On the Fundamental Group of a Generalized Lens Space

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August 13, 2007

Abstract

The purpose of this paper is to provide a brief overview of the topological properties of higher and infinite-dimensional versions of the 3-dimensional lens space $L(p, q)$. Specifically, this work focuses on determining the fundamental groups of the generalized lens space as well as the necessary conditions for lens spaces of the same dimension to be homotopy equivalent.

1 Covering Spaces

Definition 1
A continuous map between topological spaces, $p : X^c \mapsto X$, is a *covering map* if and only if
1. p is surjective
2. $\forall\ x \in X$, $\exists$ an open neighborhood U around x such that $p^{-1}(U) = \bigcup_{j \in J} U_j$ with U_j pairwise disjoint and open and $p : U_j \mapsto U$ a homeomorphism for each U_j.

Definition 2
A group action of a group G acting on a topological space X is *properly discontinuous* if and only if for each $x \in X$ there exists an open neighborhood U around x, such that $g_1 \circ U \cap g_2 \circ U = \emptyset$ for all $g_1, g_2 \in G$ with $g_1 \neq g_2$.

Definition 3
Suppose $p : X^c \mapsto X$ is a covering map and $f : Y \mapsto X$ is continuous. A *lift* of f is a continuous map $f_l : Y \mapsto X^c$ such that $p \circ f_l = f$.

Theorem 1.1
Let G act on X. Then if the action is properly discontinuous, $p : X \mapsto X/G$ under the canonical projection is a covering map.

Proof : Let U be an open neighborhood of $x \in X$ such that $g_1 \circ U \cap g_2 \circ U = \emptyset$ for all $g_1 \neq g_2$ with $g_1, g_2 \in G$. Since p is open, $p(U)$ is open, and $p^{-1}(p(U))$ are the orbits of U, which are open sets of the form $g \circ U$ with $g \in G$. Hence, because $p : g \circ U \mapsto p(U)$ is bijective, open, and continuous, it is a homeomorphism for each $g \circ U$. $\square$

Theorem 1.2
Let $p : X^c \mapsto X$ be a covering map. Then p is an open map.

Proof : Let U be an open subset of X^c and let $x \in p(U)$. Then $\exists\ O$ open around x such that O is evenly covered, and $p^{-1}(O) = \bigcup_{j \in J} O_j^c$ for $O_j^c \in X^c$ pairwise disjoint.

Then $p(O_j^c \cap U) \subseteq p(U)$ is open in O for some O_j^c with $p^{-1}(x) \subseteq O_j^c \subseteq U$, since p is a homeomorphism. Thus, $\forall x \in p \mid U$ there is an open neighborhood around x, namely $p(O_j^c \cap U)$. $\square$

Exercise 1.2
(1) Prove: X has the quotient topology with respect to the covering map p.

(2) Let G be a group and X be Hausdorff. Prove: If the action of G on X is free, then the action is properly discontinuous (Hint: Consider the intersection of the open neighborhoods of $\{g \cdot x \mid g \in G\}$ where $x \in X$).

Theorem (Path Lifting)
Let $f : [0,1] \mapsto X$ be a continuous map and $p : X^c \mapsto X$ be a covering. Suppose $x_0^c \in X^c$ such that $p(x_0^c) = f(0)$. Then there exists a unique lift f_l of f such that $f_l(0) = x_0^c$.

Proof : For each $x \in X$, let U_x be an evenly covered open neighborhood of x. Then $\bigcup_{x \in X} f^{-1}(U_x)$ covers [0,1].

Because [0,1] is compact, $\exists$ a finite subset of $\bigcup_{x \in X} f^{-1}(U_x)$ of the form $\bigcup_{i=1}^{n} I_i$ such that $[0, b_1) = I_1, (a_n, 1] = I_n, (a_i, b_i) = I_i$ with $b_{i+1} < a_i$.

Let $t_i \in [0,1]$ such that $a_{i+1} < t_i < b_i$ for $i < n$. Note that, for each i, $f([t_i, t_{i+1}]) \subset f(I_i) \in \{U_x \mid x \in X\}$ with $p \mid p^{-1}[t_i, t_{i+1}] : X^c \mapsto X$ a homeomorphism.

We proceed inductively to prove the existence/uniqueness f_l, the lift of f. Let $f_l(0) = x_0^c$. Then $f_l(s)$ is defined and unique on $s = 0$. Suppose, now, that f_l is defined and unique on $[0, t_i]$ such that $p \circ f_l(t_i) = f(t_i)$. Because p is a homeomorphism on $[t_i, t_{i+1}]$, there is a unique $\gamma : [t_i, t_{i+1}] \mapsto X^C$ such that $p \circ \gamma = f$. Let $f_l = \gamma$ on $[t_i, t_{i+1}]$, and define f_l as before on $[0, t_i]$. Then f_l is defined and unique on $[0, t_{i+1}]$. $\square$

Corollary to the Path Lifting Theorem
Let $f : [0,1] \times [0,1] \mapsto X$ be a continuous map and $p : X^c \mapsto X$ be a covering. Suppose $x_0^c \in X^c$ such that $p(x_0^c) = f(0,0)$. Then there exists a unique lift f_l of f such that $f_l(0,0) = x_0^c$.

Proof : Left as an exercise (Hint: Consider the cross products of the intervals $[t_i, t_{i+1}]$ and apply the method described in the proof of the Path Lifting Theorem).

Theorem 1.3
Let $p : X^c \mapsto X$ be a covering, and suppose that f_1 and f_2 are two lifts of $f : Y \mapsto X$ with Y connected. If $f_1(y_0) = f_2(y_0)$ for some $y_0 \in Y$, then $f_1 = f_2$.

