Riemann Surfaces

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Abstract

Riemann surface are 2-manifolds with complex analytical structure, and are thus a meeting ground for topology and complex analysis. Cohomology with coefficients in the sheaf of holomorphic functions is an important tool in the study of Riemann surfaces. This algebraic structure is interesting, because it encodes both analytical and topological properties of a Riemann surface.

1 Preliminaries

Before beginning my discussion of Riemann surfaces, I would like to remind the reader of some important definitions and theorems from topology and complex analysis. These should, for the most part, be familiar concepts; I have gathered them here primarily for reference purposes.

1.1 Topology

Definition 1.1.1 (Open Map)**.** Let X and Y be topological spaces. A map $f : X \to Y$ is called open if for all open sets $U \subseteq X$ we have that $f(U)$ is open.

Definition 1.1.2 (Open Cover)**.** Let X be a topological space. An open cover of $A \subseteq X$ is a collection $\mathcal{U}$ of open sets such that $A \subseteq \bigcup_{U \in \mathcal{U}} U$. An open cover $\mathcal{V}$ is called finer than $\mathcal{U}$ if for all $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V \subseteq U$. This is denoted $\mathcal{V} < \mathcal{U}$.

Definition 1.1.3 (Locally homeomorphic). Let X and Y be topological spaces. We say X is locally homeomorphic to Y if for all $x \in X$ there exist open sets $U \subseteq X$ and $V \subseteq Y$ such that $x \in U$ and U is homeomorphic to V.

Definition 1.1.4 (n-manifold). An n-manifold is a Hausdorff topological space with a countable basis that is locally homeomorphic to $\mathbb{R}^n$.

Definition 1.1.5 (One Point Compactification). Let X be a locally compact Hausdorff space that is not compact. The one point compactification of X is the unique compact topological space $X' \supset X$ such that $X' \setminus X$ is a single point.

Recall that the one point compactification of X can be constructed in the following way. Let $\infty \notin X$. Now let $X' = X \cup \{\infty\}$ and let

$$\{U \subset X' \mid U \text{ open in } X \text{ or } X \setminus U \text{ is compact }\}$$

be a basis for the topology on X'.

1.2 Complex Analysis

Definition 1.2.1 (Holomorphic Function). Let $U \subseteq \mathbb{C}$ be open. A function $f : U \to \mathbb{C}$ is called holomorphic if it is complex differentiable on U.

Definition 1.2.2 (Biholomorphic Function). Let $U \subseteq \mathbb{C}$ be open. A function $f : U \to V \subseteq \mathbb{C}$ is called biholomorphic if it is a holomorphic bijection with a holomorphic inverse.

Definition 1.2.3 (Meromorphic Function). Let $U \subseteq \mathbb{C}$ be open. A function $f : U \to \mathbb{C} \cup \{\infty\}$ is said to be meromorphic if $X = f^{-1}(\infty)$ is a discrete set such that f is holomorphic on $U \setminus X$ and $\lim_{z \to x} |f(z)| = \infty$ for all $x \in X$.

Definition 1.2.4 (Doubly Periodic). A function $f : \mathbb{C} \to \mathbb{C}$ is called doubly periodic if there exist $w_1, w_2 \in \mathbb{C}$ such that w_1 and w_2 are linearly independent over $\mathbb{R}$ and $f(z) = f(z \pm w_i)$ for all $z \in \mathbb{C}$ and $i \in \{1, 2\}$. w_1 and w_2 are called the periods of f.

Theorem 1.2.5 (Open mapping). *If $f : \mathbb{C} \to \mathbb{C}$ is a non-constant holomorphic map, then f is an open map.*

2 Riemann Surfaces

2.1 Definitions

Definition 2.1.1 (Complex Atlas)**.** Let X be a 2-manifold. A complex atlas on X consists of an open cover $\{U_i\}_{i\in I}$ and a collection of associated homeomorphisms $\{\phi_i : U_i \to V_i \subseteq \mathbb{C}\}_{i\in I}$ with the following property:

$$\phi_i \circ \phi_j^{-1} \text{ is biholomorphic on } \phi_j(U_i \cap U_j)\ \forall\, i, j \in I \tag{1}$$

The homeomorphisms belonging to a complex atlas are called charts. Two charts are called compatible if they satisfy property (1). Two complex atlases are considered equivalent if their union is itself an atlas.

Definition 2.1.2 (Riemann Surface)**.** A Riemann surface X is a connected 2-manifold with a complex structure given by an equivalence class of atlases on X.

Remark. Interestingly the stipulation that a manifold have a countable basis is unnecessary when considering Riemann surfaces, because the added rigidity that comes with the complex structure is enough to ensure that the topology has a countable basis. However this is not true when considering higher dimensional complex manifolds.

2.2 Examples

The following examples provide a more concrete representation of the structure of Riemann surfaces. They also afford an opportunity to explore the relationship between the study of Riemann surfaces and that of complex analysis.

Example 2.2.1 (Riemann Sphere)**.** Topologically the Riemann sphere is the one point compactification of $\mathbb{C}$. Its complex structure is defined by the following two maps:

$$\mathrm{id} : \mathbb{C} \to \mathbb{C} \text{ and } f(z) = \begin{cases} 1/z & z \in \mathbb{C}^* \\ 0 & z = \infty \end{cases}$$

The Riemann sphere is often denoted $\mathbb{P}^1$.

Proposition 2.2.2. *The Riemann sphere defined in Example 2.2.1 is a Riemann surface.*

Proof. In order to show that $\mathbb{P}^1$ is actually a Riemann surface one must prove (a) that it is a connected 2-manifold and (b) that f and id are compatible.

