Introduction to the Topology of Continuous Dynamical Systems

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1. Continuous General Dynamical Systems

Let (X, d) be a nonempty metric space with the usual basis,
$\mathfrak{B} = \{B_\epsilon(x) \,|\, x \in X,\, \epsilon \in \mathbb{R}^+\}$

1.1. **Definition.** A *continuous dynamical system* in the state or phase space X is a function Φ : X $\times$ $\mathbb{R} \to$ X, written $\Phi(x,t) = y$, or $\Phi_t(x) = y$ for $t \in \mathbb{R}$, y, $x \in$ X, where Φ has the following properties:

1. Continuity in X $\times$ $\mathbb{R}$:

$$\lim_{t \to t_0, x \to x_0} \Phi(x,t) = \Phi(x_0, t_0)$$

2. Initial condition: for any $p \in$ X,

$$\Phi(p, 0) = p$$

3. Group property: for any t_1, $t_2 \in \mathbb{R}$ and any $p \in$ X,

$$\Phi(\Phi(p, t_1), t_2) = \Phi(p, t_1 + t_2)$$

or with a different notation

$$\Phi_{t_1}(\Phi_{t_2}(p)) = \Phi_{t_1+t_2}(p)$$

This Φ_t is called the *evolution operator*. The parameter $t \in \mathbb{R}$ is called time. Fixing t, we note that Φ_t is a mapping of X into itself. The evolution operator therefore defines a one-parameter family of mappings X $\to$ X. The second notation highlights that the composition of these mappings has the some of the structure of the group $(\mathbb{R}, +)$:

Proposition 1. *The family of mappings* $G = \{\ \Phi_t \mid t \in \mathbb{R}\ \}$ *under composition is a homomorphic image of* $(\mathbb{R}, +)$*, and hence a group.*

Proof. Let a map f : $\mathbb{R} \to$ X be defined by t $\mapsto \Phi_t$. Then clearly f is a surjective homomorphism. □

[Note that Φ_t can also be seen as defining a group action of ($\mathbb{R}$, +) on X, where t acts as the map $x \mapsto \Phi_t(x)$.]

Theorem 1. *(The Integral Continuity Condition) For any point $x \in X$, any T, $\epsilon \in \mathbb{R}$, $T > 0$, $\epsilon > 0$, there exists a $\delta \in \mathbb{R}$ such that*

$$\Phi(B_\delta(x), t) \subseteq B_\epsilon(\Phi(x,t))$$

for all $t \in [0, T]$.

Proof. Assume let $\epsilon > 0$, $T > 0$, and assume there does not exist such a δ. Take $\delta_n \to 0$, a sequence in $\mathbb{R}$. We have assumed there is, for each n, some y_n such that d(x,y_n) $< \delta_n$ and a $t_n \in [0, T]$ such that d$(\Phi(x, t_n), \Phi(y_n, t_n)) > \epsilon$.

By the sequential compactness of $[0, T]$, there is a convergent subsequence $t_{n_k} \to t_0$. Clearly, $y_n \to x$ and therefore $y_{n_k} \to x$. But then by continuity,

$$\lim_{k\to\infty} \Phi(y_{n_k}, t_{n_k}) = \Phi(x, t_0), \ \lim_{k\to\infty} \Phi(x, t_{n_k}) = \Phi(x, t_0)$$

These two limits give $\epsilon \leq 0$, a contradiction. □

Equivalently, the result of this theorem states that for any given t, $\Phi(x, t)$ is a continuous function of the initial condition x.

Proposition 2. *The mappings in the group G are self-homeomorphisms of X.*

Proof. Let $\Phi_t \in$ G. Then $\Phi_t \circ \Phi_{-t} = \Phi_{-t} \circ \Phi_t = \Phi_0 = Id_G$, so Φ_t has a two-sided inverse and is hence a bijection. Moreover, all Φ_t in G are continuous on X, and hence Φ_t is a homeomorphism. □

Some notation: for $A, \subseteq$ X, $T \subseteq \mathbb{R}$,

$$\Phi(A, T) = \bigcup_{x\in A,\ t\in T} \Phi(x, t)$$

(G,+) a group, H, K $\subseteq$ G,

$$H + K = \bigcup_{k\in K, h\in H} h + k$$

Fixing a point $x_0 \in$ X, the set $\Phi(x_0, \mathbb{R})$ is called x_0's *orbit* or *trajectory*, while the function $\Phi(x_0, t)$ is called the *motion* of x_0. This function may or may not be injective. If it is, the motion is called *singular*. Motions which are not injective are contrarily *nonsingular*. Suppose $\Phi(x_0, t)$ is nonsingular. It is possible that $\Phi(x_0, t) = x_0$ for all $t \in \mathbb{R}$, in this case, x_0 is called a *rest (critical, equilibrium, or stationary)* point.

