The Grassmannian as a Projective Variety

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Abstract

This paper introduces the Grassmannian and studies it as a subspace of a certain projective space. We do this via the Plücker embedding and give specific coordinates for the image of the Grassmannian. The main result will be to show that under the Plücker embedding, the Grassmannian is a projective variety. We will do this in two ways: first, through a characterization of totally decomposable vectors, and secondly, through the Plücker relations. This will require a fair amount of linear and multilinear algebra, however most of the facts to be used will be proven when needed.

1 Prerequisites and Basic Definitions

First we will establish some conventional language: let k be an algebraically closed field, and let $k[x_1, \ldots, x_n]$ be the polynomial ring in n variables, hereafter denoted by k[X]. We define n-dimensional affine space, $\mathbb{A}^n$, to be k^n considered just as a set without its natural vector space structure. Given $f \in k[X]$, we can view f as a k-valued polynomial on affine space by evaluation, $f : (x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$. The basic objects we will be studying are a generalization of the elementary notion of a zero locus of a collection of polynomials. More precisely,

Definition 1.1 Given a subset $S \subset k[X]$, let $V(S) = \{X \in \mathbb{A}^n | f(X) = 0$ for all $f \in S\}$. If $W \subset \mathbb{A}^n$ is such that $W = V(S)$ for some $S \subset k[X]$, we say that W is an *affine variety*.

Given an affine variety W, there is in general more than one ideal $\mathfrak{I}$ for which $W = Z(\mathfrak{I})$, however, there is a unique largest ideal corresponding to W found by taking *all* the polynomials which vanish on W.

Definition 1.2 Given an affine variety $W \subset \mathbb{A}^n$, let $I(W) = \{f \in k[X] | f(a_1, \ldots, a_n) = 0$ for all $(a_1, \ldots, a_n) \in W\}$. This set is called the ideal of the variety W.

Note that $I(W)$ is indeed an ideal in $k[X]$ and it contains every other ideal on which W vanishes. We now introduce another type of space essential for our study of the Grassmannian. We define n-dimensional *projective space*, $\mathbb{P}^n$, to be the quotient of $\mathbb{A}^{n+1} \setminus 0$ by the action of $k^\times$ on $\mathbb{A}^{n+1}$ by multiplication, that is, we make the identification $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ for all nonzero $\lambda \in k$. This induces a coordinate system on the resulting quotient space $\mathbb{P}^n$ called *homogeneous coordinates.* A point in $\mathbb{P}^n$ is denoted by $[a_0 : \cdots : a_n]$, and by the definition of this space, $[a_0 : \cdots : a_n] = [\lambda a_0 : \cdots : \lambda a_n]$ for all nonzero $\lambda \in k$. We can view the identification $\sim$ geometrically by observing that two points in $\mathbb{A}^n$ are identified if and only if they lie on the same line through the origin. This allows us to view $\mathbb{P}^n$ as the space of lines in $\mathbb{A}^{n+1}$ through the origin. This construction can be generalized to any finite dimensional vector space V to form a projective space $\mathbb{P}(V)$.

In general, a polynomial $f \in k[x_0, \ldots, x_n]$ is not a function on $\mathbb{P}^n$, for $f(a_0, \ldots, a_n)$ need not equal $f(\lambda a_0, \ldots, \lambda a_n)$. For this reason, there is no well-defined notion of a zero locus of a set of such general polynomials on $\mathbb{P}^n$. Thus, on projective space, we must restrict our attention to a more specific collection of polynomials, namely the homogeneous polynomials.

Definition 1.3 A homogeneous polynomial of degree m is a polynomial $f \in k[X]$ such that $f(\lambda a_1, \ldots, \lambda a_n) = \lambda^m f(a_1, \ldots, a_n)$ for all $\lambda \in k^\times$.

Again, homogeneous polynomials are not always well-defined on $\mathbb{P}^n$ (unless they are of degree 0). The benefit of homogeneous polynomials is that sets of such polynomials have a well-defined zero locus.

Definition 1.4 A *projective variety* is a subset $W \subset \mathbb{P}^n$ such that $W = V(S)$ for some collection of homogeneous polynomials $S \subset k[X]$.

2 Grassmannians and the Plücker Embedding

Let V be a finite dimensional vector space over a field k, say of dimension n. Generalizing what we saw above for $\mathbb{P}^n$, we can view the projective space $\mathbb{P}(V)$ as the set of all lines in V that pass through the origin, that is, the set of all one dimensional subspaces of the n-dimensional vector space V. We could also view this space as follows. To every hyperplane in V we can associate a unique line through the origin in V^*, and conversely, to each line through the origin in V^* there corresponds a unique hyperplane in V. With this correspondence and the fact that $V^* \cong V$, we can view $\mathbb{P}^n$ as the set of $n-1$ dimensional subspaces of V. One could generalize this further and consider the space of all d-dimensional subspaces of V for any $1 \leq d \leq n$. This idea leads to the following definition.