(a) Clearly id is a homeomorphism, and f is a surjective bijection. For f to be a homeomorphism, it must also be continuous and open. $f|_{\mathbb{C}^*}$ is continuous so we need only consider open sets containing 0. Let $U \subseteq \mathbb{C}$ be an open set such that $0 \in U$. Also let $B_r \subseteq U$ be an open ball of radius $r > 0$ centered at 0. By the definition of f we have that $f^{-1}(B_r) = \{z \in \mathbb{C} \mid |z| > 1/r\} \cup \{\infty\}$. This set is open in $\mathbb{P}^1$, because $\{z \in \mathbb{C} \mid |z| \leq 1/r\}$ is compact. This implies that $f^{-1}(U) = f^{-1}(U \setminus \{0\}) \cup f^{-1}(B_r)$ is an open set, because $f|_{\mathbb{C}^*}$ is continuous. Therefore f is continuous.

In order to show that f is open, consider an open set $V \subseteq \mathbb{P}^1$. By the definition of $\mathbb{P}^1$ we have that $V^c = \mathbb{P}^1 \setminus V$ is compact in $\mathbb{C}$. This implies that there exists $r > 0$ such that $V^c \subseteq \{z \in \mathbb{C} \mid |z| < r\}$. Thus we have that $A = \{z \in \mathbb{C} \mid |z| > r\}$ is contained in V. By the definition of f, $f(A) = \{z \in \mathbb{C} \mid |z| > 1/r\}$ which is an open set in $\mathbb{C}$. Therefore $f(V) = f(V \setminus \{\infty\}) \cup f(A)$ is open, because $f|_{\mathbb{C}^*}$ is an open map. Thus f is a homeomorphism.

$\{\mathbb{C}\} \cup \{\mathbb{C}^* \cup \{\infty\}\}$ is an open cover of $\mathbb{P}^1$. Thus $\mathbb{P}^1$ is locally homeomorphic to $\mathbb{C}$. Recall that the topological structure of $\mathbb{C}$ is that of $\mathbb{R}^2$. This implies that the Riemann sphere is locally homeomorphic to $\mathbb{R}^2$.

Now I will show that the Riemann sphere is Hausdorff. Let $z, w \in \mathbb{P}^1$ such that $z \neq w$. If $z, w \neq \infty$, then $z, w \in \mathbb{C}$. In this case z and w can be separated, because $\mathbb{C}$ is Hausdorff. If $z = \infty$, then let $U = \{u \in \mathbb{C} \mid |w - u| < r\}$ where $r \in \mathbb{R}$ such that $r > 0$. U is an open set such that $w \in U$ and its closure $\overline{U}$ is a compact set such that $\infty \notin \overline{U}$. This implies that $V = \mathbb{P}^1 \setminus \overline{U}$ is an open set such that $\infty \in V$ and $U \cap V = \emptyset$. Therefore all $z, w \in \mathbb{P}^1$ can be separated, and thus $\mathbb{P}^1$ is Hausdorff.

Lastly I will prove that $\mathbb{P}^1$ is connected. $\mathbb{P}^1 = \mathbb{C} \cup (\mathbb{C}^* \cup \{\infty\})$ Both $\mathbb{C}$ and $\mathbb{C}^* \cup \{\infty\}$ are connected. These two sets have nonempty intersection. This implies that their union is connected. Thus $\mathbb{P}^1$ is connected. Therefore $\mathbb{P}^1$ is a connected 2-manifold.

(b) Clearly $\mathbb{C} \cap (\mathbb{C}^* \cup \{\infty\}) = \mathbb{C}^*$ and $f^{-1}(z) = f(z) = 1/z$ is holomorphic on $\mathbb{C}^*$. Thus $\mathrm{id} \circ f^{-1} = f \circ \mathrm{id}^{-1}$ is holomorphic on $\mathbb{C}^*$. Therefore $\mathbb{P}^1$ is a Riemann surface. □

Remark. The Riemann sphere is important in complex analysis, because it can be viewed as the range of meromorphic functions.

Example 2.2.3 (Torus)**.** Let $w_1, w_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. Also let $\Gamma = \{nw_1 + mw_2 \in \mathbb{C} \mid n, m \in \mathbb{Z}\}$. Topologically the torus is defined as $T = \mathbb{C}/\Gamma$ with the quotient topology. In order to understand the complex structure on T we consider an atlas of charts. Let U_1, U_2 and U_3 be defined in the following way.

$$\begin{aligned} U_1 &= \{x_1w_1 + x_2w_2 \in \mathbb{C} \mid x_1, x_2 \in (0, 1)\} \\ U_2 &= \{x_1w_1 + x_2w_2 \in \mathbb{C} \mid x_1, x_2 \in (-1/2, 1/2)\} \\ U_3 &= \{x_1w_1 + x_2w_2 \in \mathbb{C} \mid x_1, x_2 \in (-1/4, 3/4)\} \end{aligned}$$

Also let $p_i : U_i \to\ p(U_i) \subseteq T,\ i \in \{1, 2, 3\}$ be restrictions of the quotient map $p : \mathbb{C} \to \mathbb{C}/\Gamma$. The charts are then the maps $p_i^{-1},\ i \in \{1, 2, 3\}$.

Proposition 2.2.4. *The torus T as defined in Example 2.2.3 is a Riemann surface.*

Proof. As in Proposition 2.2.2 We must show that (a) T is a conected 2-manifold and (b) that p_i^{-1} are compatible for all $i \in \{1, 2, 3\}$.