A nonsingular motion of a non-rest point must be periodic. To see this, note that for such a nonsingular motion, there must exist some t_1 and t_2, with, say, $t_2 > t_1$, such that

$$\Phi(x_0, t_1) = \Phi(x_0, t_2)$$

Let $t_2 - t_1 = \tau$. Then by the group property,

$$\Phi(x_0, \tau+t) = \Phi(x_0, t_2-t_1+t) = \Phi(\Phi(x_0, t_2), -t_1+t) = \Phi(\Phi(x_0, t_1), -t_1+t)$$
$$= \Phi(x_0, t_1 - t_1 + t) = \Phi(x_0, t)$$

Proposition 3. *if $\Phi(x_0, t)$ is nonsingular and x_0 is not a rest point, there is a smallest $\tau > 0$, called the period, for which*

$$\Phi(x_0, t) = \Phi(x_0, \tau + t) \tag{1.1}$$

Proof. Let τ_1 satisfy (1.1). If there is some τ_2 satisfying (1.1) such that $\tau_2 < \tau_1$, then either there exists some τ_3 satisfying (1.1) such that $\tau_3 < \tau_2$ or not. Continuing, we end up with either some smallest $\tau_j > 0$ satisfying (1.1) or an infinite strictly decreasing sequence τ_n of elements satisfying (1.1). By the boundedness of $[t, \tau_1] \in \mathbb{R}$, τ_n converges, say to τ. Clearly $\tau < \tau_i$ for any τ_i satisfying (1.1). Since $\Phi(x_0, t+\tau_n)$ is a constant for all τ_n, and since by continuity $\Phi(x_0, t+\tau_n) \to \Phi(x_0, t+\tau)$, $\Phi(x_0, t+\tau) = \Phi(x_0, t+\tau_n) = \Phi(x_0, t)$, and so τ satisfies (1.1).

Now we show $\tau > 0$.

Suppose to the contrary that $\tau_n \to 0$. By the continuity of Φ, for any ϵ there exists a δ such that

$$\Phi(x_0, [0, \delta)) \subseteq B_\epsilon(x_0)$$

Since $\tau_n \to 0$, there exists a $\tau_i < \delta$. By the periodicity of Φ, this implies

$$\Phi(x_0, t) \subseteq B_\epsilon(x_0)$$

for all $t \in \mathbb{R}$. Since $\epsilon > 0$ is arbitrary, this implies $\Phi(x_0, t) = x_0$ for all $t \in \mathbb{R}$, a contradiction. □

This gives a complete classification of motions of points in X. Note that the trajectory or image of a periodic motion is homeomorphic to S^1. However, the image a singular motion is not necessarily homeomorphic to $\mathbb{R}$; precisely when this occurs will be investigated later.

Proposition 4. *Let x, $y \in X$. Then $y \in \Phi(x, \mathbb{R})$ if and only if $\Phi(x, \mathbb{R}) = \Phi(y, \mathbb{R})$*

Proof. This follows from the fact that orbit membership is an equivalence relation under any group action. □

A corollary of the previous proposition is that distinct trajectories never intersect. Note that this corollary furthur implies no motion can enter a rest point for a finite value of time.

More generally, the motion of any point defines the motion of all points in its trajectory: letting $y = \Phi(x, t_0) \in \Phi(x, \mathbb{R})$, the motion of y is defined by $\Phi(y, t) = \Phi(x, t_0 + t)$.

1.2. **Definition.** A set A $\subseteq$ X is *invariant* if it is mapped into itself by all maps of the family G. In other words, $\Phi(A, t) \subseteq A$ for all $t \in \mathbb{R}$, or,

$$\Phi(A, \mathbb{R}) \subseteq A$$

Φ_{-t} is in G, so that $\Phi_{-t}(A) \subseteq A$. Applying Φ_t to this containment, we get

$$\Phi_t(\Phi_{-t}(A)) = Id_G(A) = A \subseteq \Phi_t(A)$$

Which gives $\Phi(A, \mathbb{R}) = A$. It is a consequence of general properties of bijections that an arbitrary union and intersection of invariant sets is invariant.

Theorem 2. *$A \subseteq X$ is invariant if and only if it is a union of entire trajectories, i.e.*

$$A = \bigcup_{i \in I} \Phi(x_i, \mathbb{R}),\ \ x_i \in X$$

Proof. Clearly, an entire trajectory is invariant subset, and so any union of entire trajectories is an invariant subset.

Conversely, suppose A is invariant. Let $x \in$ A. Since $\Phi(A, \mathbb{R}) = A$, in particular $\Phi(x, \mathbb{R}) \subseteq A$, and so A contains the entire trajectory of each $x \in$ A and is hence is a union of entire trajectories. □

Note that a dynamical system on a space X defines a dynamical system on every invariant subset A.

