Le Chen

# An Investigation of the Planarity Condition of Grötzsch's Theorem

The University of Chicago: VIGRE REU 2007

July 16, 2007

Abstract

The idea for this paper originated from Professor László Babai's challenge to find a triangle-free graph with chromatic number 4. Professor Babai gave the following hints: the graph would have eleven vertices, the graph would be five-fold symmetric, and the graph was called Grötzsch's Graph. After independently discovering a graph which satisfied all the conditions, I checked with Professor Babai, who confirmed that it was indeed an isomorphism of Grötzsch's Graph. Independent further research led me to Grötzsch's Theorem, which I will state. The purpose of this paper is to demonstrate the need for the condition of planarity in Grötzsch's Theorem, using Grötzsch's Graph as an example. This paper is conducted from first principles; all terms will be defined along the way. No previous experience with graph theory is necessary.

## Section 1: Preliminaries

Definition 1.1: A **graph** is a pair $G = (V, E)$, where $V$ is the set of vertices and $E$ is the set of edges.

Definition 1.2: An **edge** is an unordered pair of vertices.

Definition 1.3: Two vertices are **adjacent** if they are joined by an edge. Two adjacent vertices are said to be **neighbors.**

Definition 1.4: A **walk** of length $k$ is a sequence of $k + 1$ vertices $v_o,\ldots,v_k$ such that $v_{i-1}$ and $v_i$ are adjacent for all $i$.

Definition 1.5: A **closed walk** of length k is a walk $v_0,\ldots,v_k$ where $v_k = v_o$.

Definition 1.6: A **cycle** of length k is a closed walk of length k with no repeated vertices except that $v_o = v_k$.

Definition 1.7: A graph is **triangle-free** if it does not contain a cycle of length 3.

Definition 1.8: A **legal** *k*-**coloring** of a graph is a function $c: V \rightarrow [k] = \{1,\ldots,k\}$ such that adjacent vertices receive different colors. A graph is *k*-**colorable** if there exists a legal *k*-coloring. The **chromatic number** $\chi(G)$ of a graph is the smallest k such that G is k-colorable.

Definition 1.9: A **planar graph** is a graph that can be drawn in the plane so that the lines representing the edges do not intersect except at their end vertices.

## Section 2: Grötzsch's Theorem

With the preceding definitions in mind, we are now able to state Grötzsch's Theorem without any ambiguity. Grötzsch's Theorem is named after the German mathematician Herbert Grötzsch, who proved it in 1959. However, the proof of this theorem is outside the scope of this paper.

**Theorem 2.1 (Grötzsch's Theorem):** Every triangle-free planar graph is 3-colorable.

The purpose of this paper is to prove that the condition of planarity is necessary for Grötzsch's Theorem to hold. To do this, we will use a particular graph, known, appropriately enough, as Grötzsch's Graph (see Figure 2.2). First, we will show that Grötzsch's Graph is triangle-free. Then, we will prove that it is non-planar, and finally we will show that it has a chromatic number of 4. By showing that there exists a triangle-free non-planar graph which is not 3-colorable, we will have proven that planarity is required for Grötzsch's Theorem to hold.

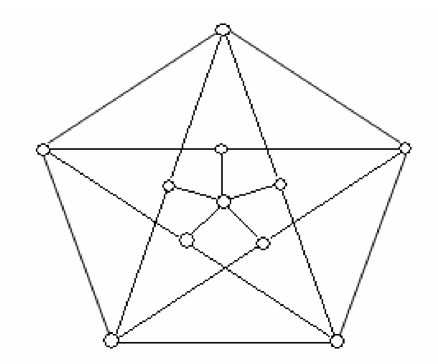


Figure 2.2: Grötzsch's Graph

## Section 3: Proof that Grötzsch's Graph is triangle-free.

In order to prove that Grötzsch's Graph is triangle-free, we must prove that it does not contain a cycle of length 3. To aid us in this endeavor, observe that Grötzsch's Graph is five-fold symmetric, that is, it is symmetric with respect to a 72° rotation. Hence, we only need to consider three possible starting vertices for any cycle: a vertex on the outer boundary, the center vertex, or one of the neighbors of the center vertex. In order to further aid us in the proof, we will assign numbers to each of the vertices in order to easily express the cycles as progressions of numbers (see Figure 3.1).