(a) First I will show that T is locally homeomorphic to $\mathbb{C}$. I claim that $\{p(U_i)\}_{i\in\{1,2,3\}}$ is an open cover of T and that $p_i,\ i \in \{1, 2, 3\}$ are homeomorphisms. The first claim is a simple result of two facts: p is an open map and $\{x_1w_1 + x_2w_2 \mid x_1, x_2 \in [-1/4, 3/4]\} \subseteq \bigcup_{i\in\{1,2,3\}} U_i$. In order to prove the second claim I only need to show that $p_i,\ i \in \{1, 2, 3\}$ are injective as they are continuous, open and surjective by definition. Let $v, z \in U_1$ such that $v \neq z$. This implies that there exist $x_1, x_2, y_1, y_2 \in (0, 1)$ such that $v = x_1w_1 + x_2w_2$ and $z = y_1w_1 + y_2w_2$. We have that $|x_1 - y_1| < 1$ and $|x_2 - y_2| < 1$ as they are all elements of the interval $(0, 1)$. One of these values must be nonzero, because $v \neq z$. This implies that $x_1 - y_1 \notin \mathbb{Z}$ or $x_2 - y_2 \notin \mathbb{Z}$. Therefore $(x_1 - y_1)w_1 + (x_2 - y_2)w_2 \notin \Gamma$. This in turn means that $p_1(v) \neq p_1(w)$. Thus p_1 must be injective. Similarly p_i is injective for all $i \in \{1, 2, 3\}$ Therefore the p_i are homeomorphisms, and as a result T is locally homeomorphic to $\mathbb{C}$.

Next I will show that T is Hausdorff. Let $z, v \in T$ such that $v \neq z$. We have that $z \in p(U_i)$ and $v \in p(U_j)$ for some $i, j \in \{1, 2, 3\}$. Let $z' = p_i^{-1}(z)$ and $v' = p_j^{-1}(v)$. Given that $\mathbb{C}$ is a two dimensional real vector spaces and $\{w_1, w_2\}$ is a linearly independent set, $\{w_1, w_2\}$ must form a basis of

$\mathbb{C}$. Therefore there are unique real numbers $x_1, x_2, y_1, y_2 \in \mathbb{R}$ such that $z' = x_1 w_1 + x_2 w_2$, and $v' = y_1 w_1 + y_2 w_2$. Without loss of generality assume that $x_1 \notin \mathbb{Z}$. Let $\delta > 0$ be the distance from x_1 to $\mathbb{Z}$, and B_z be the open ball of radius $\delta/2$ centered at z'. Clearly the closure $\overline{B_z}$ and Γ are disjoint sets. If $v' \in \Gamma$, then v and z are separated by the sets $p(B_z)$ and $p(\mathbb{C} \setminus \overline{B_z})$. If $v' \notin \Gamma$, then $y_k \notin \mathbb{Z}$ for some $k \in \{1, 2\}$. Let $\epsilon > 0$ be the distance from y_k to $\mathbb{Z}$ and B_v be the ball of radius $\epsilon/2$ centered at v'. Given that $v \neq z$ and $\mathbb{C}$ is Hausdorff, it follows that there exist open neighborhoods U of z' and V of v' such that $U \cap V = \emptyset$. In this case z and v are separated by the sets $p(U \cap B_z)$ and $p(V \cap B_v)$. Thus any pair of distinct points in T can be separated.

Now I will show that T is connected. The quotient map p is by definition continuous, and $\mathbb{C}$ is connected. This implies that $p(\mathbb{C}) = T$ is connected. Therefore T is a connected 2-manifold.

(b) It is now my task to demonstrate that p_i^{-1} and p_j^{-1} are compatible for all $i, j \in \{1, 2, 3\}$. Given that p_i and p_j are by definition restrictions of the same map, it follows that $p_i^{-1} \circ p_j$ must be the identity on $p_j^{-1}(U_i \cap U_j)$ for all $i, j \in \{1, 2, 3\}$. Thus p_i and p_j must be compatible for all $i, j \in \{1, 2, 3\}$, as the identity is a biholomorphic map. □

Functions on a torus can be extended to functions on $\mathbb{C}$ by composing with the quotient map. The resulting function is by definition doubly periodic with respect to w_1 and w_2. Conversely it is an easy exercise to show that any doubly periodic function can be considered a function on the torus where w_1 and w_2 are the periods of the function. Thus tori are the domains of doubly periodic functions and the study of doubly periodic functions is exactly the study of functions on tori.

Remark. Both of the examples given here are compact Riemann surfaces, but a Riemann surface need not be compact. Both compact and non-compact Riemann surfaces can be used in the study of complex analysis.

2.3 Holomorphic Functions

Definition 2.3.1 (Holomorphic Function)**.** Let X and Y be Riemann surfaces. A function $f : X \to Y$ is called holomorphic if for all charts $\phi : U_1 \to V_1$ on X and $\psi : U_2 \to V_2$ on Y the following holds:

$$\psi \circ f \circ \phi^{-1} \text{ is holomorphic on } \phi(U_1 \cap f^{-1}(U_2)) \tag{2}$$

We can consider $\mathbb{C}$ as a Riemann surface by simply recalling that its topological structure is that of $\mathbb{R}^2$ and giving it the complex structure associated to the atlas of identity maps. Taking $X, Y = \mathbb{C}$ in Definition 2.3.1 simply gives us that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic as a map of Riemann surfaces exactly when it is holomorphic in the usual sense. If we allow X to be any Riemann surface, but take $Y = \mathbb{C}$, then Definition 2.3.1 says that $f : X \to \mathbb{C}$ is holomorphic if and only if $f \circ \phi_i^{-1}$ is a holomorphic function in the usual sense for all charts ϕ_i on X.

