THE FROBENIUS-PERRON THEOREM

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1. Introduction

We begin by stating the Frobenius-Perron Theorem:

Theorem 1.1 (Frobenius-Perron)**.** *Let B be an $n \times n$ matrix with nonnegative entries. Then we have the following:*

(1) *B has a nonnegative real eigenvalue. The largest such eigenvalue, $\lambda(B)$, dominates the absolute values of all other eigenvalues of B. The domination is strict if the entries of B are strictly positive.*
(2) *If B has strictly positive entries, then $\lambda(B)$ is a simple positive eigenvalue, and the corresponding eigenvector can be normalized to have strictly positive entries.*
(3) *If B has an eigenvector v with strictly positive entries, then the corresponding eigenvalue λ_v is $\lambda(B)$.*

We will first illustrate the statement for 2-by-2 matrices (using very elementary arguments), and then prove the theorem for the n-by-n case. Finally, we will conclude with examples of some of the applications of the theorem.

2. The Frobenius-Perron Theorem for $n = 2$

Consider the matrix

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

with nonnegative entries. The characteristic polynomial

$$p_B(t) = \det(tI - B) = t^2 - (a+d)t + (ad - bc).$$

has discriminant

$$(a-d)^2 + 4bc \geq 0$$

and roots

$$\lambda(B) = \frac{(a+d) + \sqrt{(a-d)^2 + 4bc}}{2}, \quad \lambda'(B) = \frac{(a+d) - \sqrt{(a-d)^2 + 4bc}}{2}.$$

(1). Since $a, b, c, d \geq 0$, the discriminant is nonnegative, so the roots of the characteristic polynomial can only take on real values. Hence there exists a real eigenvalue for B. $\lambda(B)$ is nonnegative, so B has a nonnegative real eigenvalue. Since

$$t^2 - (a+d)t + (ad - bc) = \left[t - \frac{(a+d)}{2}\right]^2 - \left[\frac{(a-d)^2}{4} + (ad - bc)\right]$$

and $\frac{(a+d)}{2}$ is nonnegative, $\lambda(B) \geq |\lambda'(B)|$. If B has strictly positive entries, then $\frac{(a+d)}{2}$ is strictly positive and the domination is strict.

(2). If B has strictly positive entries, then the discriminant is greater than 0, so the characteristic polynomial must have two distinct real solutions. Of these, $\lambda(B)$ is positive and greater than $\lambda'(B)$. Hence, $\lambda(B)$ is a simple positive eigenvalue.

We now show that the eigenvector corresponding to $\lambda(B)$ can be normalized to have strictly positive entries. Define

$$D := (a-d)^2 + 4bc, \qquad \lambda := \lambda(B).$$

There exists an eigenvector x with eigenvalue λ. This eigenvector must be unique up to scaling, because there are two distinct eigenvalues, each with at least one corresponding eigenvector, and each with at most one corresponding eigenvector (up to scaling), since the number of linearly independent eigenvectors of a matrix cannot exceed its size. We have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix},$$

$$\begin{cases} ax_1 + bx_2 = \lambda x_1 \\ cx_1 + dx_2 = \lambda x_2 \end{cases}.$$

By definition, either $x_1 \neq 0$ or $x_2 \neq 0$. Suppose $x_1 \neq 0$. Then

$$\begin{aligned} a + b \cdot \frac{x_2}{x_1} &= \lambda \qquad &\Leftrightarrow \qquad \frac{x_2}{x_1} &= \frac{\lambda - a}{b}, \\ c + d \cdot \frac{x_2}{x_1} &= \lambda \cdot \frac{x_2}{x_1} \qquad &\Leftrightarrow \qquad \frac{x_2}{x_1} \cdot (\lambda - d) &= c > 0. \end{aligned}$$

We want to prove that $\frac{x_2}{x_1} > 0$. It is enough to show that either $\lambda > a$ or $\lambda > d$. This is indeed true, because $\lambda > \frac{a+d}{2}$. The same method proves the result for $x_2 \neq 0$.