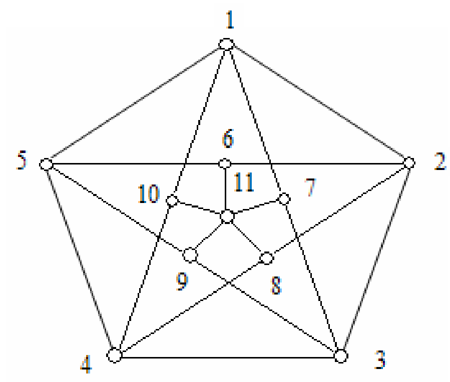



Figure 3.1: Grötzsch's Graph with numbered vertices

**Theorem 3.2:** Grötzsch's Graph is triangle-free.

Proof: As noted above, we will need to consider three cases. The proof will follow by process of exhaustion.

Case I: The cycle begins at a vertex on the outer boundary. Without loss of generality, let the cycle begin at vertex 1. From vertex 1, the cycle can proceed to vertex 2, 5, 7, or 10. From these vertices, the cycle can then proceed to vertex 3, 4, 6, 8, 9, or 11. However, observe that none of these vertices are adjacent to vertex 1; hence, a triangle cannot be formed.

Case II: The cycle begins at a neighbor of the center vertex. Without loss of generality, let the cycle begin at vertex 6. From vertex 6, the cycle can proceed to vertex 2, 5, or 11. From these vertices, the cycle can then proceed to vertex 1, 3, 4, 7, 8, 9, or 10. Again, observe that none of these vertices are adjacent to vertex 6; hence, a triangle cannot be formed.

Case III: The cycle begins at the center vertex. The first move will be to a neighbor of the center vertex. Since no two neighbors of the center point are adjacent, the second move will be to a vertex on the outer boundary. Since no vertex on the outer boundary is adjacent to the center vertex, it follows that no triangle can be formed.

As noted above, Grötzsch's Graph is five-fold symmetric, so these three cases are exhaustive. Hence, Grötzsch's Graph is triangle free.

□

## Section 4: Proof that Grötzsch's Graph is non-planar.

At first glance, it may appear obvious that Grötzsch's Graph is non-planar: the lines in the graph clearly intersect. However, the definition specifically says that a graph is planar if it *can be drawn* so that the edges do not intersect. As there are an infinite number of ways to draw Grötzsch's Graph, the definition is not very helpful. Instead, to prove that Grötzsch's Graph is non-planar, we will use an important theorem in graph theory known as Kuratowski's theorem, after the Polish mathematician Kazimierz Kuratowski, who proved it in 1930. However, to do this we must first establish some preliminaries.

Definition 4.1: A **complete graph** is a graph in which all pairs of vertices are adjacent. The complete graph on $n$ vertices is denoted by $K_n$.

Definition 4.2: A graph $H = (W, F)$ is a **subgraph** of $G = (V, E)$ if $W$ is a subset of $V$ and $F$ is a subset of $E$.

Definition 4.3: A **subdivision** $H$ of a graph $G$ is a new graph $H$ obtained by subdividing some of the edges of $G$ with additional vertices. It is also convention to regard $G$ as a subdivision of itself.

Definition 4.4: Two graphs are **homeomorphic** if they are both a subdivision of the same graph.

With these preliminaries in mind, we can now state Kuratowski's Theorem without any ambiguity. As with Grötzsch's Theorem, the proof of Kuratowski's Theorem is outside the scope of this paper.

**Theorem 4.5 (Kuratowski's Theorem)**: A graph is planar if and only if it does not contain a subgraph homeomorphic to $K_{3,3}$ or $K_5$.

