A SOLUTION TO BERLEKAMP'S SWITCHING GAME

FAN FEI CHONG

Abstract. This paper studies a game played between two players on an $n \times n$ grid. Using tools from Linear Algebra and Elementary Probability theory, we can show that the "payout" of the game is asymptotically $n^{\frac{3}{2}}$.

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1. The problem

Before stating the problem, we need some asymptotic notation.

Definition 1.1. Let a_n, b_n be sequences of real numbers, we say that

(1) $a_n = O(b_n)$ if $|a_n/b_n|$ is bounded, i.e.,$(\exists C > 0, n_0 \in \mathbb{N})(\forall n > n_0)(|a_n| \leq C|b_n|)$
(2) $a_n = \Omega(b_n)$ if $b_n = O(a_n)$, i.e., if $|b_n/a_n|$ is bounded, $(\exists c > 0, n_0 \in \mathbb{N})(\forall n > n_0)(|a_n| \geq c|b_n|)$
(3) $a_n = \Theta(b_n)$ if $a_n = O(b_n)$ and $a_n = \Omega(b_n)$, i.e. $(\exists C, c > 0, n_0 \in \mathbb{N})(\forall n > n_0)(c|b_n| \leq |a_n| \leq C|b_n|)$

Consider an $n \times n$ square grid. Each square can hold a value of ± 1. Player 1 starts the game and puts ± 1 in each cell. Player 2 selects a set of rows and a set of columns and switches the sign of each cell in the selected rows and columns. (If the sign of both the row and the column of a cell has been changed, this double change means no change for the cell.)

Let $S(n)$ be the sum of all the numbers in the grid after both players have played. This is the amount Player 1 pays to Player 2. Assuming we have both players playing with optimal strategy, what can we say about $S(n)$ as $n \to \infty$? We shall see that in fact $S(n) = \Theta(n^{3/2})$.

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2. A few observations

We observe that S is always greater than or equal to 0. If the sum of all the numbers is non-negative after Player 1 plays, Player 2 can just leave it alone. However, if the sum is negative, Player 2 can bring it back to positive by switching the signs of every row. So, now we have a trivial lower bound for S.

We will make a remark about Player 2's moves, before proceeding to improving the lower bound.

Remark 2.1. Notice the following:

(1) All possible moves for Player 2 form a group, generated by the row operations and the column operations.
(2) Each of the row operations and the column operations has order 2.
(3) Any two operations (row or column) commute.

A direct consequence of this is that in consideration of the optimal strategy for Player 2, it suffices to consider only strategies which consists of two phases: in the"first phase", Player 2 switches a set of rows, and in the "second phase", Player 2 switches a set of columns.

Once Player 2 is done with the "first phase", it is clear that the best he can do in the following phase is to switch a column **if and only if** the sum of entries in that column is negative. Now, the objective of Player 2 boils down to maximizing the sum of the absolute value of the sum of entries in the columns. If we write the numbers in the grid as a $n \times n$ matrix, and define $B_j = \sum_{i=1}^{n} a_{ij}$ to be the column sum of the j-th column, the objective of Player 2 would be to maximize $\sum_{j=1}^{n} |B_j|$ in "first phase". Similarly, Player 1's objective would be to impose an upper bound on $\sum_{j=1}^{n} |B_j|$.

3. Hadamard matrix and Player 1's optimal strategy

Before going into a discussion of Player 1's optimal strategy, we will first study the properties of a special class of square matrix - the Hadamard matrix.

Definition 3.1. A matrix H is an **Hadamard matrix** if all its entries are either 1 or -1 (we denote this $H = (\pm 1)$) and its columns are orthogonal.

Lemma 3.2. *If $A = (\pm 1)$ is an $n \times n$ matrix, then the following statements are equivalent.*

(1) A is Hadamard
(2) $A^T A = nI$
(3) $\frac{1}{\sqrt{n}} A \in O_n(\mathbb{R})$.