(3). Suppose B has an eigenvector v with strictly positive entries. We have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda_v v_1 \\ \lambda_v v_2 \end{pmatrix},$$

which we know from the proof of (2) gives us

$$a + b \cdot \frac{v_2}{v_1} = \lambda_v, \qquad c + d \cdot \frac{v_2}{v_1} = \lambda_v \cdot \frac{v_2}{v_1}.$$

¿From this, we obtain

$$\frac{v_2}{v_1} \cdot b = \lambda_v - a, \qquad \frac{v_2}{v_1} \cdot (\lambda_v - d) = c \geq 0.$$

Since $\frac{v_2}{v_1}$ is positive, we must have $\lambda_v \geq a$ and $\lambda_v \geq d$. Then $\lambda_v \geq \frac{a+d}{2}$, hence $\lambda_v = \lambda(B)$.

3. Proof of the Frobenius-Perron Theorem for n-by-n matrices

Now that we understand the theorem for $n = 2$, we will prove the general case. We will begin by proving (3), and furthermore show that if the entries of B are strictly positive, then the domination is strict. We will then show that λ_v is a simple positive eigenvalue, and the corresponding eigenvector can be normalized to have strictly positive entries. Next, we will show that the proof of (1) can be reduced to the case for B with strictly positive entries. Then by the above, it will suffice to prove the existence of an eigenvector v with strictly positive entries for B with strictly positive entries to conclude the proof of (1) and (2). We will prove the existence of such a v.

Proof of (3). Suppose B has an eigenvector v with strictly positive entries, and let λ_v denote the corresponding eigenvalue, so that $Bv = \lambda_v v$. Observe that

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 & 0 & \dots & 0 \\ 0 & v_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & v_n \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = C \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

where we denote the diagonal matrix by C in the last equality. Then

$$B \cdot C \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \lambda_v C \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \Longrightarrow C^{-1}BC \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \lambda_v \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Since

$$C^{-1} = \begin{pmatrix} v_1^{-1} & 0 & \dots & 0 \\ 0 & v_2^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & v_n^{-1} \end{pmatrix},$$

the matrix CBC^{-1} has only nonnegative entries. Similar matrices have the same eigenvalues, so we may assume without loss of generality that

$$v = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

We then have $\lambda_v = \sum_{j=1}^{n} b_{ij}$ for each $1 \leq i \leq n$. Hence λ_v is a nonnegative real number, and it is strictly positive unless $B = 0$.

Let us equip $\mathbb{C}^n$ with the ℓ^∞ norm, i.e.,

$$||z|| = \max_{i=1,\cdots,n} |z_i| \quad \text{for} \quad z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}.$$

For any $z \in \mathbb{C}^n$, the i-th entry of the vector Bz is equal to $b_{i1}z_1 + b_{i2}z_2 + \cdots + b_{in}z_n$. We have

$$|b_{i1}z_1 + \cdots + b_{in}z_n| \leq |b_{i1}||z_1| + \cdots + |b_{in}||z_n| \tag{3.1}$$

$$\leq \sum_{j=1}^{n} b_{ij} \cdot \max_{i=1,\ldots,n} |z_i| \tag{3.2}$$

$$= \lambda_v ||z||.$$

Therefore,

$$||Bz|| \leq \lambda_v ||z||.$$

Hence, if z' is an eigenvector with eigenvalue λ', then

$$||Bz'|| = |\lambda'| \cdot ||z'|| \leq \lambda_v ||z'||.$$

Therefore, $\lambda_v \geq |\lambda'|$. Hence, by definition, $\lambda_v = \lambda(B)$, as claimed.

Remark 1. Now suppose that all entries of B are strictly positive. Then $||Bz|| < \lambda_v ||z||$, unless $z_1 = z_2 = \cdots = z_n$, which is the same as saying

$$z = c \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = c \cdot v,$$

where $c \in \mathbb{C}$. This is the only case for which we have equality, hence v is the unique (up to scale) eigenvector with eigenvalue λ_v. This is because if $z_i \neq z_j$ for some $1 \leq i, j \leq n$, then one of the inequalities (3.1) or (3.2) will be strict. Then $||z||$ cancels on both sides ($||z||$ is greater than 0 by the definition of an eigenvector), and we see that λ_v strictly dominates the absolute values of all other eigenvalues of B. Hence, we have strict inequality for all eigenvalues corresponding to eigenvectors other than v.