Remark. We write $\mathcal{O}(X)$ for the set of holomorphic functions from X to $\mathbb{C}$.

Theorem 2.3.2 (Open Mapping)**.** *Let X and Y be Riemann surfaces. If $f : X \to Y$ is a non-constant holomorphic map, then f is an open map.*

Proof. Let $f : X \to Y$ be a non-constant holomorphic function. Also let $\phi : U_1 \to V_1$ and $\psi : U_2 \to V_2$ be charts on X and Y respectively. By the definition of a holomorphic function on a Riemann surface we have that $g = \psi \circ f \circ \phi^{-1}$ is a holomorphic function in the usual sense on $\phi(U_1 \cap f^{-1}(U_2))$. As f is non-constant and ϕ and ψ are homeomorphisms, we have that g is non-constant. Therefore by Theorem 1.2.5 g is open. This implies that $\psi^{-1} g \circ \phi$ is also open as the composition of open maps gives an open map. Thus $f|_{U_1}$ must be an open map. Therefore f is open on the domain of any chart on X. This implies that f must be open on the arbitrary union of domains of charts on X, because the arbitrary union of open sets is still open. Thus f must be open on all of X. □

Theorem 2.3.3. *Let X be a compact Riemann surface. If $f : X \to \mathbb{C}$ is holomorphic, then f is constant.*

Proof. Let $f : X \to \mathbb{C}$ be a holomorphic function, and assume f is not constant. This implies that f is an open map by Theorem 2.3.2. This in turn implies that $f(X)$ is open. Therefore $f(X)$ must also be compact, because X is compact and f is continuous. Thus $f(X)$ is closed, because $\mathbb{C}$ is Hausdorff. Therefore $f(X)$ is both open and closed. Thus $f(X) = \mathbb{C}$, because $\mathbb{C}$ is connected. This implies however that $\mathbb{C}$ is compact which is a contradiction. Therefore f must be constant. □

2.4 Sheaves

Definition 2.4.1 (Presheaf)**.** Let $(X, \mathcal{T})$ be a topological space. A presheaf of vector spaces on X is a family $\mathcal{F} = \{\mathcal{F}(U)\}_{U \in \mathcal{T}}$ of vector spaces and a collection of associated linear maps, called restriction maps,

$$\rho = \{\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V) \mid V, U \in \mathcal{T} \text{ and } V \subseteq U\}$$

such that

$$\rho_U^U = \mathrm{id}_{\mathcal{F}(U)} \text{ for all } U \in \mathcal{T}$$

and

$$\rho_W^V \circ \rho_V^U = \rho_W^U \text{ for all } U, V, W \in \mathcal{T} \text{ such that } W \subseteq V \subseteq U$$

Given $U, V \in \mathcal{T}$ such that $V \subseteq U$ and $f \in \mathcal{F}(U)$ one often writes $f|_V$ rather than $\rho_V^U(f)$.

Remark. A presheaf is usually just denoted by the name of its family of vector spaces, so the presheaf described above would be denoted $\mathcal{F}$.

Definition 2.4.2 (Sheaf)**.** Let $\mathcal{F}$ be a presheaf on a topological space X. We call $\mathcal{F}$ a sheaf on X if for all open sets $U \subseteq X$ and collections $\{U_i \subseteq U\}_{i \in I}$ such that $\bigcup_{i \in I} U_i = U$, $\mathcal{F}(U)$ satisfies the following two properties:

$$\text{For } f, g \in \mathcal{F}(U) \text{ such that } f|_{U_i} = g|_{U_i} \text{ for all } i \in I, \text{ it is given that } f = g. \tag{3}$$

$$\begin{gathered} \text{For all collections } \{f_i \in \mathcal{F}(U_i)\}_{i \in I} \text{ such that } f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \\ \text{for all } i, j \in I \text{ there exists } f \in \mathcal{F}(U) \text{ such that } f|_{U_i} = f_i \text{ for all } i \in I. \end{gathered} \tag{4}$$

Definition 2.4.3 (Sheaf of holomorphic functions, $\mathcal{O}$)**.** Let X be a Riemann surface. The presheaf $\mathcal{O}$ of holomorphic functions on X is made up of the complex vector spaces of holomorphic functions. For all open sets $U \subseteq X$, $\mathcal{O}(U)$ is the vector space of holomorphic functions on U. The restriction maps are the usual restrictions of functions.

Proposition 2.4.4. *If X is a Riemann surface, then $\mathcal{O}$ is a sheaf on X.*

Proof. Clearly $\mathcal{O}$ is a presheaf, so it is only necessary to show that it satisfies properties 3 and 4. Property (3) follows directly from the definition of a restriction of a function; two functions that agree on all the U_i must agree on U and hence be the same function.