Definition 2.1 Let $n \geq 2$ and consider the k-vector space V of dimension n. For $1 \leq d \leq n$, we define the Grassmannian $G_{d,V}$ to be the space of d-

dimensional vector subspaces of V.

If we make the identification $V \simeq k^n$ by choosing a basis for V, we denote the Grassmannian by $G_{d,n}$. Since n-dimensional vector subspaces of k^n are the same as $n-1$-dimensional vector subspaces of P^{n-1}, we can also view the Grassmannian as the set of $d-1$-dimensional planes in $P(V)$. Our goal is to show that the Grassmannian $G_{d,V}$ is a projective variety, so let us begin by giving an embedding into some projective space. Recall that the exterior algebra of V, $\bigwedge(V)$, is the quotient of the tensor algebra $\mathcal{T}(V)$ by the ideal generated by all elements of the form $v \otimes v$, where $v \in V$. Multiplication in this algebra (the so called wedge product) is alternating, that is, $v_1 \wedge \cdots \wedge v_m = 0$ whenever $v_i = v_{i+1}$ for any $1 \leq i < m$. Furthermore, the wedge product is anticommutative on simple tensors in the sense that $v \wedge w = -w \wedge v$ for all $v, w \in V$. We will need the following lemma to proceed.

Lemma 2.2 Let W be a subspace of a finite dimensional k-vector space V, and let $\mathcal{B}_1 = \{w_1, \ldots, w_d\}$ and $\mathcal{B}_2 = \{v_1, \ldots, v_d\}$ be two bases for W. Then $v_1 \wedge \cdots \wedge v_d = \lambda w_1 \wedge \cdots \wedge w_d$ for some $\lambda \in k$

Proof: Write $w_j = a_{1j}v_1 + \cdots + a_{dj}v_d$. Then one can compute that

$$w_1 \wedge \cdots \wedge w_d = (a_{11}v_1 + \cdots + a_{m1}v_d) \wedge \cdots \wedge (a_{1d}v_1 + \cdots + a_{dd}v_d)$$
$$= \sum_{\sigma \in S_d} \epsilon(\sigma) a_{1\sigma(1)} \cdots a_{d\sigma(d)} v_1 \wedge \cdots \wedge v_d$$

where $\epsilon(\sigma)$ is the sign of σ. Notice that $\sum_{\sigma \in S_d} \epsilon(\sigma) a_{1\sigma(1)} \cdots a_{d\sigma(d)} := \lambda$ is just the determinant of the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$.

We now define a map $p : G_{d,V} \to \mathbb{P}(\bigwedge^d(V))$: Given a subspace $W \in G_{d,V}$ and a basis $\{w_1, \ldots, w_d\}$ of W, let $p : W \mapsto w_1 \wedge \cdots \wedge w_d$. Clearly, different choices of basis for W give different wedge products in $\bigwedge^d(V)$, but Lemma 2.2 shows that this map is unique up to scalar multiplication, hence is well-defined on $\mathbb{P}(\bigwedge^d(V))$.

Proposition 2.3 $p : G_{d,V} \to \mathbb{P}(\bigwedge^d(V))$ is injective.

Proof: Define a map $\varphi : \mathbb{P}(\bigwedge^d(V)) \to G_{d,V}$ as follows. To each $[\omega] \in \mathbb{P}(\bigwedge^d(V))$, let $\varphi([\omega]) = \{v \in V | v \wedge \omega = 0 \in \bigwedge^{d+1}(V)\}$. We will show that $\varphi \circ p = id$. Let $W \in G_{d,n}$ have basis $\{w_1, \cdots, w_d\}$ so that $[w_1 \wedge \cdots \wedge w_d] = p(W)$. Then for each $w \in W$, it is clear that $w \wedge w_1 \wedge \cdots \wedge w_d = 0$, so $W \subset \varphi \circ p(W)$. Moreover, if $v \in \varphi \circ p(W)$, $v \wedge w_1 \cdots \wedge w_d = 0$. Extend the linearly independent set $\{w_1, \ldots, w_d\}$ to a basis $\{w_1, \ldots, w_n\}$ for V. Then writing $v = \sum a_i w_i$, we see that $(\sum a_i w_i) \wedge w_1 \wedge \cdots \wedge w_n = 0$. After distributing and using the properties of the wedge product, one sees that all the $a_i = 0$ for $i > d$ and thus $v = a_1 w_1 + \cdots + a_d w_d$. Therefore, $v \in W$ and $\varphi \circ p(W) \subset W$, completing the proof that $\varphi \circ p = id$.