Remark 2. We will now prove by contradiction that the "algebraic" multiplicity of λ_v (i.e., the multiplicity of λ_v as a root of the characteristic polynomial of B) is exactly 1. Suppose the multiplicity of λ_v is greater than 1. By the Jordan theorem, there exists an invertible matrix C such that CBC^{-1} is upper triangular and looks like the following matrix:

$$\begin{pmatrix} \ddots & & & * \\ & \lambda_v & 1 & \\ & & \lambda_v & \\ 0 & & & \ddots \end{pmatrix},$$

with a Jordan block of size at least 2. Note that we may exclude the case with two 1-by-1 Jordan blocks with the same λ_v, because then we would have two independent eigenvectors for λ_v, but we proved in Remark 1 that v is unique up to scalar multiple. We make the following claim:

Claim 1.

(i) There exist entries of $(\frac{1}{\lambda_v}CBC^{-1})^n$ such that the absolute values of these entries approach ∞ as $n \to \infty$.
(ii) Hence, the same is true for $(\frac{1}{\lambda_v}B)^n$.

Proof. (i)

$$\frac{1}{\lambda_v} \cdot \begin{pmatrix} \lambda_v & 1 & & 0 \\ & \lambda_v & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_v \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{\lambda_v} & & 0 \\ & 0 & \ddots & \\ & & \ddots & \frac{1}{\lambda_v} \\ 0 & & & 0 \end{pmatrix}.$$

Let

$$\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} = I, \qquad \begin{pmatrix} 0 & \frac{1}{\lambda_v} & & 0 \\ & 0 & \ddots & \\ & & \ddots & \frac{1}{\lambda_v} \\ 0 & & & 0 \end{pmatrix} = A.$$

We need only look at this particular Jordan block, because when we multiply the Jordan block decomposed matrix CBC^{-1} by itself, each Jordan block is only affected by its corresponding Jordan block. Furthermore, for $k > 1$, each A^k affects only its particular diagonal line of entries, from $a_{1(k+1)}$ to $a_{n(n-k)}$. By the binomial theorem, we have that

$$\begin{aligned}(I+A)^n &= \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} I^{n-k}A^k \\ &= \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} A^k \\ &= I + nA + \begin{pmatrix} n \\ 2 \end{pmatrix} A^2 + \cdots \\ &= \begin{pmatrix} 1 & \frac{n}{\lambda_v} & & * \\ & 1 & \ddots & \\ & & \ddots & \frac{n}{\lambda_v} \\ 0 & & & 1 \end{pmatrix}.\end{aligned}$$

This shows that $(\frac{1}{\lambda_v}CBC^{-1})^n$ has entries whose absolute values approach ∞ as $n \to \infty$.

(ii) To show that $||B^n|| \to \infty$ as $||(CBC^{-1})^n|| \to \infty$, note that $(CBC^{-1})^n = CB^nC^{-1}$. Think of these n-by-n matrices as elements of $\mathbb{C}^{n^2}$. Consider the function $f : \mathbb{C}^{n^2} \to \mathbb{C}^{n^2}$, where $f(X) = CXC^{-1}$. This function is continuous. Its inverse, $f^{-1} : \mathbb{C}^{n^2} \to \mathbb{C}^{n^2}$ is also continuous, where $f^{-1}(X) = C^{-1}XC$. Therefore, the entries of $(CBC^{-1})^n$ are bounded iff B^n is bounded.