$K_{3,3}$ is the complete bipartite graph on two sets of three vertices, which does not concern us. We will focus on $K_5$, the complete graph on five vertices. The purpose of this section is to show that Grötzsch's Graph is non-planar. We will do this by finding a subgraph of Grötzsch's Graph that is homeomorphic to $K_5$. This, by the properties of the contrapositive, will prove that Grötzsch's Graph is non-planar.

**Theorem 4.6**: Grötzsch's Graph is non-planar.

Proof: We must find a subgraph of Grötzsch's Graph that is homeomorphic to $K_5$, the complete graph on five vertices. The subgraph that we are searching for is simply the subgraph where we omit the central vertex and all its associated edges (see Figure 4.7). Let's compare that with the complete graph on five vertices (see Figure 4.8). Clearly, our subgraph is a subdivision of $K_5$, obtained by inserting a vertex onto each of the edges joining the outer points. $K_5$ in turn is a subdivision of itself, which means that both our subgraph and $K_5$ are subdivisions of $K_5$, which by definition means that our subgraph and $K_5$ are homeomorphic. Hence, by Kuratowski's Theorem, Grötzsch's Graph is non-planar.

□

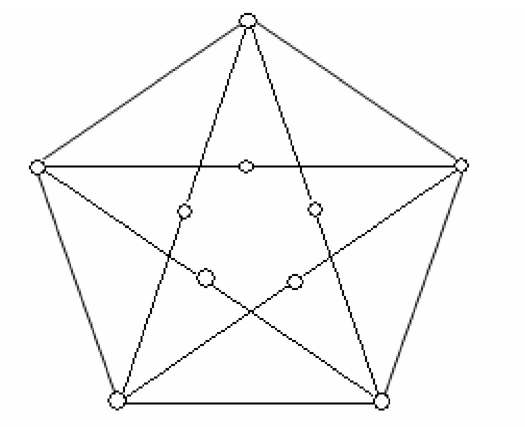


Figure 4.7: Subgraph of Grötzsch's Graph without the central vertex

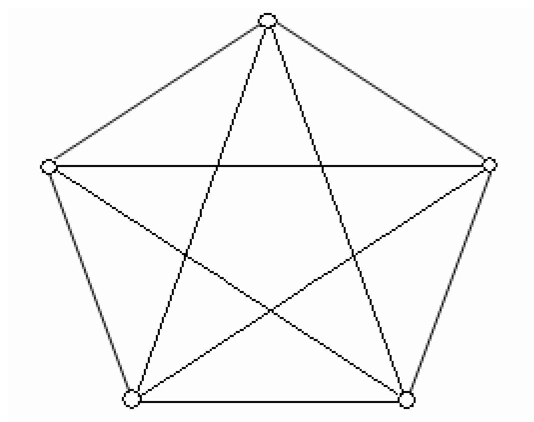


Figure 4.8: The complete graph on five vertices

## Section 5: Proof that Grötzsch's Graph has chromatic number 4.

Since there exist adjacent points in Grötzsch's Graph, it is clearly not 1-colorable. In order to prove that it has chromatic number 4, we will also have to prove that Grötzsch's Graph is neither 2-colorable nor 3-colorable, but that it is in fact 4-colorable. To do this, I will utilize an isomorphism of Grötzsch's Graph where adjacent vertices are more easily visualized. In solving Professor Babai's challenge, this was the graph I came up with.

Definition 5.1: An **isomorphism** between the graphs $G = (V, E)$ and $H = (W, F)$ is a bijection $f: V \rightarrow W$ which preserves adjacency.

Observe that since isomorphisms preserve adjacency, the coloring properties of a graph will be the same as those of its isomorphisms.

My isomorphism is pictured below (see Figure 5.2). The vertices of my isomorphism are numbered so that they are a bijection of Grötzsch's Graph using the numbered vertices from Figure 3.1. From this point forward I will refer to my isomorphism as *G* and refer to its vertices by number based off of Figure 5.2. Observe that *G* is five-fold symmetric and preserves the adjacency of Grötzsch's Graph.