The map p in the previous proposition is known as the *Plücker embedding*, and allows us to view the Grassmannian as a subset of the projective space $\mathbb{P}(\bigwedge^d(V))$. Furthermore, we can identify this space with some $\mathbb{P}^N$ as follows. Fix a basis $\{v_1, \ldots, v_n\}$ for V, and consider the set $\{v_{i_1} \wedge \cdots \wedge v_{i_d} | 1 \leq i_1 < \cdots < i_d \leq n\}$. This set forms a basis for $\bigwedge^d(V)$, which shows $\bigwedge^d(V)$ is a vector space of dimension $\binom{n}{d}$. Thus we can let $N = \binom{n}{d} - 1$ and embed the Grassmannian in $\mathbb{P}^N$.

Let $I_{d,n} = \{\bar{i} = (i_1, \ldots, i_d) | 1 \leq i_1 < \cdots < i_d \leq n\}$ and index the coordinates of $\mathbb{P}^N$ by $I_{d,n}$. More specifically, let the basis vector of $\mathbb{P}^N$ indexed by $\bar{i} = (i_1, \ldots, i_d)$ be $v_{i_1} \wedge \cdots \wedge v_{i_d}$. Given a subspace $W \in G_{d,V}$, we will now explicitly find $p_{\bar{i}}(W)$, that is, the $\bar{i}^{th}$ coordinate of the image of the Grassmannian under the Plücker embedding. Choose a basis $\{w_1, \ldots, w_d\}$ for the subspace W. We can then write each w_j in terms of the basis vectors for V (which were chosen above) as $w_j = a_{1j}v_1 + \cdots + a_{nj}v_n$. Define an $n \times d$ matrix M_W by $M_W = (a_{ij})$. Note that the j^{th} column of M_W will then be the coordinates of w_j. Then $p : W \mapsto [w_1 \wedge \cdots \wedge w_d]$ and one can compute that

$$\begin{aligned} w_1 \wedge \cdots \wedge w_d &= (a_{11}v_1 + \cdots + a_{n1}v_n) \wedge \cdots \wedge (a_{1d}v_1 + \cdots + a_{nd}v_n) \\ &= \sum_{\bar{i} \in I_{d,n}} \sum_{\sigma \in S_d} \epsilon(\sigma) a_{i_1\sigma(1)} \cdots a_{i_d\sigma(d)} v_{\bar{i}} \end{aligned}$$

where $\epsilon(\sigma)$ denotes the sign of the permutation σ. Observe that the $\bar{i}^{th}$ coordinate for $p(W)$ is $p_{\bar{i}} = \det(M_{\bar{i}})$ where $M_{\bar{i}}$ is the $d \times d$ submatrix formed from the $i_1, \ldots, i_d^{th}$ rows of M_W. Some care must be taken regarding the uniqueness of these coordinates; recall that a different choice of basis would have mapped to a scalar multiple of $w_1 \wedge \cdots \wedge w_d$ by Lemma 2.2. However, if we view these coordinates as homogeneous, the Plücker coordinates are well-defined on $\mathbb{P}^N$. Thus, we have shown the following proposition.

Proposition 2.4 The $\bar{i}^{th}$ homogeneous coordinate for $p(W) \in \mathbb{P}^N$ is given by the corresponding $d \times d$ minor of M_W: $\det(M_{\bar{i}})$.

3 The Grassmannian and Decomposable Vectors

While we have seen that we can embed the Grassmannian in projective space, we wish to show in this section that it is actually a projective variety. In order to show this result, we will first need some results on divisors of vectors.

Definition 3.1 Let V be a finite dimensional vector space, $v \in V$, and ω a multivector in $\bigwedge^d(V)$. Then v is said to *divide* ω if there exists $\varphi \in \bigwedge^{d-1}(V)$ such that $\omega = v \wedge \varphi$.

Lemma 3.2 Let V, v, and ω be as above. Then v divides ω if and only if $v \wedge \omega = 0$.