1.3. **Definition.** A set A $\subseteq$ X is *positively invariant* or *negatively invariant* if, respectively,

$$\Phi(A, \mathbb{R}^+) \subseteq A, \; \Phi(A, \mathbb{R}^-) \subseteq A$$

Theorem 3. *Let $x \in X$. If for all $\epsilon > 0$, there exists a subset some trajectory which is the image of a connected subset of $\mathbb{R}$ and which is completely contained in the ϵ-neighborhood of x, i.e. if there exists a $y \in X$ and $[0, T]$ $\in \mathbb{R}$ such that*

$$\Phi(y, [0, T]) \subset B_\epsilon(x)$$

then x is a rest point.

Proof. Assume x is not a rest point. Since the trajectory of x cannot enter a rest point, for any $[0, T]$ there is some $t_0 \in [0, T]$ such that $\Phi(x, t_0) = x_0$ $\neq x$. Let

$$m = \frac{d(x_0, x)}{2} > 0$$

By the continuity of Φ_{t_0} on X, there is a δ' such that

$$\Phi(B_{\delta'}(x), t_0) \subseteq B_m(x_0). \tag{1.2}$$

We let δ = min (δ', m). Clearly (1.2) holds if we replace δ' with δ. Then by the triangle inequality, $B_\delta(x) \cap B_m(x_0) = \emptyset$. Therefore, for any $\epsilon < \delta$, all $y \in B_\epsilon(x)$ have $\Phi(y, t_o) \in B_m(x_0)$, which implies $\Phi(y, t_o) \notin B_\delta(x)$. We have thus found an ϵ such that no connected subset of the domain of any trajectory is entirely contained in the ϵ-neighborhood of x, and hence the conditions of the theorem do not hold, a contradiction. □

Corollary 1. *If $\lim_{t \to +\infty} \Phi(x, t) = p$, then p is a rest point.*

Proof. Translating the meaning of this limit by the group property, we get that for any ϵ, there is a t_0 such

$$\Phi(\Phi(x, t_0), \mathbb{R}^+)) \subseteq B_\epsilon(p)$$

Then p is a rest point by Theorem 2. □

The same result clearly also holds for $\lim_{t \to -\infty} \Phi(x, t) = p$.

Proposition 5. *If every ϵ-neighborhood of a point $x \in X$ has a point $y \in X$ such that the motion $\Phi(y, t)$ has an arbitrarily small period, then x is a rest point.*

Proof. By continuity of $\Phi(x,t)$, there exists a $[0,T]$ such that for $\epsilon' > 0$,

$$\Phi(x,[0,T]) \subseteq B_{\epsilon'}(x)$$

By the Integral Continuity Condition, there is a $\delta > 0$ such that for $y \in B_\delta(x)$, $t \in [0,T]$,

$$\Phi(y,t) \subseteq B_{\epsilon'}(\Phi(x,t))$$

. We may assume $\delta < \epsilon'$. Then by the triangle inequality,

$$d(\Phi(y,t),x) < 2\epsilon',\ \ t \in [0,T]$$

We may assume by the conditions of the theorem that y has an arbitrarily small period, say $\tau < T$. Then letting $2\epsilon' = \epsilon$, we have $\Phi(y,\mathbb{R}) \subseteq B_\epsilon(x)$, so x is a rest point by the previous theorem. □

Theorem 4. *Every positively invariant set $A \subseteq X$ which is homeomorphic to a closed ball in $\mathbb{R}^n$ contains a rest point.*

Proof. We identify A with a closed ball in $\mathbb{R}^n$ by the homeomorphism. Since A is positively invariant, Φ_t, $t > 0$ is a map of A into A which has a fixed point by Brouwer's theorem. For a sequence of functions Φ_{t_k}, $t_k \to 0$, let this fixed point be x_k. Then by sequential compactness, there exists a convergent subsequence $x_{k_l} \to x \in$ A, and x is then a rest point by Proposition 5. □

1.4. **Definition.** Two dynamical systems, $\Phi\colon$ X $\times \mathbb{R} \to$ X and $\Psi\colon$ Y $\times \mathbb{R} \to$ Y are said to be *isomorphic* or *topologically equivalent* if there exists a homeomorphism $f\colon$ X $\to$ Y such that

$$f(\Phi(x,t)) = \Psi(f(x),t)$$

If the map f is not a homeomorphism but at least continuous, then the systems are *homomorphic* dynamical systems.

It is easy to see that a homomorphism (and hence an isomorphism) of dynamical systems maps trajectories to trajectories, rest points to rest points, periodic motions to periodic motions, and invariant sets to invariant sets. In particular, these facts allow the complete classifcation of the isomorphism type of a dynamical system with X $= \mathbb{R}$ by its rest points and the 'direction' of the interval trajectories.