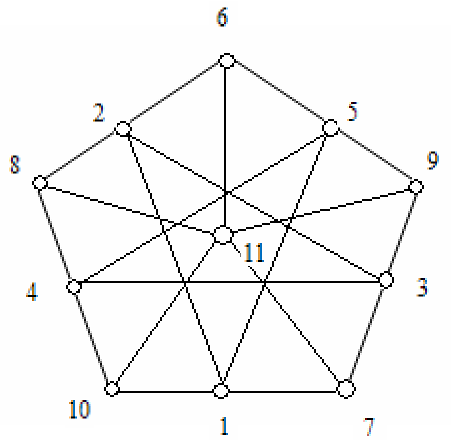



Figure 5.2: *G,* an isomorphism of Grötzsch's Graph

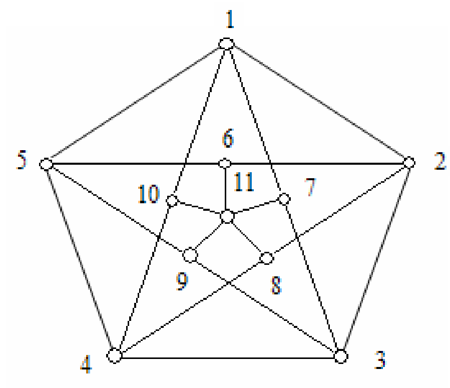



Figure 5.3: Grötzsch's Graph with numbered vertices

We will start by proving that *G*, and therefore Grötzsch's Graph, is not 2-colorable.

**Theorem 5.4**: Grötzsch's Graph is not 2-colorable.

Proof: By contradiction. Using *G*, assume that *G* is 2-colorable. Fix the color of the center vertex (red in Figure 5.5). All the neighbors of the center vertex must be of a different color. Since we are assuming that we can completely color *G* with 2 colors, all the neighbors of the center vertex must be the same color (blue in Figure 5.5). Hence, vertices 1-5 cannot be blue, since they are all adjacent to blue vertices. We are assuming that *G* is 2-colorable; therefore, vertices 1-5 must be red. However, without loss of generality, let vertex 5 be red. This means that vertices 1 and 4 are adjacent to both red and blue vertices, and therefore can be neither red nor blue, which contradicts the assumption. Therefore, by contradiction, *G* is not 2-colorable. Since *G* is an isomorphism of Grötzsch's Graph, it follows that Grötzsch's Graph is not 2-colorable.

□

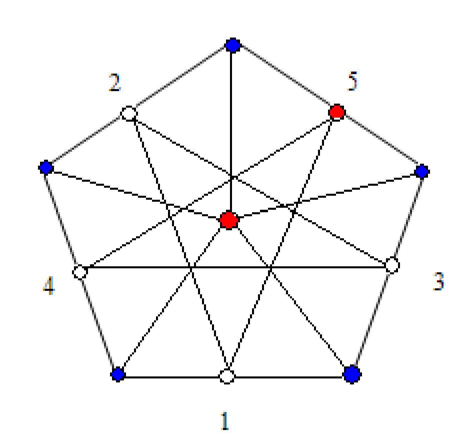



Figure 5.5: Attempted 2-coloring of *G*

We will now prove that Grötzsch's Graph is not 3-colorable. We will do this by assuming that Grötzsch's Graph is in fact 3-colorable, fixing the color of the center vertex and exhausting the possible cases for color combinations of the neighbors of the center vertex, then showing that contradictions will follow in each case. We will use the isomorphism *G* in the proof. Since we are assuming that *G* is 3-colorable, there are four possible cases for the coloring of the center neighbors: all of the neighbors are the same color, four are of one color and one of is a different color, three consecutive neighbors are of one color and the other two are of a different color, and three neighbors of one color are split up by two neighbors of a different color. These cases are illustrated below (see Figure 5.6). Observe that it does not matter which specific vertices are assigned which colors, since *G*, and Grötzsch's Graph, are five-fold symmetric.