Proof. (1) $\Leftrightarrow$ (2): Let $\underline{a}_i$ be the i^{th} column vector of A. Then $A^T A = (\underline{a}_i^T \underline{a}_j)$, i.e. the $(i,j)^{th}$ entry of $A^T A$ is the dot product of the i^{th} with the j^{th} column vectors of A. This implies that (2) is true if and only if $\underline{a}_i^T \underline{a}_i = n$ for all i (which is true because all the entries of A are either 1 or -1) and $\underline{a}_i^T \underline{a}_j = 0$ for all $i \neq j$. The latter is true if and only if all the columns of A are orthogonal, which is precisely the definition of Hadamard matrix.
(2) $\Leftrightarrow$ (3): (2) implies that $\frac{1}{\sqrt{n}} A^T$ is a left inverse of $\frac{1}{\sqrt{n}} A$. Since determinant is multiplicative and $det(I) = 1$, we then know that $det(A) \neq 0$ and A is invertible. By the unique existence of inverse, $\frac{1}{\sqrt{n}} A^T$ is the inverse of $\frac{1}{\sqrt{n}} A$. Notice that the

two expressions above are transpose of each other, and therefore $\frac{1}{\sqrt{n}}A \in O_n(\mathbb{R})$. The other direction of the implication follows immediately from the definition of orthogonal matrix. □

Lemma 3.3. *If A is Hadamard, then $\begin{pmatrix} A & A \\ A & -A \end{pmatrix}$ is also Hadamard.*

The proof is clear from the definition of Hadamard matrix and left to the readers as an exercise. Notice that since the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is Hadamard, this fact allows us to find an $n \times n$ Hadamard matrix for every $n = 2^k$ where $k \in \mathbb{N}$.

Lemma 3.4. *If $A = [\underline{a}_1, ..., \underline{a}_n]$ is an Hadamard matrix, then switching the signs (1 to -1, -1 to 1) of a whole row or a whole column would still give an Hadamard matrix.*

Proof. The case for switching signs in a column is clear by definition of Hadamard matrix and the fact that inner product is bilinear. By Lemma 3.2, it follows from (1) $\Leftrightarrow$ (3) that A is Hadamard if and only if the rows of A are orthogonal. And now, the same argument (as in the case of switching signs in a column) gives us that switching signs in a row would still keep the matrix Hadamard. □

We are only one lemma away from proving that $S(n) \leq cn^{3/2}$ for some constant c. Let's first remind ourselves of an elementary inequality which we will use in the proof of the next lemma.

Proposition 3.5. *(Cauchy-Schwarz) For $a, b \in \mathbb{R}^n$, we have $|a \cdot b| \leq ||a||\,||b||$.*

Now, we are ready to prove the key lemma which relates Hadamard matrix to the Switching Game (and hence Player 1's strategy).

Lemma 3.6. *(Lindsey's Lemma) If A is an $n \times n$ Hadamard matrix and T is a $k \times l$ submatrix, then $|\sum_{(i,j)\in T} a_{ij}| \leq \sqrt{nkl}$.*

To improve the readibility of the proof, we introduce the notion of incidence vector.

Notation 3.7. Let N be the set {1, 2, ..., n} and K be a subset of it. Then the **incidence vector** of set K is the $n \times 1$ column vector I_K of 0's and 1's where the i^{th} column is 1 if and only if $i \in K$.

Proof. Let $K \subset N, L \subset N$ be the set of rows and the set of columns that defines T. Note that $||I_K|| = \sqrt{k}$ and $||I_L|| = \sqrt{l}$. Now the sum of all entries of T is $I_K^T A I_L$. We need to estimate $|I_K^T A I_L|$. Applying the Cauchy-Schwarz Inequality (Proposition 3.5), setting $\vec{a} = I_K$ and $\vec{b} = AI_L$, we have $|I_K^T A I_L| \leq ||I_K||\,||AI_L|| = \sqrt{k}\,||AI_L||$.