Proof: ($\Rightarrow$) is clear. To show ($\Leftarrow$), choose a basis $\{v_1, \ldots, v_n\}$ of V where $v_1 = v$. Write the basis vector $v_{i_1} \wedge \cdots \wedge v_{i_d}$ for $\bigwedge^d(V)$ as $v_{\bar{i}}$, where $\bar{i} = (i_1, \ldots, i_d) \in I_{d,n}$. Then $\omega = \sum_{\bar{i} \in I_{d,n}} a_{\bar{i}} v_{\bar{i}}$ for some $a_{\bar{i}} \in k$. By assumption,

$$v \wedge \left(\sum_{\bar{i} \in I_{d,n}} a_{\bar{i}} v_{\bar{i}} \right) = \sum_{\bar{i} \in I_{d,n}} a_{\bar{i}} (v \wedge v_{\bar{i}}) = 0$$

Because all of the vectors of the form $v \wedge v_{\bar{i}}$ for which $i_1 > 1$ are independent, we have all the $a_{\bar{i}} = 0$ for which $i_1 > 1$. This shows that ω is a linear combination of basis vectors of the form $v_{\bar{i}} = v \wedge v'_{\bar{i}}$ where $i_1 = 1$ in $\bar{i}$, and this allows us to write $\omega = \sum a'_{\bar{i}} (v \wedge v'_{\bar{i}}) = v \wedge (\sum a'_{\bar{i}} v'_{\bar{i}})$.

The set of all $v \in V$ dividing $\omega \in \bigwedge^d(V)$ forms a subspace of V, which we will denote by D_ω.

Definition 3.3 Let $\omega \in \bigwedge^d(V)$. We say that ω is *totally decomposable* if we can write $\omega = v_1 \wedge \cdots \wedge v_d$ where $\{v_1, \ldots, v_d\} \subset V$ is linearly independent.

As we will later see, the image of the Grassmannian under the Plücker embedding can be expressed in terms of totally decomposable vectors. The following proposition characterizes the totally decomposable vectors in terms of their spaces of divisors.

Proposition 3.4 A multivector $\omega \in \bigwedge^d(V)$ is totally decomposable if and only if $\dim(D_\omega) = d$.

Proof: Suppose ω is totally decomposable, say $\omega = v_1 \wedge \cdots \wedge v_d$. The space of all vectors that divide ω, D_ω, is precisely $\{v \in V | v \wedge v_1 \wedge \cdots \wedge v_d = 0\}$ by Lemma 3.2. Clearly $v_1, \ldots, v_d$ are d linearly independent elements of D_ω. Also, any vector $v \in D_\omega$ can be written as a linear combination of the v_i. To see this, extend the set of $\{v_i\}_{i=1}^d$ to a basis of V, say $\{v_1, \ldots, v_n\}$. Then we can write $v = \sum_{i=1}^n a_i v_i$ and we must have

$$0 = v \wedge \omega = \left(\sum_{i=1}^n a_i v_i \right) \wedge v_1 \wedge \cdots \wedge v_d.$$

All the terms with $i \leq d$ will vanish after distributing, so all the $a_i = 0$ for $d < i \leq n$. So v is a linear combination of the $\{v_1, \cdots, v_d\}$, which shows they span D_ω and also form a basis. Conversely, suppose $\dim(D_\omega) = d$ and let $\{v_1, \ldots, v_d\}$ be a basis. Again, we can extend this to a basis $\{v_1, \ldots, v_n\}$ for V. Write the basis elements of $\bigwedge^d(V)$ as in the proof of Lemma 3.2. Then $\omega = \sum_{\bar{i} \in I_{d,n}} a_{\bar{i}} v_{\bar{i}}$ for some $a_{\bar{i}} \in k$. Since $v_j \wedge \omega = 0$ for all $1 \leq j \leq d$, we have

$$0 = v_j \wedge \left(\sum_{\bar{i} \in I_{d,n}} a_{\bar{i}} v_{\bar{i}} \right) = \sum_{\bar{i} \in I_{d,n}} a_{\bar{i}} (v_j \wedge v_{\bar{i}})$$

This equation shows that for all the $\bar{i} = (i_1, \ldots, i_d)$ for which j does not appear as any of the $i_1, \ldots, i_d$, we must have $a_{\bar{i}} = 0$ since $v_j \wedge v_{\bar{i}} \neq 0$ in this case. This holds for all $1 \leq j \leq d$, so all the $a_{\bar{i}} = 0$ for which $\{i_1, \ldots, i_d\} \neq \{1, \ldots, d\}$. This shows that $\omega = a' v_1 \wedge \cdots \wedge v_d$ for some scalar a', and thus, ω is decomposable.

For $\omega \in \bigwedge^d(V)$, define a map $\varphi(\omega) : V \to \bigwedge^{d+1}(V)$ by $\varphi(\omega)(v) = \omega \wedge v$. We can further classify the totally decomposable vectors in terms of this map.

Corollary 3.5 ω is totally decomposable if and only if $\varphi(\omega)$ has rank $n - d$, or equivalentaly, if and only if $\ker(\varphi(\omega))$ has dimension d.

Proof: Observe that $\ker(\varphi(\omega))$ is precisely D_ω. The result then easily follows from Proposition 3.4.