2. Limiting Properties and Stability

2.1. **Definition.** A point $p \in$ X is an ω *or* α*-limit point* of a motion $\Phi(x,t)$, respectively, if there exists a sequence t_k tending to positive or negative infinity, respectively, such that $\Phi(x,t_k) \to p$.

Equivalently, p is an $\omega(\alpha)$-limit point of $\Phi(x,t)$ if for any $\epsilon > 0$ and $T \in \mathbb{R}$, there is a $t' > T$ $(t' < T)$ such that $\Phi(x,t') \in B_\epsilon(p)$.

[Note that all points of nonsingular motions are both ω and α-limit points.]

2.2. **Definition.** The *dynamical limit sets* of a point $x \in$ X, written Ω_x and A_x are the sets of all ω and α-limit points, respectively, of the motion $\Phi(x,t)$.

Theorem 5. *For any $x \in X$, Ω_x and A_x are invariant.*

Proof. We prove the theorem for Ω_x. The proof for A_x is analogous. Let $x \in$ X, $t_k \to +\infty$, $\Phi(x,t_k) \to p$, and $t_0 \in \mathbb{R}$. Φ_{t_0} is a continuous map, so that in the image $\Phi(X,t_0)$, $\Phi(\Phi(x,t_k),t_0)$ converges to $\Phi(p,t_0)$, which gives, by the group property, $\Phi(x,t_0+t_k) \to \Phi(p,t_0)$. But then $T+t_0 \to +\infty$, so $\Phi(p,t_0) \in \Omega_x$.

Hence $\Phi(p,\mathbb{R}) \subseteq \Omega_x$, finishing the proof. □

Some more notation:

$$\Sigma_A = \overline{\Phi(A,\mathbb{R})},\ \Sigma_A^+ = \overline{\Phi(A,\mathbb{R}^+)},\ \Sigma_A^- = \overline{\Phi(A,\mathbb{R}^-)}$$

$$\Delta_x = \Omega_x \cup A_x$$

Clearly, ω and α-limit points of a motion are limit points of the trajectories, and hence $\Delta_x \subseteq \Sigma_x$, etc. In fact, it is easy to see that

$$\Sigma_x = \Phi(x,\mathbb{R}) \cup \Delta_x,\ \Sigma_x^+ = \Phi(x,\mathbb{R}^+) \cup \Omega_x,\ \Sigma_x^- = \Phi(x,\mathbb{R}^-) \cup A_x$$

2.3. **Definition.** A point x and the motion $\Phi(x,t)$ are said to be *positively* or *negatively Lagrange stable* (*stable* L,$^+$ *stable* L^-), respectively, if Σ_x^+ or Σ_x^- are compact. If Σ_x is compact, the motion is simply *Lagrange stable* (*stable* L).

Note that rest points and periodic motions are always Lagrange stable, as is any motion on a compact space, and, if X $= \mathbb{R}^n$, any motion contained within a bounded subset.

Theorem 6. *A motion $\Phi(x,t)$ is positively Lagrange stable if and only if all of the following conditions hold:*

1. *Ω_x is nonempty.*
2. *Ω_x is compact.*
3. $\lim_{t\to+\infty} d(\Phi(x,t),\Omega_x) = 0$.

Proof. Assume $\Phi(x,t)$ is postively Lagrange stable. We first show this implies 1, 2, and 3.

1. Let $t_k \to +\infty$. By compactness of Σ_x^+, $\Phi(x,t_k)$ must have a convergent subsequence $\Phi(x,t_{k_l}) \to y$,, which gives $y \in \Omega_x$, so Ω_x is nonempty.

2. Ω_x is a subset of the compact space Σ_x^+, so all we have to show is closure. Let q be a limit point of Ω_x. Then any open neighborhood of q, $B_{\epsilon'}(q)$ contains a $y \in \Omega_x$, which, as an interior point of $B_{\epsilon'}(q)$, has an open neighborhood $B_\epsilon(y) \subseteq B_{\epsilon'}(q)$. As y is an ω-limit point of a sequence $\Phi(x,t_k)$, $B_\epsilon(y)$ contains $\Phi(x,t)$, $t > T$ for arbitrarily large T. But by the inclusions above, this means any neighborhood of q contains such a $\Phi(x,t)$, which gives $q \in \Omega_x$.

3. Suppose $\lim_{t\to+\infty} d(\Phi(x,t),\Omega_x) \neq 0$. This gives that there exists a sequence $t_k \to +\infty$ such that for some $\epsilon > 0$,

$$d(\Phi(x,t_k),\Omega_x) > \epsilon, \forall k \in \mathbb{N} \tag{2.1}$$

Since Σ^+ is compact the sequence $\Phi(x,t_k)$ contains a convergent subsequence $\Phi(x,t_{k_l}) \to q$. But then $q \in \Omega_x$, a contradiction of (2.1) above.