In order to show that $\mathcal{O}$ satisfies property (4) I will construct the desired function from an arbitrary collection. Let U be an open subset of a Riemann surface X and $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of U such that $U_i \subseteq U$ for all $i \in I$. Also let $\{f_i \in \mathcal{O}(U_i)\}_{i \in I}$ be a collection of holomorphic functions such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. Now I must show that there is a function $f \in \mathcal{O}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$. For $x \in U$ define $f(x) = f_i(x)$ where $i \in I$ such that $x \in U_i$. To show that f is well defined consider $x \in U$ and $i, j \in I$ such that $x \in U_i$ and $x \in U_j$. Clearly $x \in U_i \cap U_j$. This implies that $f_i|_{U_i \cap U_j}(x) = f_j|_{U_i \cap U_j}(x)$ by the definition of the f_i. This in turn implies $f_i(x) = f_j(x)$, because the restriction map is the standard function restriction. Therefore f is a well defined function. As all the f_i are holomorphic, it follows that given any $x \in U$ there exists a neighborhood of x, namely some $U_i \in \mathcal{U}$, where f is holomorphic. Therefore $f \in \mathcal{O}(U)$. □

2.5 Cohomology

Definition 2.5.1 (Cochain)**.** Let X be a Riemann surface and $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Also let $\mathcal{F}$ be a sheaf of complex vector spaces on X and $n \in \mathbb{N} \cup \{0\}$. The n^{th} cochain group of $\mathcal{F}$ with respect to $\mathcal{U}$ is defined as follows:

$$C^n(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \dots i_n) \in I^{n+1}} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_n})$$

An n-cochain is simply an element of the n^{th} cochain group. $C^n(\mathcal{U}, \mathcal{F})$ is a complex vector space under component-wise addition and scalar multiplication.

Definition 2.5.2 (Coboundary map)**.** Let X,$\mathcal{U}$, $\mathcal{F}$ and n be defined as in Definition 2.5.1. The n^{th} coboundary map is given by

$$\delta_n : C^n(\mathcal{U}, \mathcal{F}) \to C^{n+1}(\mathcal{U}, \mathcal{F})$$

such that $(f_{i_0, \dots, i_n})_{i_0, \dots, i_n \in I} \in C^n(\mathcal{U}, \mathcal{F})$ maps to $(g_{i_0, \dots, i_{n+1}})_{i_0, \dots, i_{n+1} \in I} \in C^{n+1}(\mathcal{U}, \mathcal{F})$

$$\text{where } g_{i_0, \dots, i_{n+1}} = \sum_{m=0}^{n+1} (-1)^m f_{i_0, \dots, \widehat{i}_m, \dots, i_{n+1}}|_{U_0 \cap \cdots \cap U_{n+1}}$$

Lemma 2.5.3. *For all $n \in \mathbb{N}$ we have that $\delta_n \circ \delta_{n-1} = 0$.*

Proof. Let X be a Riemann surface, $n \in \mathbb{N}$ and $h \in C^{n-1}(X)$. Also let $f = \delta_{n-1}(h)$ and $g = \delta_n(f)$. This gives us that

$$f_{i_0, \dots, i_n} = \sum_{k=0}^{n} (-1)^k h_{i_0, \dots, \widehat{i}_k, \dots, i_n}|_{U_0 \cap \cdots \cap U_n}$$

and

$$\begin{aligned}
g_{i_0,\dots,i_{n+1}} &= \sum_{m=0}^{n+1}(-1)^k f_{i_0,\dots,\widehat{i}_m,\dots,i_{n+1}}|_{U_0\cap\dots\cap U_{n+1}} \\
&= \sum_{\substack{k<m \\ 0\le m\le n+1 \\ 0\le k\le n}} (-1)^{m+k} h_{i_o,\dots,\widehat{i}_k,\dots,\widehat{i}_m,\dots,i_{n+1}}|_{U_0\cap\dots\cap U_{n+1}} + \\
&\quad + \sum_{\substack{m\le k \\ 0\le m\le n+1 \\ 0\le k\le n}} (-1)^{m+k} h_{i_o,\dots,\widehat{i}_m,\dots,\widehat{i}_{k+1},\dots,i_{n+1}}|_{U_0\cap\dots\cap U_{n+1}} \\
&= \sum_{\substack{k<m \\ 1\le m\le n+1 \\ 0\le k\le n}} (-1)^{m+k} h_{i_o,\dots,\widehat{i}_k,\dots,\widehat{i}_m,\dots,i_{n+1}}|_{U_0\cap\dots\cap U_{n+1}} + \\
&\quad + \sum_{\substack{m<k \\ 0\le m\le n \\ 1\le k\le n+1}} (-1)^{m+k-1} h_{i_o,\dots,\widehat{i}_m,\dots,\widehat{i}_k,\dots,i_{n+1}}|_{U_0\cap\dots\cap U_{n+1}} \\
&= 0
\end{aligned}$$

Therefore $\delta_n(\delta_{n-1}(h)) = \delta_n(f) = g = 0$. □

Definition 2.5.4 (Cocyle, Coboundary)**.** Let $X, \mathcal{U}, \mathcal{F}$ and n be defined as in Definition 2.5.1. The space of n-cocyles is defined as

$$Z^n(\mathcal{U}, \mathcal{F}) = \text{Ker}(\delta_n)$$

The space of n-coboundaries is defined as

$$B^n(\mathcal{U}, \mathcal{F}) = \text{Im}(\delta_{n-1})$$

Definition 2.5.5 (Cohomology, $H^n(\mathcal{U}, \mathcal{F})$)**.** Let $X, \mathcal{U}, \mathcal{F}$ and n be defined as in Definition 2.5.1. The n^{th} cohomology with coefficients in $\mathcal{F}$ with respect to the cover $\mathcal{U}$ is then defined as

$$H^n(\mathcal{U}, \mathcal{F}) = Z^n(\mathcal{U}, \mathcal{F})/B^n(\mathcal{U}, \mathcal{F}) \tag{5}$$

The cohomology group defined above is dependent on the open cover $\mathcal{U}$. To construct a cohomology group that varies only with the choice of sheaf and Riemann surface one takes a limit using increasingly fine covers. Due to the cumbersome nature of the notation I will only give the limit definition for the first cohomology group. The given method can however be easily extend to higher cohomology groups.