$\square$

However, observe that the entries of $(\frac{1}{\lambda_v}B)^n$ cannot approach ∞, since

$$||Bz|| \leq \lambda_v||z|| \quad \Leftrightarrow \quad \frac{1}{\lambda_v}||Bz|| = \left|\left|\frac{1}{\lambda_v}Bz\right|\right| \leq ||z||,$$

and therefore

$$\left|\left|\left(\frac{1}{\lambda_v}B\right)^n z\right|\right| = \left|\left|\frac{1}{\lambda_v}\left[\left(\frac{1}{\lambda_v}\right)^{n-1} B^n\right] z\right|\right| \leq ||z|| \tag{3.3}$$

for any $||z||$, since $(\frac{1}{\lambda_v})^{n-1}B^n$ is just another matrix with strictly positive entries and therefore can be substituted for B in the inequality (3.3). For any matrix A that has the property $||Az|| \leq ||z||$, if $a \in \mathbb{C}^n$ is any row vector of A, then $||a \cdot z|| \leq ||z||$. Then we have that the entries of A are bounded, since $|a_{ij}| \leq \frac{2||z||}{|z_j|}$, Therefore, the entries of $(\frac{1}{\lambda_v}B)^n$ must be bounded. We have a contradiction, which shows that the algebraic multiplicity of λ_v cannot be greater than 1.

Proof of (1) and (2). We will reduce the proof of (1) to the case where all entries of B are strictly positive. The idea is that we may "approximate" B by matrices with strictly positive entries. Consider B with nonnegative entries. Define B_r to be the same matrix with the 0 entries replaced by $\frac{1}{r}$, where $r \in \mathbb{R}$ and $r > 0$. We will:

(i) show that the eigenvalues of B_r approach the eigenvalues of B as $r \to \infty$;
(ii) prove the existence of an eigenvector for B_r with strictly positive entries, and hence a positive eigenvalue for B_r, by our proof of (3) and the remarks; and
(iii) prove that this positive eigenvalue is precisely $\lambda_v = \lambda(B)$ and satisfies the properties stated in parts (1) and (2) of Theorem 1.1.

Since the eigenvalues of a matrix are the roots of its characteristic polynomial, if we show that as polynomials approach polynomials, roots approach roots, then we will have proved (i). We will use the following lemma:

Lemma 1. *Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a^n, a_i \in \mathbb{C}$, $|a_1|, \cdots, |a_n| < M$. If $z \in \mathbb{C}$ is a root of f, then $|z| < 1 + nM$.*

Proof. Suppose $|z| \geq 1 + nM$. If $f(x) = 0$, then

$$x^n = -a_{n-1}x^{n-1} - a_{n-2}x^{n-2} - \cdots - a_0.$$

Taking the absolute values of both sides,

$$\begin{aligned} |x^n| &= |-a_{n-1}x^{n-1} - a_{n-2}x^{n-2} - \cdots - a_0| \\ &= |a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0|. \end{aligned}$$

We divide by x^n and obtain

$$\begin{aligned} 1 &= |a_{n-1}x^{n-1}/x + a_{n-2}x^{n-2}/x + \cdots + a_0/x| \\ &\leq \left|\frac{a_{n-1}}{x}\right| + \left|\frac{a_{n-2}}{x^2}\right| + \cdots + \left|\frac{a_0}{x_n}\right| \\ &< \left|\frac{M}{x}\right| + \left|\frac{M}{x^2}\right| + \cdots + \left|\frac{M}{x^n}\right| \\ &\leq \left|\frac{M}{1+nM}\right| + \left|\frac{M}{(1+nM)^2}\right| + \cdots + \left|\frac{M}{(1+nM)^n}\right| \\ &\leq n\left|\frac{M}{1+nM}\right| \\ &< 1. \end{aligned}$$

Contradiction.