Now assume 1, 2, and 3, hold. We show the sequential compactness of Σ_x^+. Assume without loss of generality that t_k is monotonically increasing. If it is bounded, it converges, and hence so does $\Phi(x,t_k)$. If $t_k \to +\infty$ By 3, there exists a sequence r_k in Ω_x such that

$$\lim_{k\to+\infty} d(\Phi(x,t_k),r_k) = 0.$$

By 2, r_k has a convergent subsequence $r_{k_n} \to r \in \Omega_x$. But then again by 3, $\Phi(x,t_k) \to r \in \Sigma_x$ so x is Lagrange stable.

□

If Ω_x or A_x is empty, respectively, then the motion $\Phi(x,t)$ is called *positively* or *negatively departing*, and in case Δ_x is empty simply *departing*. If Ω_x is nonempty but $\Omega_x \cap \Phi(x,\mathbb{R}^+)$ is empty, the motion is *positively asymptotic*. *Negatively asymptotic* is defined analogously, and a motion which is both positively and negatively asymptotic is *asymptotic*.

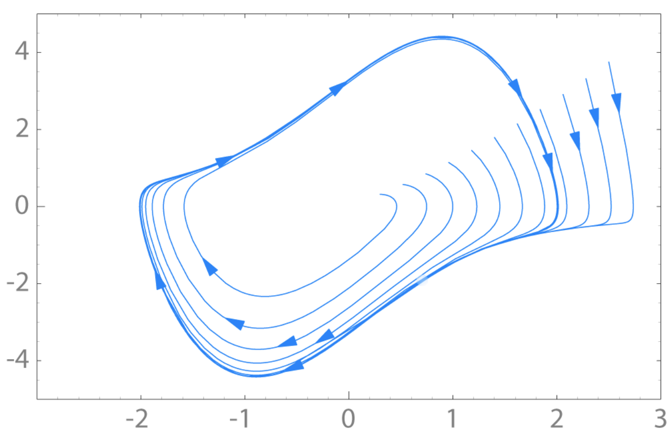

FIGURE 1. **The Van Der Pol dynamical system. The thick blue line is a periodic, Poisson stable motion. It is also the positive dynamical limit set of both the interior and exterior motions, which are both positively asymptotic. Such a dynamical limit set which is also a periodic motion is called a limit cycle. The exterior motions are negatively departing.**

The third possibility, that $\Omega_x \cap \Phi(x,\mathbb{R}^+)$ (or $A_x \cap \Phi(x,\mathbb{R}^-)$) is nonempty, is the subject of the following definition:

2.4. **Definition.** A motion $\Phi(x,t)$ is *positively* or *negatively Poisson stable*(*stable* $P,^{+}$ *stable* P^{-}) if, respectively, $\Omega_x \cap \Phi(x,\mathbb{R}^+)$ or $A_x \cap \Phi(x,\mathbb{R}^-)$ are nonempty. If both of these intersections are nonempty, the motion is simply *Poisson stable* (*stable* P).

Proposition 6. *If* $\Phi(x,t)$ *is a positive Poisson stable motion,* $\Phi(x,\mathbb{R}) \subseteq \Omega_x$.

Proof. Ω_x is invariant, so if $y \in \Omega_x$, for any t, $\Phi(y,t) \in \Omega_x \cap \Phi(x,\mathbb{R})$, $\Phi(y,\mathbb{R}) \in \Omega_x \cap \Phi(x,\mathbb{R})$, and hence $\Phi(y,\mathbb{R}) = \Phi(x,\mathbb{R}) \in \Omega_x$ □

Clearly, all periodic and rest motions are Poisson stable. We now give an example of a singular Poisson stable motion on the torus, $\{(\theta,\phi)\,|\,\theta,\phi \in [0,1] \subset \mathbb{R}\}$ where $(\theta+l,\phi+k) = (\theta,\phi)$ for any integers l,k. Define

$$\frac{d\theta}{dt} = 1,\ \frac{d\phi}{dt} = a,\ a \in \mathbb{R}$$

This defines a dynamical system $\Phi((\theta,\phi),t) = (\theta+t,\phi+at)$. It is easy to see that if a $\in \mathbb{Q}$, $a = \frac{p}{q}$ for p,q relatively prime, this motion is periodic with period q. If a is irrational, the motion is not periodic, because otherwise the period would give a solution to the equations $\tau = l, a\tau = k$ for $l,k \in \mathbb{Z}$ and then $a = \frac{k}{l}$, a contradiction. In fact, for a irrational this motion is everywhere dense on the torus, so that $\Delta_{(\theta,\phi)}$ is the entire torus for any (θ,ϕ), which gives that the motion is Poisson stable.