Proof : Let Ω be the set of all $y \in Y$ such that $f_1(y) = f_2(y)$. We prove Ω is both open and closed.

Let $y \in \Omega$. Then there is an open neighborhood U of $f(y)$ such that $p^{-1}(f(y)) = \bigcup_{j\in J} O_j$ where the O_j are pairwise disjoint, open sets in X^c mapped homeomorphically into U by p.

Then $f_1(y) = f_2(y)$ and $p \circ f_1(y) = p \circ f_2(y) = f(y) \Rightarrow \exists\ O_i \in \bigcup_{j\in J} O_j$, $f_1(y) = f_2(y) \in O_i \Rightarrow {f_1}^{-1}(O_i) \cap {f_2}^{-1}(O_i)$ is an open cover of y.

Let $b \in {f_1}^{-1}(O_i) \cap {f_2}^{-1}(O_i)$. Then $f_1(b)$ and $f_2(b)$ are both in O_i and p is a homeomorphism on O_i. Thus $f(b) = p \circ f_1(b) = p \circ f_2(b)$, $f_1(b) = f_2(b) \Rightarrow b \in \Omega \Rightarrow {f_1}^{-1}(O_i) \cap {f_2}^{-1}(O_i)$ is an open neighborhood around y that is contained in Ω. Ω is open.

Suppose $y \notin \Omega$. Then, $\exists\ O_m, O_n \in X^c$ such that $f_1(y) \in O_m$ and $f_2(y) \in O_n$. Then ${f_1}^{-1}(O_i) \cap {f_2}^{-1}(O_i)$ is an open neighborhood around y that is contained in the complement of Ω. Thus, Ω is closed. $\square$

2 The Generalized Lens Space

Construction
Consider the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ where $n \in \mathbb{N}$. In this case, $S^{2n+1} = \left\{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^{n} |z_n|^2 = 1\right\}$

The Generalized n-Lens Space, denoted $L(p, q_1, q_2, \ldots, q_n)$ where $p \in \mathbb{N}$ and p is prime to q_i for $i \leq n$, is the quotient space $S^{2n+1}/\mathbb{Z}_p$ where $\mathbb{Z}_p$ acts on S^{2n+1} via the following:

Let $g \in \mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ and let $(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$.

Then $g \cdot (z_0, z_1, \ldots, z_n) = (e^{(2\pi gi/p)} z_0, e^{(2\pi giq_1/p)} z_1, e^{(2\pi giq_2/p)} z_2, \ldots, e^{(2\pi giq_n/p)} z_n)$.

Exercise 2.0
(1) Prove: The above action is free and well-defined.

(2) Prove: $L(2, \overbrace{1, 1, \ldots, 1}^{n}) = \mathbb{R}P^{2n+1}$.

Theorem 2.1
Let X be a path connected topological space, and let G be a group whose action on X is properly discontinuous. Also, let $x_0 \in X$ and let $p : X \mapsto X/G$ be the canonical projection from X to X/G, with $p(x_0) \in X/G$. Define $\phi : \pi(X/G, p(x_0)) \mapsto G$ by: $\phi(f) = g \in G$ such that $g \cdot x_0 = l_{x_0} f(1)$, where $f \in \pi(X/G, p(x_0))$ and $l_{x_0} f$ is the homotopy class of lifts from $[0,1]$ to X of representatives of f based at x_0. Then ϕ is a homomorphism.

Proof : Suppose $[f_1]$, $[f_2] \in X/G$ are based at $p(x_0)$, and let f_1, f_2 be representatives of their respective homotopy classes. Suppose $\phi(f_1) = x_1 = g_1 \cdot x_0$ and $\phi(f_2) = x_2 = g_2 \cdot x_0$, where $g_1, g_2 \in G$.

Then,

$$\begin{gathered} p(g_1 \cdot l_{x_0}(f_2)) = f_2 \\ \Rightarrow l_{x_1}(f_2) = g_1 \cdot l_{x_0}(f_2) \\ \Rightarrow l_{x_0}(f_1 \circ f_2) = l_{x_0}(f) \circ l_{x_1}(f_2) \end{gathered}$$

which further implies

$$\begin{gathered} \phi([f_1] \cdot [f_2]) = \phi([f_1 \circ f_2]) \\ = k \in G \text{ such that: } k \cdot x_0 = g_1 \cdot x_2 = g_1 \cdot (g_2 \cdot x_0) = (g_1 \cdot g_2) \cdot x_0. \\ = (g_1 \cdot g_2) \\ = \phi([f_1]) \cdot \phi([f_2]). \end{gathered}$$

□

Corollary 2.1
Let p_* be the induced homomorphism of the fundamental groups $\pi(X, x_0)$ and $\pi(X/G, p(x_0))$. Then $\pi(X/G, p(x_0))/p_*\pi(X, x_0) \cong G$.

Proof : Left as an exercise (Hint: Consider the kernel of ϕ, and show that ϕ is a surjective map).

Theorem 2.2
$L(p, q_1, q_2, \ldots, q_n) \cong \mathbb{Z}_p$.

Proof : By Exercises 1.2.2 and 2.0.1, the canonical map $m : S^{2n+1} \mapsto S^{2n+1}/\mathbb{Z}_p$ is a covering map. Thus, by Corollary 2.1, $\pi(S^{2n+1}/\mathbb{Z}_p, m(x_0))/m_*\pi(X, x_0) \cong \pi(S^{2n+1}/\mathbb{Z}_p, m(x_0)) \cong G$. □

3 Bibliography

1. A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2001.
2. C. Kosniowski, *A First Course in Algebraic Topology*, Cambridge University Press, Cambridge, 1980.

With Special Thanks to : Dr. Benson Farb, Dr. Peter May, Dr. Andy Putnam, and Mr. Khalid Bou-Rabee of the University of Chicago.