□

This lemma establishes an upper bound for the absolute values of the roots. Arrange the eigenvalues of B_r in any order; call them $\lambda_1^{(r)}, \cdots, \lambda_n^{(r)}$. Since the sequence $\{(\lambda_1^{(r)}, \cdots, \lambda_n^{(r)}) \in \mathbb{C}^n\}_{r=1}^{\infty}$ is bounded, it has a convergent subsequence. Call it $\{(\lambda_1^{(r_j)}, \cdots, \lambda_n^{(r_j)})\}_{r=1}^{\infty}$. Then

$$B_{r_j} \to B \quad \Longrightarrow \quad p(B_{r_j}) \to p(B)$$

as $j \to \infty$, where $p(B)$ denotes the characteristic polynomial of B. If we put

$$\lambda_k := \lim_{j\to\infty} \lambda_k^{(r_j)},$$

this implies that

$$p(B) = \prod_{k=1}^{n}(t - \lambda_k),$$

because

$$p(B_{r_j}) = \prod_{k=1}^{n}(t - \lambda_k^{(r_j)})$$

for every j. Therefore, there exists a subsequence such that the n-tuple of roots $\lambda_1^{(r)}, \cdots, \lambda_n^{(r)}$ converge to the n-tuple of roots of B (i.e., the eigenvalues). Hence, the λ_k's are the eigenvalues of B.

We will now prove the existence of an eigenvector with strictly positive entries for B with strictly positive entries. We will use the following claim:

Claim 2. *If $Bv' = \lambda_{v'}v'$ with the entries of v' being nonnegative, $v' \neq 0$, and the entries of B being strictly positive, then each entry of v' must be positive.*

Proof. Since all entries of v' are nonnegative, the same is true of Bv'. Furthermore, all entries of B are positive, so the entries of Bv' are all positive, since there is at least one nonnegative, non-zero entry in v'. However, v' is an eigenvector, so Bv' is a scalar of multiple v', which requires it to have zero entries in the same locations as v'. Hence, none of the entries of v' can be zero.

□

So we may prove the existence of an eigenvector v' with nonnegative entries for B with strictly positive entries, which by the claim is equivalent to proving the existence of an eigenvector v with strictly positive entries.

Let us consider the cube $D : \{d \in D \mid 0 \leq d_i \leq 1, \ \forall i = i, \cdots, n\}$. For the matrix B, we will write $||B|| = \max_{i=1,\ldots,n} \sum_{j=1}^{n} |b_{ij}|$. Consider the function $f(d) = ||dBd^{-1}||$, where we consider $d = (d_1, \cdots, d_n) \in D$ as a diagonal matrix

$$d = \begin{pmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_n \end{pmatrix}.$$

Note that f is only defined in the "interior", D^{int}, of the cube D, where $d \in D^{int}$ if $d_i \neq 0$ for all $i = 1, \ldots, n$. Moreover, f is clearly continuous on D^{int}.

For each $\epsilon > 0$, let us define a set D^ϵ by $d \in D^\epsilon$ iff $d_i \geq \epsilon$ for all i and $\sum_{i=1}^n d_i = 1$. Note that $D^\epsilon \subset D^{int}$ and is closed and bounded, hence compact. We claim that there exists $d \in D^{int}$ such that $f(d') \geq f(d)$ for all $d' \in D^{int}$.

Claim 3. *For any sufficiently small $\epsilon > 0$, there exists $d'' \in D^\epsilon$ such that $f(d') \geq f(d'')$ for all $d' \in D^{int}$.*

Proof. Without loss of generality, we can assume $\sum_{i=1}^n d'_i = 1$, since we can rescale the sum of the coordinates of the vector to equal 1. Assume d' does not lie in D^ϵ. (If it does, then since D^ϵ is compact, the function f achieves a minimum on D^ϵ and we are done.) Then for some i, we have $d'_i < \epsilon$ and $d'_1 + \cdots + d'_{i-1} + d'_{i+1} + \cdots + d'_n > 1 - \epsilon$. So there exists $j \neq i$ such that $d'_j > \frac{1-\epsilon}{n-1}$. Take the ji-th entry of $f(d') = ||d'Bd'^{-1}||$:

$$f(d') \geq d'_j {d'}_i^{-1} b_{ji} > \frac{(1-\epsilon)b_{ji}}{(n-1)\epsilon}.$$

If we take $\epsilon \to 0$, i.e., if one of the coordinates of $d \in D$ approaches 0, then $f(d) \to \infty$, so f achieves its smallest value on some $d'' \in D^\epsilon \subset D^{int}$. Therefore, $f(d) \geq f(d'')$, $\forall d \in D^{int}$.

