SURFACE GROUPS

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1. Introduction

Definition 1.1. We define a *surface group* to be the group given the by presentation

$$\Gamma_g = \langle a_1, b_1, a_2, b_2 \dots a_g, b_g | [a_1, b_1] \dots [a_g, b_g] = 1 \rangle.$$

Question 1.2. *For what integers g and h is Γ_h a subgroup of Γ_g?*

As a first step, we observe the connection between the surface group Γ_g and the closed, orientable genus-g surface Σ_g.

Computation 1.3. *The fundamental group $\pi_1(\Sigma_g)$ of the closed, orientable genus-g surface Σ_g is the surface group Γ_g.*

Proof. First, recall *Proposition 1.26* from Hatcher's *Algebraic Topology* regarding applications of van Kampen's Theorem to cell complexes: If we attach 2-cells to a path connected space X via maps ϕ_α, making a space Y, and $N \subset \pi_1(X, x_0)$ is the normal subgroup generated by all loops $\lambda_\alpha \phi_\alpha \lambda_\alpha^{-1}$, then the inclusion $X \hookrightarrow Y$ induces a surjection $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ whose kernel is N. Thus $\pi_1(Y) \approx \pi_1(X)/N$.

Consider the wedge-sum of $2g$ circles, labeled $a_1, a_2, \dots, a_g$, and $b_1, b_2, \dots, b_g$. This will be the 1-skeleton of a cell-decomposition of the genus-g surface. By gluing a single 2-cell along the word $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$, we obtain the closed, orientable genus-g surface. This can be visualized, as in Figure 1 below, by the familiar identification space of a genus-g surface as a $4g$-gon with pairs of edges, and all of the vertices, identified.

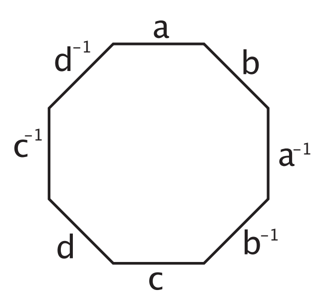

Figure 1. Genus-2 surface as a quotient of an octogon

By van Kampen's theorem, $\pi_1(\bigvee_{i=1}^{2g} S_i^1)$ of $2g$ circles is the free group on $2g$ generators, and by *Proposition 1.26*, $\pi_1(\Sigma_g)$ is the quotient of $\pi_1(\bigvee_{i=1}^{2g} S_i^1)$ by the normal subgroup generated

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by the word $[a_1, b_1] \dots [a_g, b_g]$, which is precisely the surface group Γ_g. This can be explained by the observation that the loop $[a_1, b_1] \dots [a_g, b_g]$ (or any of its conjugates) is nullhomotopic as a result of having added the 2-cell. □

2. Preliminaries

Proposition 2.1. *Σ_h is a covering space of Σ_g if and only if $\chi(\Sigma_g) | \chi(\Sigma_h)$.*

Proof. It is well known, and easy to verify, that $\chi(\Sigma_g) = 2 - 2g$.

Suppose $\chi(\Sigma_g)|\chi(\Sigma_h)$, then for some $n \in \mathbb{N}$, $n(2-2g) = 2-2h$, so $h = n(g-1)+1$, and we see by cutting along the loops L, as in figure 2, and identifying them, that Σ_h is a covering space of Σ_g.