Furthermore, the following proposition shows that the rank of $\varphi(\omega)$ is never strictly less than $n-d$ and allows us to strengthen the statement of Corollary 3.5.

Corollary 3.6 $\omega \in \bigwedge^d(V)$ is totally decomposable if and only if $\varphi(\omega)$ has rank $\leq n - d$.

Proof: By corollary 3.5, it suffices to show that $\varphi(\omega)$ never has rank $< n - d$. Suppose it did, so that the dimension of $\ker(\varphi(\omega))$ was $l > d$. The argument for $(\Leftarrow)$ direction of Proposition 3.4 shows that we can then decompose ω as $\lambda v_1 \wedge \cdots \wedge v_l$ for some scalar λ, contradicting the assumption that $\omega \in \bigwedge^d(V)$.

We will also need the following result from linear algebra.

Lemma 3.7 The rank of a matrix $M \in M_{n \times m}(k)$ is the largest integer r such that some $r \times r$ minor does not vanish.

Proof: Let r be as in the proposition, and let $\operatorname{rank}(M)$ be ρ. Since there is a nonzero $r \times r$ minor of M, the columns of M used in this minor must be linearly independent, and thus, $\rho \geq r$. To prove the other inequality, let M' be the submatrix of M consisting of ρ linearly independent columns of M. M' also has rank ρ, and so it has row rank ρ. Therefore, the rows of M' span k^ρ and must contain a basis. Taking these r rows of M' for this basis then gives a nonzero

$\rho \times \rho$ minor of M, showing that $\rho \leq r$.

We are now in a position to show that the Grassmannian is a projective variety through the following identification.

Lemma 3.8 $[\omega] \in \mathbb{P}(\bigwedge^d V)$ lies in the image of the Grassmannian under the Plücker embedding if and only if ω is totally decomposable.

Proof: If ω is totally decomposable as $\omega = v_1 \wedge \cdots \wedge v_d$, then the subspace of V spanned by $\{v_1, \ldots, v_d\}$ is d-dimensional, hence is some $U \in G_{d,V}$ and $p(U) = [\omega]$. Conversely, suppose $[\omega] = p(U)$ for some $U \in G_{d,V}$. Choose a basis $\{u_1, \ldots, u_d\}$ for U. Then $[\omega] = [u_1 \wedge \cdots \wedge u_d]$, so ω is totally decomposable as $\lambda u_1 \wedge \cdots \wedge u_d$ for some scalar λ.

Theorem 3.9 $p(G_{d,V}) \subset \mathbb{P}(\bigwedge^d(V))$ is a projective variety.

Proof: The map $\varphi : \bigwedge^d(V) \to Hom(V, \bigwedge^{d+1}(V))$ sending $\omega \mapsto \varphi(\omega)$ is easily seen to be linear. For $\omega \in \bigwedge^d(V)$, we view $\varphi(\omega) \in Hom(V, \bigwedge^{d+1}(V))$ as an $n \times \binom{n}{d+1}$ matrix where the entries are functions of ω. The linearity of φ implies that $\varphi(\lambda\omega) = \lambda\phi(\omega)$ and shows that these functions are homogeneous of degree one. By Corollary 3.6 and Lemma 3.8, a particular $[\omega']$ lies in $p(G_{d,V})$ if and only if $\varphi(\omega')$ has rank $\leq n-d$. We must now show that $\varphi(\omega')$ has rank $\leq n-d$ if and only if all of its $(n-d+1) \times (n-d+1)$ minors vanish. This follows from Lemma 3.8, for if all all the $n-d+1$ minors vanish it follows that $\text{rank}(\varphi(\omega')) \leq n-d$. Conversely, if $\varphi(\omega')$ has rank $r \leq n-d$, then by the maximality of r in the Lemma, all the $n-d+1$ minors must vanish. Therefore, a point $[\omega']$ lies in $p(G_{d,V})$ if and only if all of the $n-d+1$ minors of $\varphi(\omega')$ vanish, that is, if ω' is in the zero locus of the $n-d+1$ minors of the matrix $\varphi(\omega)$.

4 The Plücker Relations

The homogeneous polynomials found above have one drawback: they do not generate the homogeneous ideal of the Grassmannaian. In order to find a set of polynomials that do generate this ideal, we must introduce the Plücker relations. First we will need a general result about the exterior algebra.

Proposition 4.1 Let V be a vector space of dimension n and let V^* be its dual space. Then for any $0 < d < n$, $\bigwedge^d(V) \cong \bigwedge^{n-d}(V^*)$.