$\square$

Claim 4. *d' is an eigenvector for B with eigenvalue $f(d') = \lambda$.*

Proof. Replacing B by $d'Bd'^{-1}$, we may assume without loss of generality that $d' = (1, \cdots, 1)$, by the same line of reasoning as given in the proof of (3). We have

$$\max_i \sum_{j=1}^n b_{ij} = \lambda$$

$$\max_i \sum_{j=1}^n d_i d_j^{-1} b_{ij} \geq \lambda, \quad \text{such that } d_k > 0, \ \forall k. \tag{1.1}$$

Let $S = \{i \mid \sum_{j=1}^n b_{ij} < \lambda\}$. We only need to show that $S = \varnothing$. We will prove this by contradiction. By our condition, $\forall\ d_1, \cdots, d_n$, there exists $i \in \{1, \cdots, n\}$ such that $d_i \sum_{j=1}^n d_j^{-1} b_{ij} \geq \lambda$, and at the very least $\sum_{j=1}^n d_j^{-1} b_{ij} \geq \lambda$. So if we take $(d_1, \cdots, d_n)$ "very close" to $(1, \cdots, 1)$, then $i \notin S$. For instance, we could take

$$d_i = \begin{cases} 1 - \epsilon & (i \notin S) \\ 1 & (i \in S). \end{cases}$$

If $S \neq \varnothing$ and $\epsilon > 0$ is sufficiently small then $\max_i \sum_{j=1}^n d_i d_j^{-1} b_{ij} < \lambda$, contrary to (1.1).

$\square$

Since we have shown that an eigenvector v with strictly positive entries indeed exists for B with strictly positive entries, and we know it has the corresponding eigenvalue λ_v by (3), we have that $\lim_{r\to\infty} B_r = B$ has a nonnegative real eigenvalue. By our previous lemma, there exists a subsequence such that the eigenvalues of B_r converge to the eigenvalues of B. For every r, B_r has a positive real eigenvalue that strictly dominates all the other ones, by (3'). Call this eigenvalue $\lambda_1^{(r)}$, and arrange the other eigenvalues of B_r in any order; call them $\lambda_2^{(r)}, \cdots, \lambda_n^{(r)}$. We know the sequence $\{(\lambda_1^{(r)}, \cdots, \lambda_n^{(r)}) \in \mathbb{C}^n\}_{r=1}^{\infty}$ converges to $\{(\lambda_1^{(r_j)}, \cdots, \lambda_n^{(r_j)})\}_{r=1}^{\infty}$

We conclude that:

(1) $\lambda_1^{(r_j)} > 0$ by construction $\Rightarrow \lambda_1$ is real and nonnegative; and

(2) for any $2 \le k \le n$, $\lambda_1^{(r_j)} > |\lambda_k^{(r_j)}|$, $\forall j$. By passing to the limit as $j \to \infty$, we obtain $\lambda_1 \ge |\lambda_k|$, $\forall 2 \le k \le n$.

This completes the proof of Theorem 1.1.

4. Applications

The Frobenius-Perron theorem has a natural interpretation in the theory of Markov chains, which in turn has applications in population modeling and biophysics, to name but a few. We will illustrate a few of these.

Suppose we have any vector $u_0 = (x, 1-x)$, which we multiply over and over by the "transition matrix"

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix},$$

Then $u_1 = Au_0$, $u_2 = Au_1$, $\cdots$, $u_k = A^k u_0$. The claim is that the vectors $u_0, u_1, \ldots$ will approach a "steady state", i.e., multiplying A will eventually cease to change the vector. The limit state for this particular example is $u_\infty = (.6, .4)$. Observe that $Au_\infty = u_\infty$:

$$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}.$$

u_∞ is an eigenvector with eigenvalue 1, and this makes it steady. But what is significant is that the final outcome *does not depend on the starting vector*; for *any* u_0, $A^k u_0$ will *always* converge to $(.6, .4)$ as $k \to \infty$.