If $\Phi(x,t)$ is a nonsingular positively Poisson stable motion, we can fix x and write $\Phi(x,t) = f(t)$ for $t \in \mathbb{R}$. There exists a sequence $t_k \to +\infty$, $f(t_k) \to x$. In addition, $f(t)$ has a left inverse; call it g. Note that the image sequence of this inverse function is just the original sequence t_k, i.e. $g(f(t_k)) = t_k$, and it diverges. But $g(x) = 0$, so g is not continuous and hence $f : \mathbb{R}^+ \to \Phi(x,R^+)$ is not a homeomorphism, which motivates the following theorem. But first we need a lemma.

Lemma 1. *If X is a space with connected open basis and f is a continuous bijection from the half-line to X* $f : \mathbb{R}, \to X$, *then f is a homeomorphism.*

Proof. : We show f is an open map.

Let B be a basic open set in $\mathbb{R}$. B is contained in a compact subset, K, of $\mathbb{R}$. The restriction of f to K is an open mapping onto its image, so that f(B) is open in f(K). But then there is an open set, U, of X, for which the intersection U $\cap$ f(K) $=$ f(B). But in a space with connected basis, the intersection of a compact set and an open set is open, so that f(B) is open in X. □

Theorem 7. *If* $x \in X$, *x and* $\Phi(x,t)$ *are positively Poisson stable if and only if* $\Phi(x,\mathbb{R}^+) \ncong \mathbb{R}^+$

Proof. We fix x and write $\Phi(x,t) = f(t)$ as above. Then $f\ \mathbb{R}^+ \to \Phi(x,\mathbb{R}+)$ is a continuous bijection. If $\Phi(x,\mathbb{R}+) \cong \mathbb{R}+$, there exists a homeomorphism $\Psi : \Phi(x,\mathbb{R}). \to \mathbb{R}^+$. This gives $\Psi \circ f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous bijection of the half real line, which implies by the lemma that it is a homeomorphism.

But then $\Psi^{-1} \circ \Psi \circ f = f$ is a homeomorphism, which is a contradiction. $=\Psi(f(t))$ □

2.5. **Definition.** A point $x \in$ X is called *wandering* if there exists an ϵ-neighborhood $B_\epsilon(x)$ and a $T \in \mathbb{R}$ such that for all $t > T$,

$$B_\epsilon(x) \cap \Phi(B_\epsilon(x), t) = \emptyset \tag{2.2}$$

A point is *nonwandering* if it is not a wandering point.

All singular motions are nonwandering, and if x is positively (or negatively) Poisson stable, it is *a fortiori* nonwandering, since then there exists a sequence $t_k \to +\infty$ with $\Phi(x, t_k) \to x$. The converse is not true, since there exist nonsingular, nonwandering motions that are not Poisson stable. For example, if we "inserted" a rest point into the periodic motion in Figure 1, the leftover nonsingular motion would be nonwandering (thanks to the fact that it's a limit set of nearby motions) but not Poisson stable, since its own dynamical limit set would just be the inserted rest point.

Proposition 7. *The set of wandering points, W, is open.*

Proof. Suppose x satisfies (2.3) above. If $q \in B_\epsilon(x)$, then since it is an interior point, there is an open neighborhood $B_\delta(q) \subseteq B_\epsilon(x)$. Yet q with this $B_\delta(q)$ satisfies the conditions in (2.3), so q is also a wandering point. This gives that the set of wandering points is open. □

Proposition 8. *The set of wandering points, W, is invariant.*

Proof. Since Φ_{t_0} is a bijection for any t_0,

$$\Phi(B_\epsilon(x), t_0) \cap \Phi(\Phi(B_\epsilon(x), t)t_0) = \emptyset.$$

which gives that if x is any wandering point, $\Phi(x, t_0)$ is a wandering point for any t_0, hence the set of wandering points is invariant. □

The set of nonwandering points, M $= W^c =$ X $\setminus$ W, is therefore closed and invariant.

A point x is wandering in an invariant set A $\subseteq$ X if it is wandering in (A,d) defined as a dynamical system by the restriction $\Phi\ |_A :$ A $\times\, \mathbb{R} \to$ A. It is easy to show that all wandering points in X are wandering points in A. The contrapositive gives that all points that are *nonwandering* in A are nonwandering in X. The converse of this statement is not necessarily true; take for example the point x in Figure 1, which is wandering in L, but nonwandering in X.