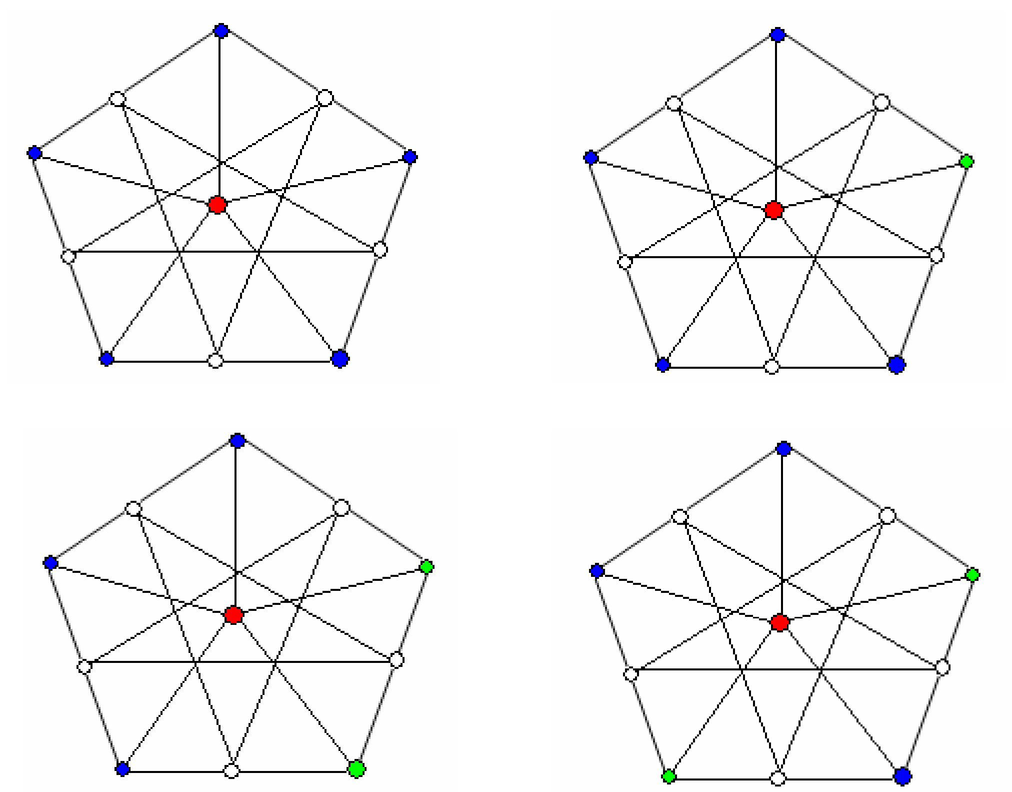


Figure 5.6: The possible color configurations of the center and center neighbors under the assumption of 3-colorability (not specific to vertices).

**Theorem 5.7:** Grötzsch's Graph is not 3-colorable.

Proof: Assume that Grötzsch's Graph is 3-colorable. Fix the color of the center vertex (red in all Figures). The proof will proceed by exhausting cases for the possible colors of the center neighbors. Each case will proceed by contradiction.

Case I: The five center neighbors are all the same color (blue in Figure 5.8). Observe that vertices 1-5 cannot be blue, since they are all adjacent to blue vertices. We are assuming 3-colorability, hence, vertices 1-5 must be either red or green. Without loss of generality, let vertex 5 be red. This forces vertices 1 and 4 to be green. Since vertices 2 and 3 are adjacent to vertices 1 and 4, respectively, neither vertex 2 nor vertex 3 can be green. Without loss of generality, let vertex 2 be red. But now observe that vertex 3 is adjacent to red, green, and blue vertices, so it must be a fourth color. But this is a contradiction. Now, without loss of generality, let vertex 5 be green. This forces vertices 1 and 4 to be red. By a similar argument to the above, letting vertex 2 be green forces vertex 3 to be a fourth color, again a contradiction. Hence, by contradiction, *G* is not 3-colorable in this case.