Having a steady state does not alone imply that all vectors u_0 lead to u_∞. For example,

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

has the steady state

$$B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

but the starting vector $u_0 = (0, 1)$ will give $u_1 = (0, 2)$ and $u_2 = (0, 4)$. B has $\lambda = 1$ but it also has $\lambda = 2$. Any $|\lambda| > 1$ means blowup.

The explanation for the phenomenon that for some matrices, all vectors u_0 lead to u_∞, forms the basis for the theory of Markov chains. There are two special properties of A that guarantee a steady state u_∞. These properties define what is called a Markov matrix, and A above is just one particular example:

1. Every entry of A is nonnegative.
2. Every column of A adds to 1.

Two facts are immediate for any Markov matrix A:

(i) multiplying a nonnegative u_0 by A produces a nonnegative $u_1 = Au_0$; and
(ii) if the components of u_0 add to 1, so do the components of $u_1 = Au_0$.

Statement (ii) follows from the fact that the components of u_0 add to 1 when $[1, \cdots, 1]\, u_0 = 1$. This is true for each column of A by Property 2. Then by matrix multiplication, it is true for Au_0:

$$[1, \cdots, 1]\, Au_0 = [1, \cdots, 1]\, u_0 = 1.$$

The same applies to $u_2 = Au_1$, $u_3 = Au_2$, etc. Hence, every vector $u_k = A^k u_0$ is nonnegative with components adding to 1. These are "probability vectors". The limit u_∞ is also a probability vector, but first we must prove that a limit exists. We will show that $\lambda = 1$ is an eigenvalue of A and estimate the other eigenvalues.

Theorem 4.1. *If A is a positive Markov matrix, then $\lambda_1 = 1$ is larger than any other eigenvalue. The eigenvector x_1 is the steady state: $u_k = x_1 + c_2(\lambda_2)^k x_2 + \ldots + c_n(\lambda_n)^k x_n$ always approaches $u_\infty = x_1$.*

Every column of $A - I$ adds to $1 - 1 = 0$. The rows of $A - I$ add up to the zero row. Those rows are linearly dependent, so $A - I$ is singular. Its determinant is zero, hence $\lambda_1 = 1$ must be an eigenvalue of A. Strict domination, and hence uniqueness, follows from (2) of the Frobenius-Perron theorem. The other eigenvalues gradually disappear because $|\lambda| < 1$. The more steps we take, the closer we come to u_∞.

Example. The fraction of Illinois's wild raccoons in Chicago starts at $\frac{1}{50} = .2$. The fraction outside Chicago is .98. Every month 80% of raccoons in Chicago leave Chicago, while 20% of raccoons in Chicago remain in Chicago. Furthermore, 5% of raccoons outside Chicago arrive in Chicago, while 95% of raccoons outside of Chicago remain outside Chicago. Hence, the probability vector is multiplied by the Markov matrix

$$A = \begin{bmatrix} .80 & .05 \\ .20 & .95 \end{bmatrix},$$

which gives us

$$u_1 = Au_0 = A \begin{bmatrix} .02 \\ .98 \end{bmatrix} = \begin{bmatrix} .065 \\ .935 \end{bmatrix}.$$

In one month, the fraction of raccoons in Chicago is up to .065. What is the eventual outcome?

Since every column of A adds to 1, nothing is gained or lost - we are simply moving a fixed number of raccoons. The fractions add to 1 and the matrix A keeps them that way. We want to know how they are distributed after k time periods - which leads us to A^k.