Definition 2.5.6 (Refining map)**.** Let X be a Riemann surface and $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. Also let $\mathcal{V} = \{V_j\}_{j\in J}$ be an open cover of X such that $\mathcal{V}$ is finer than $\mathcal{U}$. A refining map is a map $\tau : J \to I$ such that $V_j \subseteq U_{\tau j}$ for all $j \in J$.

Definition 2.5.7 (Refinement induced maps)**.** Let X,$\mathcal{U}$ and $\mathcal{V}$ be as in Definition 2.5.6. Also let $\mathcal{F}$ be a sheaf on X. A refining map $\tau : J \to I$ induces a map

$$t^{\mathcal{U}}_{\mathcal{V}} : Z^1(\mathcal{U},\mathcal{F}) \to Z^1(\mathcal{V},\mathcal{F}) \text{ such that}$$
$$(f_{i,l}) \in Z^1(\mathcal{U},\mathcal{F}) \text{ maps to } (g_{j,k}) \in Z^1(\mathcal{V},\mathcal{F})$$
$$\text{where } g_{j,k} = f_{\tau j,\tau k}|_{V_j\cap V_k} \text{ for all } j,k \in J.$$

This map commutes with the coboundary maps and thus induces a map

$$t^{\mathcal{U}}_{\mathcal{V}} : H^1(\mathcal{U},\mathcal{F}) \to H^1(\mathcal{V},\mathcal{F}).$$

Lemma 2.5.8. *The induced map*

$$t^{\mathcal{U}}_{\mathcal{V}} : H^1(\mathcal{U},\mathcal{F}) \to H^1(\mathcal{V},\mathcal{F}).$$

is independent of the choice of refining map.

Proof. Let X,$\mathcal{U}$, $\mathcal{V}$ and $\mathcal{F}$ be as defined in Definition 2.5.7. Also let $\tau,\sigma : J \to I$ be refining maps, and $t^{\mathcal{U}}_{\mathcal{V}}$ and $s^{\mathcal{U}}_{\mathcal{V}}$ be the maps that they induce on the first cohomology groups with coefficients in $\mathcal{F}$. Finally let $(f_{i,l}) \in Z^1(\mathcal{U},\mathcal{F})$. In order to show that $t^{\mathcal{U}}_{\mathcal{V}}((f_{i,l}))$ and $s^{\mathcal{U}}_{\mathcal{V}}((f_{i,l}))$ are equivalent in $H^1(\mathcal{V},\mathcal{F})$ I must prove that their difference is in $B^1(\mathcal{V},\mathcal{F})$.

Consider $g_{j,k} = f_{\tau j,\tau k}|_{V_j\cap V_k}$ and $\bar{g}_{j,k} = f_{\sigma j,\sigma k}|_{V_j\cap V_k}$ for all $j,k \in J$. Clearly $V_j \subseteq U_{\tau j} \cap U_{\sigma j}$ by the definition of a refining map. Define $h_j = f_{\tau j,\sigma j}|_{V_j}$ for all $j \in J$. Now on $V_j \cap V_k$ we have

$$\begin{aligned}
g_{j,k} - \bar{g}_{j,k} &= f_{\tau j,\tau k} - f_{\sigma j,\sigma k} && \text{by definition}\\
&= f_{\tau j,\tau k} + f_{\tau k,\sigma j} - f_{\tau k,\sigma j} - f_{\sigma j,\sigma k} && \text{adding zero}\\
&= f_{\tau j,\sigma j} - f_{\tau k,\sigma k} && \text{because } (f_{i,l}) \text{ is a cocycle}\\
&= h_j - h_k && \text{by definition}
\end{aligned}$$

Therefore $(g_{j,k}) - (\bar{g}_{j,k}) = \delta((h_j))$, where δ is the coboundary map from $C^0(\mathcal{V},\mathcal{F})$ to $C^1(\mathcal{V},\mathcal{F})$. Thus $(g_{j,k}) - (\bar{g}_{j,k}) \in B^1(\mathcal{V},\mathcal{F})$. By the definition of the induced maps $t^{\mathcal{U}}_{\mathcal{V}}$ and $s^{\mathcal{U}}_{\mathcal{V}}$ we have that $t^{\mathcal{U}}_{\mathcal{V}}((f_{i,l})) = (g_{j,k})$ and $s^{\mathcal{U}}_{\mathcal{V}}((f_{i,l})) = (\bar{g}_{j,k})$. This of course implies that $t^{\mathcal{U}}_{\mathcal{V}} - s^{\mathcal{U}}_{\mathcal{V}} \in B^1(\mathcal{V},\mathcal{F})$. Therefore the two maps are the same. □

Lemma 2.5.9. *The induced map*

$$t^{\mathcal{U}}_{\mathcal{V}} : H^1(\mathcal{U},\mathcal{F}) \to H^1(\mathcal{V},\mathcal{F}).$$

is injective

Proof. Let X,$\mathcal{U}$, $\mathcal{V}$ and $\mathcal{F}$ be as defined in Definition 2.5.7. Also let $\tau : J \to I$ be a refining map and $t^{\mathcal{U}}_{\mathcal{V}}$ be the induced map on the first cohomology groups. In order to prove that $t^{\mathcal{U}}_{\mathcal{V}}$ is injective, it is enough to show that $\mathrm{Ker}(t^{\mathcal{U}}_{\mathcal{V}}) = \{0\}$.