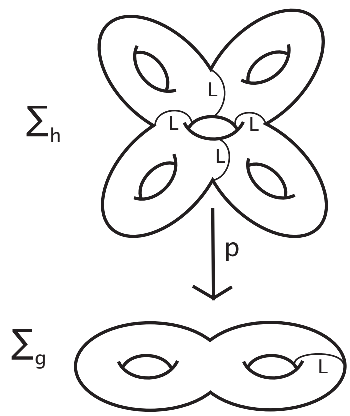

Figure 2. Σ_5 covering Σ_2 with 4-fold symmetry

Now suppose that Σ_h is an n-sheeted covering space of Σ_g. Recall that if T_g is a triangulation of the surface Σ_g, and V, E, and F are the number of vertices, edges, and faces of T respectively, then we can calculate the Euler characteristic of Σ_g by $\chi(\Sigma_g) = V - E + F$. Let T_g be a triangulation of Σ_g. Then T_g "lifts" to a triangulation T_h of Σ_h. Note, by *Proposition 1.33, suppose given a covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X_0, x_0)$ and a map $f : (Y, y_0) \to (X, x_0)$ with Y path-connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ of f exists iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.* Any contractible space has trivial fundamental group, hence any map from a contractible space to Σ_g lifts. Then, for each vertex e^0_α in T_g, there are n pre-images of e^0_α in T_h. Hence, if V is the number of vertices in T_g, nV is the number of vertices in T_h. Similarly, since the edges and the faces of the triangulation are contractible, each one lifts to n distinct pre-image edges and and faces. Thus, if T_g has E edges and F faces, T_h has nE edges and nF faces. Thus $\chi(\Sigma_g) = V - E + F$, and $\chi(\Sigma_h) = nV - nE + nF = n(V - E + F) = n\chi(\Sigma_g)$. □

If the covering space $\widetilde{\Sigma}_g$ of Σ_g is an "infinite genus" surface, then the statement $\chi(\widetilde{\Sigma}_g)|\chi(\Sigma_g)$ is not well defined. In general, for an infinite-sheeted cover, it may not be true that $\chi(\widetilde{\Sigma}_g)|\chi(\Sigma_g)$. But this does not create any problems because the fundamental group of an infinite genus surface is not a surface group.

Proposition 2.2. *The fundamental group of a non-compact surface is free.*

We shall use this fact without proof.

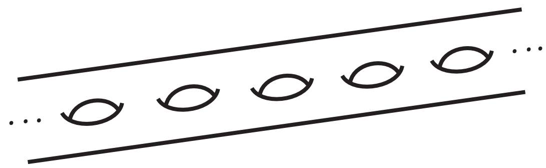

Figure 3. Example of an infinite-genus surface

3. The Answer

Theorem 3.1. *Given integers g and h, $\Gamma_h < \Gamma_g$ if and only if $g-1|h-1$.*

Proof. This proof hinges on the Fundamental Theorem of Covering Spaces, quoted here from from Hatcher:

Let X be path-connected, and semilocally simply-connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\widetilde{X}, \widetilde{x}_0) \to (X_0, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$, obtained by associationg the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ to the covering space $(\widetilde{X}, \widetilde{x}_0)$. If basepoints are ignored, this correspondences gives a bijection between isomorphism classes of path-connected covering spaces $p : \widetilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

Now, suppose $g-1|h-1$. Then, $2-2g|2-2h$, so $\chi(\Sigma_g)|\chi(\Sigma_h)$ for the genus-g and -h surfaces Σ_g and Σ_h. Hence, by Proposition 2.1, Σ_h is a covering space of Σ_g, so $\Gamma_h = \pi_1(\Sigma_h) < \pi_1(\Sigma_g) = \Gamma_g$.

Next, suppose $\Gamma_h < \Gamma_g$. By the Fundamental Theorem of Covering Spaces, there is a covering space $\widetilde{\Sigma}_g$ of Σ_g such that $\pi_1(\widetilde{\Sigma}_g) = \Gamma_h$. By proposition 2.2, $\widetilde{\Sigma}_g$ is not an infinite-sheeted cover since its fundamental group is Γ_h. Therefore, it must be some closed, orientable surface with finite genus. However, since only the surface Σ_h has the fundamental group Γ_h, $\widetilde{\Sigma}_g = \Sigma_h$. Thus, Σ_h is a covering space of Σ_g, hence $g-1|h-1$. $\square$

References

[1] A. Hatcher. Algebraic Topology. Cambridge University Press. 2002.