Theorem 8. *Let $x \in X$. Every $y \in \Omega_x$ is nonwandering in Σ_x.*

Proof. Since $y \in \Omega_x$, there exists a sequence $\Phi(x, t_k) \to y$, $t_k \to +\infty$. Let $\epsilon > 0$, $T > 0$. Then there is a point $x_0 = \Phi(x, t_i) \in B_\epsilon(y)$. In addition it is clear that y is an ω-limit point of x_0, so there is a sequence p_k with element p_i such that $\Phi(x_0, p_i) \in B_\epsilon(y)$, $p_i > T$. This gives

$$x_0 \in B_\epsilon(y) \cap \Phi(B_\epsilon(y), p_i),\ p_i > T.$$

which means y is nonwandering. □

Corollary 2. *If at least one motion in X is either positively or negatively Lagrange stable, the set of nonwandering points, M, is nonempty.*

Corollary 3. *In a compact metric space the set of nonwandering points, M, is nonempty.*

Corollary 4. *In a compact metric space every motion tends toward the set of nonwandering points.*

Proof. This follows from the fact that all motions in a compact metric space are Lagrange stable, and the fact that for a Lagrange stable motion $\Phi(x,t)$,

$$\lim_{t\to+\infty} d(\Phi(x,t), \Omega_x) = 0$$

□

2.6. **Definition.** A set A $\subseteq$ X is *minimal* if it is nonempty, closed, and invariant and contains no nonempty, closed and invariant proper subset.

Examples of compact minimal sets include periodic and rest trajectories, while the trajectory of any departing motion is a noncompact minimal set.

Theorem 9. *Every closed, invariant, compact set A contains a minimal subset.*

Proof. If A contains a proper closed invariant subset, call it F_1. Otherwise set F_1 = A. If F_1 contains any closed invariant proper subset, call it F_2, otherwise set $F_1 = F_2$. We continue to get a nested sequence of closed, invariant subsets; A $\supseteq F_1 \supseteq F_2 \ldots$

Let

$$F_\alpha = \bigcap_{k=1}^{\infty} F_k$$

F_α is clearly closed, invariant, and is nonempty by the finite intersection property of A. We continue the procedure above to get a transfinite nested sequence of closed, invariant subsets;

$$A \supseteq F_1 \supseteq F_2 \supseteq \ldots \supseteq F_\alpha \supseteq F_{\alpha+1} \ldots \supseteq F_\beta \supseteq \ldots \tag{2.3}$$

Since A is a compact metric space, it has a countable basis. By the Baire-Hausdorff theorem, this well-ordered sequence of nested closed sets has at most a countable number of elements, and so the successive elements must coincide after some F_γ. Then $F_\gamma \subseteq$ A is a minimal set. □

Note that in a compact metric space X, the set of nonwandering points is closed, invariant, and nonempty. We can repeat the above argument for M, and add the requirement that each motion in M_{i+1} be *nonwandering in* M_i, since each M_i is itself a compact invariant set and a dynamical system under Φ. This results in an M_γ, and since $M_{\gamma+1} = M_\gamma$, M_γ is nonwandering in itself. M_γ is also the maximal such set (because a subset nonwandering in itself would have to be open in M_γ).

2.7. **Definition.** The set M_γ (or simply Z) is called the *center* or *set of central motions* of a dynamical system.

Theorem 10. *Z contains an everywhere dense set of Poisson stable points.*

Proof. Let $x_0 \in$ Z, $\epsilon \in \mathbb{R}^+$. We show $B_\epsilon(x_0)$ contains a Poisson stable point. It follows from the fact that x_0 is nonwandering that there exists a $t_1 > 1$ and an ϵ_1 with $\epsilon_1 < \frac{\epsilon}{6}$ such that by the triangle inequality,

$$\Phi(B_{\epsilon_1}(x_0), t_1) \subseteq B_{\frac{\epsilon}{2}}(x)$$

In particular,

$$\Phi(\overline{B_{\epsilon_1}(x_0)}, t_1) \subseteq B_\epsilon(x)$$

Since x_0 nonwandering in Z, there is a point $x_1 \in$ Z, with

$$x_1 \in \Phi(B_{\epsilon_1}(x), t_1) \cap B_{\epsilon_1}(x)$$

We then "start over" with x_1 in the role of x. Since x_1 is a wandering point, there is a $t_2 <$ -2 and an ϵ_2 with

$$\epsilon_2 < \frac{d(\Phi(B_{\epsilon_1}(x), t_1)^c, x_1)}{6}$$

such that

$$\Phi(B_{\epsilon_2}(x_1), t_2) \cap B_{\epsilon_2}(x_1) \neq \emptyset$$

Then there is some x_3 in the intersection above, and

$$x_3 \in \Phi(\overline{B_{\epsilon_2}(x_1)}, t_2) \subseteq \Phi(\overline{B_{\epsilon_1}(x)}, t_1) \subseteq B_\epsilon(x)$$