Proof: Consider the nondegenerate pairing $\bigwedge^d(V) \times \bigwedge^{n-d}(V) \to \bigwedge^n(V)$ given by $\omega \times \sigma \mapsto \omega \wedge \sigma$. Since V is of dimension n, we can find an isomorphism $\eta : \bigwedge^n(V) \to k$. Thus, we can define a map $\Psi : \bigwedge^d(V) \to (\bigwedge^{n-d}(V))^*$ which takes $\omega \in \bigwedge^d(V)$ to the map $\omega^* : \bigwedge^{n-d}(V) \to k$ defined by $\sigma \mapsto \eta(\omega \wedge \sigma)$, and one can check that this is an isomorphism. If we also make the identification $(\bigwedge^d(V))^* = \bigwedge^d(V^*)$, this gives us the desired isomorphism of $\bigwedge^d(V)$ with

$\bigwedge^{n-d}(V^*)$.

It must be pointed out that the isomorphism in the above proposition is only unique up to a scalar multiple as it depends on our choice of η, which is non-canonical. We can explicitly compute the action of this isomorphism on a totally decomposable $\omega = w_1 \wedge \cdots \wedge w_d \in \bigwedge^d(V)$ for a given choice of η, as we will now do. Extend the set $\{w_1, \ldots, w_d\}$ to a basis $\{w_1, \ldots, w_n\}$ for V. Then the unique top dimensional basis element in $\bigwedge(V)$ will be $w_1 \wedge \cdots \wedge w_n$, and the isomorphism $\eta : \bigwedge^n(V) \to k$ is completely determined by the element in k to which we send this top form. Say $\eta(w_1 \wedge \cdots \wedge w_n) = \lambda \in k$. The following lemma gives a formula for ω^*, the image of ω under the above isomorphism.

Lemma 4.2 Let $\omega = w_1 \wedge \cdots \wedge w_d \in \bigwedge^d(V)$ and extend the set $\{w_1, \ldots, w_d\}$ to a basis $\{w_1, \ldots, w_n\}$ for V. Let $\{w_1^*, \ldots, w_n^*\}$ be the dual basis for V^*. Then under the isomorphism in Proposition 4.1, $\omega \mapsto \omega^*$ where $\omega^* = \lambda\, w_{d+1}^* \wedge \cdots \wedge w_n^*$ and λ is defined as above.

Proof: For $1 \leq i_1, \ldots i_d \leq n$, we have

$$\omega \wedge w_{i_1} \wedge \cdots \wedge w_{i_d} = \begin{cases} 0 & \text{if some } i_j \in \{1, \ldots, d\} \\ \pm w_1 \wedge \cdots \wedge w_n & \text{otherwise} \end{cases}$$

Notice that

$$[\lambda\, w_{d+1}^* \wedge \cdots \wedge w_n^*](w_{i_1} \wedge \cdots \wedge w_{i_d}) = \begin{cases} 0 & \text{if some } i_j \in \{1, \ldots, d\} \\ \pm\lambda & \text{otherwise} \end{cases}$$

is precisely $\eta(\omega \wedge w_{i_1} \wedge \cdots \wedge w_{i_d})$. Furthermore, from the construction of the isomorphism in Proposition 4.1 we see that $\eta(\omega \wedge w_{i_1} \wedge \cdots \wedge w_{i_d}) = \omega^*(w_{i_1} \wedge \cdots \wedge w_{i_d})$, and therefore, it follows that $\omega^* = \lambda\, w_{d+1}^* \wedge \cdots \wedge w_n^*$.

Let $\psi : (\bigwedge^{n-d}(V))^* \to Hom(V, (\bigwedge^{n-d+1}(V))^*)$ be defined as follows. For each $\omega^* \in \bigwedge^{n-d}(V^*)$, $\psi_{\omega^*} : V^* \to \bigwedge^{n-d+1}(V^*)$ is the map sending $v^* \mapsto v^* \wedge \omega^*$. We have the following analogue to Corollary 3.6 for $\psi(\omega^*)$, the proof of which we omit as it is similar to what was shown in Section 3 above.

Lemma 4.3 For any $\omega \in \bigwedge^d(V)$, let ω^* be the image of ω under the isomorphism in Proposition 4.1. Then $\psi(\omega^*)$ has rank $\geq d$ with equality if and only if ω is totally decomposable.

Before proceeding with the development of the Plücker relations, we will pause to review some linear algebra.

Definition 4.4 We define a pairing $\langle \cdot, \cdot \rangle : V^* \otimes V \to k$ by $\langle w^*, v \rangle = w^*(v)$ for all $v \in V$, $w^* \in V^*$. We will also write $\langle v, w^* \rangle$ to denote this pairing.

