^2$) is a tessellation in the hyperbolic plane and that it is a tessellation of the same polygon we began with. Because of these delicacies, we must proceed carefully.

**Theorem 7.4.** *Theorem 5.12 holds for polygons,* $\Pi$ *satisfying the conditions listed in Proposition 7.2.*

This Theorem will be proven in two parts

(1) Construct a hyperbolic surface, S as follows: Without loss of generality, a generating set $(g_1 \dots g_n)$ may be chosen for the group $G^+$ under which $\Pi$ is a fundamental region so that each $g_i$ is an isometry associated with a side-pairing transformation. The sides of $\Pi$ may also be numbered $s_1 \dots s_{2n}$ so that $g_i(s_i) = s_{n+i}$ and $g_i^{-1}(s_{n+i}) = s_i$. Identify the edges and vertices of each polygon $h\Pi$ so that two points respectively on the boundaries of $h_1\Pi$ and $h_2\Pi$ are the same, $p_{h_1\Pi} = p_{h_2\Pi}$ if for some $i$, $g_i(p_{h_1\Pi}) = p_{h_2\Pi}$. That is the polygons generated by $G^+$ are sewn together exactly as you would expect them to be.

*Proof.* In order to prove this we must verify that each point in $S$ has a neighborhood $U$ which is isomorphic to a neighborhood of the hyperbolic plane. Clearly any point on the interior of a polygon has such a neighborhood. A neighborhood around a point on a line segment which is not a vertex can be expressed as the union of two open half-discs which are isomorphic to open half-discs in the hyperbolic plane and a line segment isomorphic to a segment of a geodesic in the hyperbolic plane with the same length as the diameter of the two half-discs. An open neighborhood in the hyperbolic plane can be expressed in the same way, so there is an isometry between the two. It really only remains to verify that vertices have such neighborhoods.

Suppose $V$ is a vertex in $S$ with vertex cycle $v_1, \dots v_l$ induced by the side-pairing transformations. This vertex cycle is related to the cycle of side-pairing isometries $g_{i1} \dots g_{il}$ such that $g_{ij}$ identifies $v_j$ with $v_{j+1}$ and

$g_{il}$ identifies $v_l$ with $v_1$. Let $\theta_j$ be the angle at $v_j$. The cyclic nature of the vertices means that:

$$g_l(g_{l-1}(\ldots g_2(g_1(\Pi)))) = r_{\sum_{j=1}^{l} \theta_j}(\Pi)$$

where $r_\theta$ is rotation through the angle theta. We know from the angle conditions that $\sum_{j=1}^{l} = 2\pi/p$, where $p$ is a positive integer. This means that $(g_l \ldots g_1)^p = 1$. So after exactly $p$ repetitions of the vertex cycle, the polygons close up and the same logic which was used to prove that points on the boundary have neighborhoods isometric to neighborhoods in $\mathbb{D}^2$.

Observe that $S$ is complete. □

(2) The tessellation of $S$ may be lifted to $\mathbb{D}^2$ to form a tessellation with the polygon $\Pi$.

*Proof.* By the path lifting lemma, there is a unique lift of the system of paths which makes up the tessellation on $S$. It is enough to show that the lift of the boundary of $\Pi$ and the interior of that region is isomorphic to $\Pi$ and that there is only one polygon in the pre-image of each polygon in S.

In order to prove the first statement, it is enough to observe the lift at one vertex of $\Pi$, say $V$, which, without loss of generality, can be assumed to $p(O)$ where $p$ is the pencil map from $\mathbb{H}^2$ to $S$. In $S$ there are two sides of $\Pi$ which meet at $V$ and the pre-image of these sides confined to a small neighborhood around the origin for which $p$ is an isometry, necessarily has the same angle as the two sides in $S$. The line segments which make up the polygons in $S$ are, by construction, isomorphic to polygons in $\mathbb{D}^2$, so the lifts of their line segments will be geodesics in $\mathbb{D}^2$. From section 3, we know that the angles of a polygon uniquely determine the polygon in the hyperbolic plane, so the lift of $\Pi$ is isomorphic to $\Pi$.

Now we examine the lift of the polygon $g\Pi$ for some $g \in G^+$. $g$ can be expressed as the composition of generators $g_{i1}^{\pm1} g_{i2}^{\pm1} \ldots g_{in}^{\pm1}$. These generators determine a path from $\Pi$ to $g\Pi$ in $S$. The lift of this path to $\mathbb{D}^2$ must terminate in the location of the lift of $g\Pi$. Because each point in $\mathbb{D}^2$ is determined uniquely by the geodesic from itself to the origin, there may only be one image of $g\Pi$ in $\mathbb{D}^2$. Furthermore $g(\text{lift}\Pi) = \text{lift}(g\Pi$, using the same path previously mentioned. □

As a result of this theorem and proposition 7.2, we now know not only that there is a correlation between being a fundamental region and tessellating the hyperbolic plane, but we also know what kinds of polygons are fundamental regions. What follows is one example of tessellations of the hyperbolic plane.

**Example 7.5.** The image in figure 5 is a tessellation of the disc model with triangles which have angles $\pi/2$, $\pi/3$, and $\pi/7$. This triangle is a fundamental region for the group generated by reflections across the sides of the triangle. A consequence of the angle condition from proposition 7.2 is that all triangles which tessellate have angles $\pi/p$, $\pi/q$, $\pi/r$ where $p, q, r$ are integers such that the sum of the three angles is less than $\pi$.

## References

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