Continuing indefinitely we obtain two divergent sequences $\{t_{2k}\}$ and $\{t_{2k+1}\}$, a sequence $\epsilon_k \to 0$, and an infinite sequence of nested, closed neighborhoods. Since compact spaces are complete, there is some unique q such that

$$q = \bigcap_{k=1}^{\infty} \overline{B_{\epsilon_k}(x_{k-1})}$$

Then for any δ, there is an n such that for all $k > n$

$$\Phi(B_{\epsilon_k}(x_{k-1}), t_k) \subseteq B_{\epsilon_n}(x_{n-1}) \subseteq B_\delta(q)$$

Since $q \in B_{\epsilon_n}(x_{n-1})$, in particular $\Phi(q, t_k) \subseteq B_\delta(q)$ for all $k > n$. Since t_k diverges in both directions, this gives $q \in \Omega_q \cap A_q \cap \Phi(q, \mathbb{R})$, so that q is Poisson stable. □

Corollary 5. *Z is the closure of the set of all Poisson stable points.*

3. Differential Equations as Dynamical Systems

The general form of for a system of differential equations $f : \mathrm{G} \to \mathbb{R}^n$ (G $\subseteq \mathbb{R}^n$), which define a dynamical system $\Phi_t(x_1, \ldots, x_n)$ by their solutions is:

$$\frac{dx_i(t)}{dt} = f_i(x_1, \ldots, x_n),\ 1 \leq i \leq n \tag{3.1}$$

Where f is a continuous function on G, which is closed and bounded. The autonomy of f is necessary if x(t) is to induce the group property. In fact, letting x(t) be a solution, x(t_0) $= x_0 \in \mathbb{R}^n$, x($t_0 + t$) is a solution only if

$$\frac{dx_i(t_0+t)}{dt} = \frac{x_i(y)}{dy}\frac{dy}{dt} = f_i(x_1, \ldots, x_n, t),\ 1 \leq i \leq n$$

where $y = t_0 + t$. this is equivalent to

$$f_i(x_1, \ldots, x_n, t) = f_i(x_1, \ldots, x_n, t_0 + t),\ 1 \leq i \leq n$$

and since t_0 was arbitrary, $f_i(x_1, \ldots, x_n, t)$ must be a constant function of t for all i, i.e. f must be autonomous.

The conditions for the existence of a solution to (3.1) with domain all of $\mathbb{R}$ are given by

Theorem 11. *If the functions f_i in (3.1) are continuous on $\mathbb{R}^n$ and*

$$f_i(x_1, \ldots, x_n) = O(|x_1| + ...|x_n|)$$

then parametrized solution $x : \mathbb{R} \to \mathbb{R}^n$ exists on all of $\mathbb{R}$

The proof will only be outlined: it depends on the construction of arbitrarily close approximations, called ϵ-solutions, by Euler polygons.

3.1. **Definition.** A parametrization $x(t) : \mathbb{R} \to \mathbb{R}^n$ with $x(0) = x_0$ is called an ϵ- solution of (2.1) on $[a, b] \subseteq \mathbb{R}$ if each $x_i(t)$ is continuous, sectionally smooth, and satisfies:

$$x_i(t) = p_i(x_0) + \int_{t_0}^{t} f_i(x_1, \ldots, x_n)dt + \int_{t_0}^{t} \Theta_i(t)dt,\ |\Theta_i(t)| < \epsilon$$

where the error function $\Theta : \mathbb{R} \to \mathbb{R}^n$ is piecewise continuous, and $p_i : \mathbb{R}^n \to \mathbb{R}$ is the i-th projection function.

We use these ϵ-solutions to give a sequence of uniformly bounded, equicontinuous functions x^{ϵ_n} which converge, by Arzelà's theorem, to a continuous function $x(t)$ satisfying (3.1). The condition on the asymptotic behavior of f is negligible due to the fact that a differential equation on G can simply be extended to an equation which is continuous and bounded on $\mathbb{R}^n$.

A clear requirement for Φ_t to be well defined is that the trajectories be distinct, or equivalently that the sets of pairs $(t, x(t))$ defining distinct solutions intersect trivially (these sets will be equivalence classes, after all). This is equivalent to the uniqueness of the solution $x(t)$ passing through an arbitrary point $y \in \mathbb{R}^n$. The conditions for this is given by:

Theorem 12. *If the functions f_i in (3.1) are Lipschitz on G, then there exists a unique solution in G satisfying a given initial condition.*

For this and the previous proof, see [1]. We have seen any solution $x(t)$ of (3.1) will be continuous and satisfy the group property. Letting $x(t)$ be the solution defined by the initial condition $x(0) = x \in \mathbb{R}^n$, $\Phi(x,t) = x(t)$ defines a dynamical system.

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