This pairing is a nondegenerate bilinear form. For instance, to see bilinearity in the second coordinate, $\langle w^*, av + bv' \rangle = w^*(av + bv') = aw^*(v) + bw^*(v')$ by the linearity of w^*. Linearity of $\langle \cdot, \cdot \rangle$ in the first coordinate is an even more trivial computation. That it is nondegenerate is also easy to see: if $\langle w^*, v \rangle = 0$ for all $v \in V$ then w^* has to be the zero map. Also, if $\langle w^*, v \rangle = 0$ for all $w^* \in V^*$, then if we choose a basis $\{v_1, \dots, v_n\}$ for V and write $v = \sum a_i v_i$, then in particular, every element of the dual basis for V^* evaluates to 0 at v, that is, $v_i^*(v) = a_i = 0$, and it follows that $v = 0$.

Definition 4.5 Let $a : V \to W$ be linear and suppose $v \in V$ and $w^* \in W^*$. The map ${}^t a : W^* \to V^*$ defined by $\langle {}^t a(w^*), v \rangle = \langle w^*, a(v) \rangle$ is called the *transpose* of a.

Since the maps $\varphi(\omega)$ and $\psi(\omega)$ defined above are linear, we can also define their transposes ${}^t\varphi(\omega) : \bigwedge^{d+1}(V^*) \to V^*$ and ${}^t\psi(\omega) : \bigwedge^{n-d+1}(V) \to V$. For each $\alpha \in \bigwedge^{d+1}(V^*)$ and $\beta \in \bigwedge^{n-d+1}(V)$, let

$$\Xi_{\alpha,\beta}(\omega) = \langle {}^t\varphi(\omega)(\alpha), {}^t\psi(\omega^*)(\beta) \rangle.$$

An expression of the form $\Xi_{\alpha,\beta}(\omega) = 0$ is called a Plücker relation.

Theorem 4.6 $\omega \in \bigwedge^d(V)$ is totally decomposable if and only if $\Xi_{\alpha,\beta}(\omega) = 0$ for every $\alpha \in \bigwedge^{d+1}(V^*)$ and $\beta \in \bigwedge^{n-d+1}(V)$.

Proof: ($\Rightarrow$) Suppose ω is totally decomposable, say as $\omega = w_1, \dots, w_d$. Let $W = \langle w_1, \dots, w_d \rangle$ and choose a basis $\{v_1^*, \dots, v_{n-d}^*\}$ for $W^\perp$. By Lemma 4.2, $\chi(\omega) = \omega^*$ is of the form $\omega^* = v_1^* \wedge \cdots \wedge v_{n-d}^*$, where $\{v_1, \dots, v_{n-d}\}$ forms a basis for $W^\perp$. We then have,

$$\begin{aligned}
&\Xi_{\alpha,\beta}(\omega) = 0 \text{ for all } \alpha, \beta \\
&\qquad \Leftrightarrow \langle {}^t\varphi(\omega)(\alpha), {}^t\psi(\omega^*)(\beta) \rangle = 0 \text{ for all } \alpha, \beta \\
&\qquad \Leftrightarrow \langle \alpha, \varphi(\omega){}^t\psi(\omega^*)(\beta) \rangle = \langle \alpha, \omega \wedge {}^t\psi(\omega^*)(\beta) \rangle = 0 \text{ for all } \alpha, \beta \\
&\qquad \Leftrightarrow \omega \wedge {}^t\psi(\omega^*)(\beta) = 0 \text{ for all } \beta \text{ (since } \langle \cdot, \cdot \rangle \text{ is nondegenerate)} \\
&\qquad \Leftrightarrow im({}^t\psi(\omega^*)) \subset \ker(\varphi(\omega))
\end{aligned}$$

We now show that if $\langle v_i^*, {}^t\psi(\omega^*)(\beta) \rangle = 0$ for all β, then $im({}^t\psi(\omega^*)) \subset \ker(\varphi(\omega))$. Given some $\beta \in \bigwedge^{n-d+1}(V)$, suppose $\langle {}^t\psi(\omega^*)(\beta), v_i^* \rangle = 0$ for each $i = 1, \dots, n-d$. Then any $v^* \in V^*$ sends ${}^t\psi(\omega^*)(\beta)$ to zero, and hence, ${}^t\psi(\omega^*)(\beta) \in (W^\perp)^\perp = W$. We can then write ${}^t\psi(\omega^*)(\beta) = \sum a_i w_i$, and it follows that $\omega \wedge {}^t\psi(\omega)(\beta) = 0$, that is, ${}^t\psi(\omega^*)(\beta) \in \ker(\varphi(\omega))$. This condition that $\langle v_i^*, {}^t\psi(\omega^*)(\beta) \rangle = \langle \psi(\omega^*) v_i^*, \beta \rangle = \langle \omega^* \wedge v_i^*, \beta \rangle = 0$ clearly holds for all β, as $\omega^* = v_1^* \wedge \cdots \wedge v_{n-d}^*$.