Let $(f_{i,l}) \in Z^1(\mathcal{U},\mathcal{F})$ be a cycle such that $t^{\mathcal{U}}_{\mathcal{V}}((f_{i,l})) = 0$. This implies that $(f_{\tau j,\tau k}) \in B^1(\mathcal{V},\mathcal{F})$. Which in turn implies that there exists a $(g_j) \in C^0(\mathcal{V},\mathcal{F})$ such that $f_{\tau j,\tau k} = g_j - g_k$ on $V_j \cap V_k$ for all $j,k \in J$. On $U_i \cap V_k \cap V_l$ we have

$$\begin{aligned} g_j - g_k &= f_{\tau j,\tau k} && \text{by definition} \\ &= f_{\tau j,i} + f_{i,\tau k} && \text{because } (f_{j,k}) \text{ is a cocycle} \\ &= -f_{i,\tau j} + f_{i,\tau k} && \text{because } (f_{j,k}) \text{ is a cocycle} \end{aligned}$$

for all $j,k \in J$ and $i \in I$. This implies that $f_{i,\tau j} + g_j = f_{i,\tau k} + g_k$ on $(U_i \cap V_j) \cap (U_i \cap V_k)$ for all $j,k \in J$ and $i \in I$. Because $\mathcal{V}$ is an open cover of X, $\{U_i \cap V_j\}_{j\in J}$ must be an open cover of U_i. Therefore by property (4) of a sheaf, there exists $h_i \in \mathcal{F}(U_i)$ such that

$$h_i|_{U_i\cap V_j} = f_{i,\tau j} + g_j \text{ for all } j \in J.$$

For each $i \in I$ let $h_i \in \mathcal{F}(U_i)$ be the element described above. On $U_i \cap U_l \cap V_j$ we then have

$$\begin{aligned} f_{i,l} &= f_{i,\tau j} + f_{\tau j,l} && \text{because } (f_{i,l}) \text{ is a cocycle} \\ &= f_{i,\tau j} - f_{l,\tau j} && \text{because } (f_{i,l}) \text{ is a cocycle} \\ &= f_{i,\tau j} + g_j - f_{l,\tau j} - g_j && \text{adding zero} \\ &= h_i - h_l && \text{by definition} \end{aligned}$$

for all $i,l \in I$ and $j \in J$. Therefore by property (3) of a sheaf, we have that $f_{i,l} = h_i - h_l$ on $U_i \cap U_l$ for all $i,l \in I$. This implies that $(f_{i,l}) = \delta((h_i))$ where δ is the coboundary map from $C^0(\mathcal{U},\mathcal{F})$ to $C^1(\mathcal{U},\mathcal{F})$. This in turn implies that $(f_{i,j}) = 0$ in $H^1(\mathcal{U},\mathcal{F})$. Therefore $\mathrm{Ker}(t^{\mathcal{U}}_{\mathcal{V}}) = \{0\}$, and $t^{\mathcal{U}}_{\mathcal{V}}$ is injective. □

Definition 2.5.10 (Cohomology, $H^1(X, \mathcal{F})$)**.** Let X be a Riemann surface, $\mathcal{F}$ be a sheaf on X and $\mathbf{U}$ be the set of all open covers of X. Define a relation $\sim$ on the disjoint union of $H^1(\mathcal{U}, \mathcal{F})$ where $\mathcal{U} \in \mathbf{U}$ in the following way. Given $\mathcal{U}, \mathcal{V} \in \mathbf{U}$, $\alpha \in H^1(\mathcal{U}, \mathcal{F})$ and $\beta \in H^1(\mathcal{V}, \mathcal{F})$ we say $\alpha \sim \beta$ if there exists $\mathcal{W} \in \mathbf{U}$ such that $\mathcal{W} < \mathcal{U}$, $\mathcal{W} < \mathcal{V}$ and $t^{\mathcal{U}}_{\mathcal{W}}(\alpha) = t^{\mathcal{V}}_{\mathcal{W}}(\beta)$. This is an equivalence relation by Lemmas 2.5.8 and 2.5.9. The first cohomology group of X with coefficients in the sheaf $\mathcal{F}$ is defined as the set of equivalence classes of $\sim$ with addition given by adding representatives.

$$H^1(X, \mathcal{F}) = \left(\coprod_{\mathcal{U} \in \mathbf{U}} H^1(\mathcal{U}, \mathcal{F}) \right) \Big/ \sim$$

For the zeroth cohomology the limit definition is not necessary this is due to the following theorem.

Theorem 2.5.11 (Zeroth Cohomology)**.** *If X is a Riemann surface and $\mathcal{F}$ is a sheaf on X, then $H^0(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X)$ for all open covers $\mathcal{U}$ of X.*

Proof. Let X be a Riemann surface, $\mathcal{F}$ be a sheaf on X and $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Consider

$$H^0(\mathcal{U}, \mathcal{F}) = Z^0(\mathcal{U}, \mathcal{F}) / B^0(\mathcal{U}, \mathcal{F})$$

By definition $B^0(\mathcal{U}, \mathcal{F}) = 0$, because there are no nontrivial -1-cochains. Therefore

$$H^0(\mathcal{U}, \mathcal{F}) = Z^0(\mathcal{U}, \mathcal{F}) = \mathrm{Ker}(\delta_0 : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}))$$

Let $(f_i) \in Z^0(\mathcal{U}, \mathcal{F})$. This implies that $f_i = f_j$ on $U_i \cap U_j$ for all $i, j \in I$ by the definition of δ_0. Therefore by property (4) of a sheaf there exists $f \in \mathcal{F}(X)$ such that $f|_{U_i} = f_i$ for all $i \in I$. By property (3) of a sheaf this f is unique. Thus there is a bijection from $Z^0(\mathcal{U}, \mathcal{F})$ to $\mathcal{F}(X)$. Property (3) of a sheaf gives us that this is an isomorphism. Therefore $H^0(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X)$. □

Remark. Due to theorem 2.5.11, we can simply define $H^0(X, \mathcal{F})$ to be $\mathcal{F}(X)$.