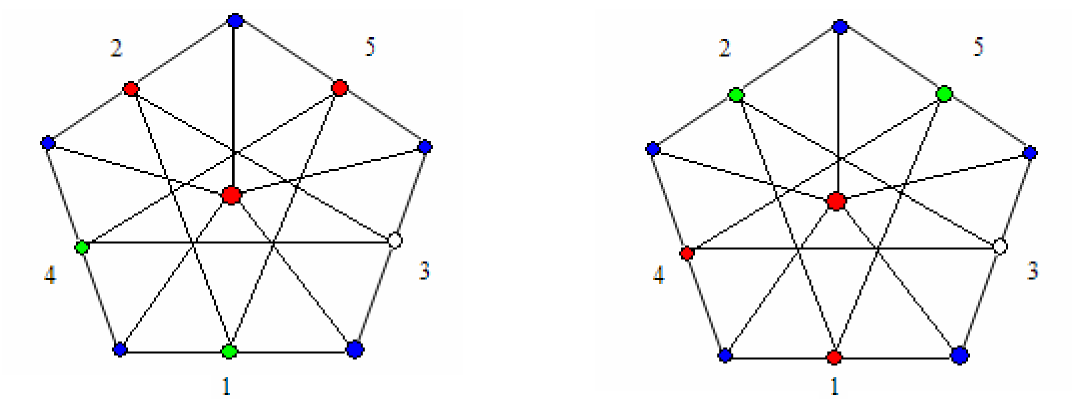



Figure 5.8: Attempted 3-colorings of *G* in case I.

Case II: Four of the center neighbors are one color (blue in Figure 5.9), and one is of a different color (green in Figure 5.9). Without loss of generality, let vertex 9 be the green vertex. We are assuming 3-colorability; hence, vertices 5 and 3 must be red. This in turn forces vertices 1 and 4 to be green. But now observe that vertex 2 is adjacent to red, green, and blue vertices, so it must be a fourth color, which is a contradiction. Hence, *G* is not 3-colorable in this case.

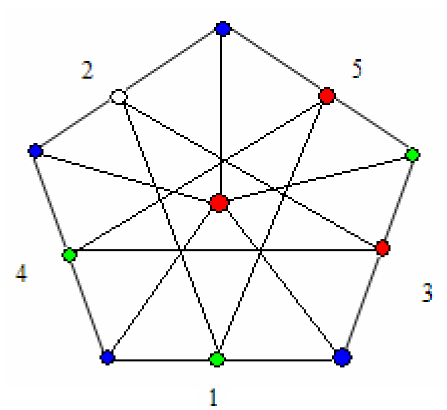



Figure 5.9: Attempted 3-coloring of *G* in case II.

Case III: Three consecutive center neighbors are of one color (blue in Figure 5.10), and the other two are of a different color (green in Figure 5.10). Without loss of generality, let vertices 7 and 9 be the green vertices. We are assuming 3-colorability; hence vertices 1 and 5 must be red. But this is a contradiction, as vertices 1 and 5 are adjacent and therefore cannot be the same color. Hence, *G* is not 3-colorable in this case.

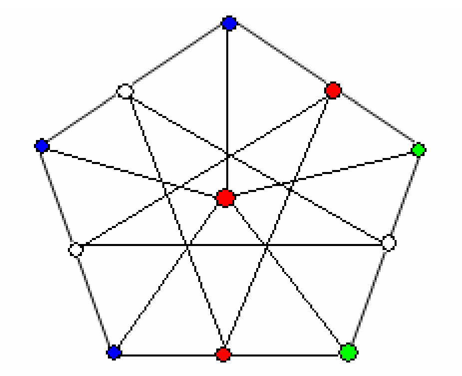


Figure 5.10: Attempted 3-coloring of *G* in case III.