($\Leftarrow$) Now suppose that ω is not totally decomposable. By Corollary 3.7, $\varphi(\omega)$ has rank $> n - d$. Therefore, $\ker(\varphi(\omega))$ has dimension $< d$. Moreover, by Lemma 4.3, $\psi(\omega^*)$ has rank $> d$. We have shown above that $\Xi_{\alpha,\beta}(\omega) = 0$ for all α, β if and only if $im({}^t\psi(\omega^*)) \subset \ker(\varphi(\omega))$. By dimension considerations, we cannot have $im({}^t\psi(\omega^*)) \subset \ker(\varphi(\omega))$, and thus $\Xi_{\alpha,\beta}(\omega) \neq 0$ for some α and β.

In light of Lemma 3.8, this theorem tells us that $[\omega]$ lies in the Grassmannian if and only if $\Xi_{\alpha,\beta}(\omega) = 0$ for every $\alpha \in \bigwedge^{d+1}(V^*)$ and $\beta \in \bigwedge^{n-d+1}(V)$. To complete this proof that the Grassmannian is a projective variety, we must show that the Plücker relations are homogeneous polynomials. In fact, we will show that they are quadratic forms, that is, homogeneous of degree two. This will follow from the following more general result.

Proposition 4.7 Let $f : V \to W$ and $g : V \to W^*$ be linear maps, and suppose V and W are finite dimensional vector spaces over a field k. Then the pairing $\langle f(v), g(v)\rangle : V \to k$ is a quadratic form.

Proof: Fix a basis $\{e_1, \ldots, e_n\}$ for V. Let $v \in V$ and write $v = \sum a_i e_i$. Then by the linearity of f and g and the bilinearity of $\langle \cdot, \cdot\rangle$, we have

$$\langle f(v), g(v)\rangle = \langle f(\sum a_i e_i), g(\sum a_i e_i)\rangle = \sum a_i a_j \langle v(e_i), v(e_j)\rangle$$

Note that each $\langle v(e_i), v(e_j)\rangle \in k$. So if we let $b_{i,j} = \langle v(e_i), v(e_j)\rangle$, then $\langle f(v), g(v)\rangle = \sum b_{i,j} a_i a_j$, and we see that $\langle f(v), g(v)\rangle$ is a quadratic form in the coordinates of v.

To apply this proposition to $\Xi_{\alpha,\beta}(\omega) = \langle {}^t\varphi(\omega)(\alpha), {}^t\psi(\omega^*)(\beta)\rangle$, we will show each entry in this pairing is linear. We have already observed that $\varphi(\omega)$ and $\psi(\omega^*)$ are linear. Furthermore, the transpose map sending a linear transformation $T \mapsto {}^tT$: for $T, S : V \to W$, $a \in k$, $w^* \in W^*$ and $v \in V$, ${}^t(aT + S) : W^* \to V^*$ is defined by

$$\begin{aligned}
&\langle {}^t(aT+S)(w^*), v\rangle \\
&\qquad = \langle w^*, (aT+S)(v)\rangle = a\langle w^*, T(v)\rangle + \langle w^*, S(v)\rangle \\
&\qquad = a\langle {}^tT(w^*), v\rangle + \langle {}^S(w^*), v\rangle = \langle a{}^tT(w^*) + {}^tS(w^*), v\rangle \\
&\qquad = \langle (a{}^tT + {}^tS)(w^*), v\rangle
\end{aligned}$$

And lastly, the isomorphism $\bigwedge^d(V) \cong \bigwedge^{n-d}(V^*)$ and evaluation at the vectors α, β are clearly linear, so both ${}^t\varphi(\omega)(\alpha)$ and ${}^t\psi(\omega^*)(\beta)$ are linear functions of ω. This completes our second proof that the Grassmannian is a projective variety and that

$$p(G_{d,V}) = Z(\{\Xi_{\alpha,\beta}(\omega) | \alpha \in \bigwedge^{d+1}(V^*), \beta \in \bigwedge^{n-d+1}(V)\})$$

The importance of studying the Plücker relations is in that they also generate the homogeneous ideal of the Grassmannian, which we state without proof.

Theorem 4.8 $I(Z(\{\Xi_{\alpha,\beta}(\omega) | \alpha \in \bigwedge^{d+1}(V^*), \beta \in \bigwedge^{n-d+1}(V)\})) = (\Xi_{\alpha,\beta}(\omega))$

References

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