Solution. To study the powers of A we diagonalize it:

$$|A - \lambda I| = \begin{vmatrix} .80 - \lambda & .05 \\ .20 & .95 - \lambda \end{vmatrix} = \lambda^2 - 1.75\lambda + .75 = (\lambda - 1)(\lambda - .75).$$

$$A \begin{bmatrix} .2 \\ .8 \end{bmatrix} = \begin{bmatrix} .2 \\ .8 \end{bmatrix}, \qquad A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = .75 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

We have eigenvalues $\lambda_1 = 1$ and $\lambda_2 = .75$ with corresponding eigenvectors $x_1 = (.2, .8)$ and $x_2 = (-1, 1)$. The eigenvectors are the columns of S, where S is the eigenvector matrix, $A^k = S\Lambda^k S^{-1}$. The starting vector u_0 is a combination of x_1 and x_2:

$$u_0 = \begin{bmatrix} .02 \\ .98 \end{bmatrix} = \begin{bmatrix} .2 \\ .8 \end{bmatrix} + .18 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Now multiply by A to find u_1. The eigenvectors are multiplied by $\lambda_1 = 1$ and $\lambda_2 = .75$:

$$\begin{aligned} u_1 &= 1 \begin{bmatrix} .2 \\ .8 \end{bmatrix} + (.75)(.18) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ u_k &= A^k u_0 = \begin{bmatrix} .2 \\ .8 \end{bmatrix} + (.75)^k (.18) \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \end{aligned}$$

The eigenvector x_1 with $\lambda = 1$ is the steady state u_∞. The other eigenvector x_2 gradually disappears because $|\lambda| < 1$. In the limit, $\frac{2}{10}$ of the raccoons are in Chicago and $\frac{8}{10}$ are outside.

Although we arrived at this particular conclusion using diagonalization, Jordan decomposition can be used to justify the statement for non-diagonalizable matrices. With a positive Markov matrix, the powers A^k approach the rank one matrix that has the steady state x_1 in every column.

It is of interest to biophysicists to derive approximate analytic expressions for the fraction of mutant proteins that fold stably to their native structure as a function of the number of amino acid substitutions, and estimate the asymptotic behavior of this fraction for a large number of amino acid substitutions. Using Markov chain approximation, it is possible to model how such a fraction decays.

5. Appendix

There is also an alternate proof of the existence of an eigenvector with strictly positive entries for B with strictly positive entries, which is faster than the one we gave in §3, but uses a rather nontrivial result, namely, Brauer's fixed point theorem, stated as Theorem 5.1 below. (Note that the proof we presented in §3 is much more elementary.)

Proof. Let us consider the subset $\Delta \subset \mathbb{R}^n$ defined by $z \in \Delta$ iff $z_i \geq 0$ for all $i = 1, \cdots, n$ and $\sum_{i=1}^{n} z_i = 1$. This is what is called an $(n-1)$-dimensional simplex (for $n = 2$, we get an interval, for $n = 3$, a triangle, and so on). Then, let us consider the map $\Phi : \Delta \to \Delta$, defined as follows:

$$\Phi(z) = \frac{Bz}{(Bz \cdot (1, 1, \cdots, 1))},$$

where $Bz \cdot (1, 1, \cdots, 1)$ denotes the dot product of the vectors Bz and $(1, 1, \cdots, 1)$ (i.e., the sum of the coordinates of the vector Bz). Clearly, Φ is a continuous map, so by Brauer's fixed point theorem, there exists $z \in \Delta$ such that $\Phi(z) = z$. Hence, $z = \frac{Bz}{Bz \cdot (1, \cdots, 1)} \Rightarrow Bz = (Bz \cdot (1, \cdots, 1))z$, so z is an eigenvector with nonnegative entries.

□

Theorem 5.1 (Brauer's fixed point theorem)**.** *Let $f : \Delta^n \to \Delta^n$ be a continuous map from an n-dimensional simplex to itself. Then, it has a fixed point (i.e., there exists $z \in \Delta^n$ such that $f(z) = z$).*

References

[1] Gilbert Strang. Introduction to Linear Algebra, 3rd Edition. Wellesley-Cambridge Press. 2003.

The proof of the Frobenius-Perron Theorem given in this paper is based on the presentation contained in unpublished notes of Vladimir Drinfeld distributed to the participants of the Geometric Langlands seminar in January 2006. The examples of Markov matrices were taken from the source named above.

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