Corollary 2.5.12. *If X is a compact Riemann surface, then $H^0(X, \mathcal{O}) \cong \mathbb{C}$.*

Proof. Let X be a compact Riemann surface. By Theorem 2.3.3 f is constant for all $f \in \mathcal{O}(X)$. This implies that $\mathbb{C} \cong \mathcal{O}(X) \cong H^0(X, \mathcal{O})$, by Theorem 2.5.11. □

The limit definition of cohomology is good, because it provides a structure that is only dependent on the choice of sheaf and Riemann surface. However calculations straight from the definition can be very cumbersome. Happily there are multiple theorems that make the task of computing cohomology groups much more approachable. I will now give two such theorems without proof so that I can more easily compute the first cohomology of the Riemann sphere with coefficients in the sheaf of holomorphic functions.

Theorem 2.5.13 (Cohomology of a disk)**.** *If $D = \{z \in \mathbb{C} \mid r > |z|\}$ is a disk of radius $0 < r \leq \infty$ in the complex plane, then $H^1(D, \mathcal{O}) = 0$.*

Theorem 2.5.14 (Leray)**.** *Let X be a Riemann surface and $\mathcal{U}$ be an open cover of X. If $H^1(U, \mathcal{O}) = 0$ for all $U \in \mathcal{U}$, then $H^1(\mathcal{U}, \mathcal{O}) \cong H^1(X, \mathcal{O})$.*

Remark. A more general version of this theorem holds for all cohomology groups with respect to a sheaf of abelian groups on an arbitrary topological space.

Theorem 2.5.15 (Cohomology of the Riemann Sphere)**.** $H^1(\mathbb{P}^1, \mathcal{O}) = 0$

Proof. Let $U_1 = \mathbb{P}^1 \setminus \{\infty\}$ and $U_2 = \mathbb{P}^1 \setminus \{0\}$. Also let $\mathcal{U} = \{U_1, U_2\}$ and $(f_{i,j}) \in Z^1(\mathcal{U}, \mathcal{O})$. Because $(f_{i,j})$ is a cocycle, we have that $f_{1,1} = f_{2,2} = 0$ and $f_{1,2} = -f_{2,1}$. This means that $(f_{i,j})$ is determined by its value at $f_{1,2}$. By the definition of the Riemann sphere, $U_1 \cap U_2 = \mathbb{C}^*$. This implies that $f_{1,2}$ is an element of $\mathcal{O}(\mathbb{C}^*)$ and hence is a holomorphic function with a Laurent expansion on $\mathbb{C}^*$. Let

$$f_{1,2}(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

be its Laurent expansion. Also let

$$f_1(z) = \sum_{n=0}^{\infty} c_n z^n \text{ and } f_2(z) = -\sum_{n=-\infty}^{-1} c_n z^n.$$

Clearly $f_i \in \mathcal{O}(U_i)$ for all $i \in \{1, 2\}$, and $f_{1,2} = f_1 - f_2$. This implies that $(f_{i,j}) = \delta((f_i))$ where δ is the coboundary map from $C^0(\mathcal{U}, \mathcal{O})$ to $C^1(\mathcal{U}, \mathcal{O})$. Therefore for all $(f_{i,j}) \in Z^1(\mathcal{U}, \mathcal{O})$ we have that $(f_{i,j}) \in B^1(\mathcal{U}, \mathcal{O})$, and thus $H^1(\mathcal{U}, \mathcal{O}) = 0$.

By the definition of the Riemann sphere $U_1 = \mathbb{C}$ and $U_2 = \mathbb{C}^* \cup \{\infty\}$. Thus U_2 is biholomorphic to $\mathbb{C}$ under the map $z \mapsto 1/z$. Therefore by Theorem 2.5.13

$$H^1(U_i, \mathcal{O}) = H^1(\mathbb{C}, \mathcal{O}) = 0 \text{ for all } i \in \{1, 2\}.$$

Thus by Theorem 2.5.14

$$H^1(\mathbb{P}^1, \mathcal{O}) = H^1(\mathcal{U}, \mathcal{O}) = 0.$$

□

2.6 Further Study

This paper has provided an introduction to the study of Riemann surfaces. As motivation for further study here are two beautiful and powerful theorems relating to the cohomology of Riemann Surfaces.

Theorem 2.6.1. *If X is a compact Riemann surface, then*

$$\dim H^1(X, \mathcal{O}) < \infty$$

.

It turns out that for a compact Riemann surface X we have that $H^1(X, \mathcal{O}) \cong \mathbf{C}^g$ where g is the genus of X. This is interesting, because it shows an algebraic structure built from analytic objects reflecting a topological property.

Theorem 2.6.2 (Serre Duality)**.** *If X is a compact Riemann surface, then $H^1(X, \mathcal{O}) \cong H^0(X, \Omega)$ where Ω is the sheaf of holomorphic one forms on X.*

Remark. A more general version of this theorem holds for any Riemann surface and can be applied to different pairs of sheaves.

This theorem is interesting, because it shows that there is a strong relationship between holomorphic one forms on Riemann surfaces and holomorphic functions. It is also useful for calculating cohomology groups, because it reduces calculations of the first cohomology to calculations of the zeroth.

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