Case IV: Three center neighbors are of one color (blue in Figure 5.11), and they are split up by two neighbors of a different color (green in Figure 5.11). Without loss of generality, let vertices 9 and 10 be the green vertices. We are assuming 3-colorability; hence vertices 1 and 5 must be red. But this is a contradiction, as vertices 1 and 5 are adjacent and therefore cannot be the same color. Hence, *G* is not 3-colorable in this case.

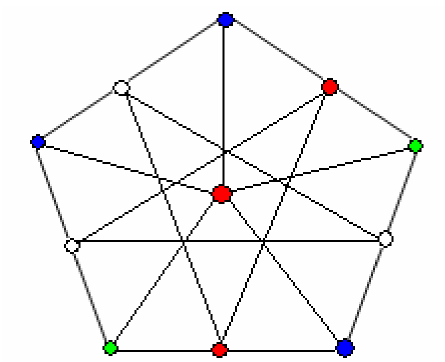
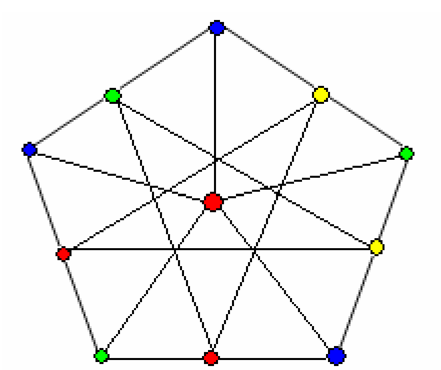


Figure 5.11: Attempted 3-coloring of *G* in case IV.

The five-fold symmetry of *G* means that these four cases are exhaustive. All four cases result in contradictions of the main assumption, which is that *G* is 3-colorable. Hence, by contradiction, *G* is not 3-colorable. Since *G* is an isomorphism of Grötzsch's Graph, and isomorphisms preserve adjacency, it follows that Grötzsch's Graph is not 3-colorable.

□

**Theorem 5.12:** Grötzsch's Graph has chromatic number 4.



Proof: Taking Figure 5.11 and adding a fourth color (yellow in Figure 5.13), we see that *G* can indeed be legally 4-colored. This, along with Theorems 5.4 and 5.7 imply by definition that *G*, and therefore Grötzsch's Graph, has chromatic number 4.

□

Figure 5.13: 4-coloring of *G*

## Section 6: Conclusion

For reference, Grötzsch's Theorem is reprinted here, along with the three key theorems of the paper.

**Theorem 2.1 (Grötzsch's Theorem):** Every triangle-free planar graph is 3-colorable
**Theorem 3.2:** Grötzsch's Graph is triangle-free.
**Theorem 4.6**: Grötzsch's Graph is non-planar.
**Theorem 5.12:** Grötzsch's Graph has chromatic number 4.

Taking the validity of Grötzsch's Theorem as a given, and having proven that Grötzsch's Graph is triangle-free, non-planar, and has chromatic number 4 (i.e. is not 3-colorable), we have shown that planarity is a necessary condition to Grötzsch's Theorem. Therefore, we have fulfilled the purpose of this paper.

## Section 7: Acknowledgments

1) Definitions 1.1-1.9, Definitions 4.1-4.4, Theorem 4.5, and Definition 5.1 all come from Professor László Babai, *Discrete Mathematics: Lecture Notes, Incomplete Preliminary Version,* 2003. An exercise posed in class, mentioned in the Abstract, was also the inspiration for this paper.

2) Grötzsch's Theorem and Grötzsch's Graph are both taken from the Wikipedia article on the Grötzsch Graph, <http://en.wikipedia.org/wiki/Grotzsch_graph>. Last accessed July 16, 2007.

3) The idea that a graph is a subdivision of itself (mentioned in Definition 4.3) was not in Professor Babai's definition. This property of subdivisions was confirmed by multiple articles following a Google search with the search prompt "a graph is a subdivision of itself."