By Lemma 3.2, we have $\frac{1}{\sqrt{n}}A \in O_n(\mathbb{R})$. Hence $||(\frac{1}{\sqrt{n}}A)I_L|| = ||I_L|| = \sqrt{l}$ and $||AI_L|| = \sqrt{nl}$. This gives us the desired inequality $|I_K^T A I_L| \leq \sqrt{nkl}$. □

Now, we have finally achieved enough understanding of the Hadamard matrix to devise a strategy for Player 1 to sufficiently constrain the payout of the game.

Theorem 3.8. *For all $n \geq 1$, $S(n) \leq \sqrt{2n^3}$.*

Proof. For a given n, pick $k \in \mathbb{N}$ such that $2^{k-1} \leq n < 2^k$. By Lemma 3.3, there exists an Hadamard matrix of dimension $2^k \times 2^k$. Pick one of such and call it A. Now Player 1 needs only pick any $n \times n$ submatrix of A, call it T and copy the distribution of 1's and -1's onto the $n \times n$ grid. Then no matter what Player 2 does, the payout is guaranteed to be $\leq \sqrt{2n^3}$. How does this work?

We can view the $n \times n$ grid (identified with T) as embedded in A. Now say Player 2 has decided to switch the signs in Row 1. We can extend this "row move" of T to a "row move" of A, involving the corresponding row of A which contains Row 1 of T. By Lemma 3.4, A remains an Hadamard matrix after the move. The "modified" T would still be an $n \times n$ submatrix of an Hadamard matrix, by taking the same rows and columns as before from the "modified" A. The same argument would work if Player 2 decides to change multiple rows and columns. Extending the "row moves" and "column moves" of T to the corresponding ones in A, we see that the "modified" T remains an $n \times n$ submatrix of an Hadamard matrix of dimension $2^k \times 2^k$.

Therefore we may conclude using Lindsey's Lemma (Lemma 3.6) that after Player 2's turn, the sum of the entries in the grid is $\leq \sqrt{2^k n^2} \leq \sqrt{2n^3}$. □

This gives an upper bound for S(n). We proceed to show that this bound is tight (asymptotically) by demonstrating that the optimal strategy of Player 2 would guarantee a "payout" which is greater than or equal to $cn^{\frac{3}{2}}$ for some $c \in \mathbb{R}$.

4. Probabilistic approach and Player 2's optimal strategy

We are in fact not going to write out explicitly the optimal strategy, as it suffices to show that there exists a strategy for Player 2 which gives a "payout"$\geq cn^{\frac{3}{2}}$. We will show this with a probablistic approach (which often yields elegant solutions in unexpected situations in combinatorics). Let's recall a result in elementary probability theory.

Proposition 4.1. *Let X be a discrete random variable. Then* $\exists\ k \in \mathbb{R}$ *such that* $k \geq E(X)$ *and* $P(X = k) > 0$.

We will discuss the implication of this result in light of a particular probabilistic strategy of Player 2. In the "first phase", for each of the n rows, Player 2 tosses a fair coin. If the outcome is Head, he switches the signs in that particular row and does nothing if the coin-toss turns out otherwise. After this process, fix a j. We have each of the a_{ij}'s in that column **independently and identically distributed**(i.i.d.) as Bernoulli distribution with success probability 0.5 (i.e. Bin(1, 0.5)). It is easy to see that each entry has exactly 0.5 probability to stay the same, and 0.5 to be changed; this corresponds to 0.5 for +1 and 0.5 for -1, regardless of what the initial value is. Different entries at the same column are independently distributed because they depend on the outcome of different coin-tosses. Note that $E(a_{ij}) = 0$ and $Var(a_{ij}) = 1$ for all $1 \leq i, j \leq n$. Now, let's recall a special case of the Central Limit Theorem for a sequence of i.i.d. variables.

Theorem 4.2. *(Central Limit Theorem for i.i.d. Random Variables) Let* $X_1, X_2, ...$ *be a sequence of independently identically distributed random variables having mean* μ *and variance* $\sigma^2 \neq 0$. *If additionally there exists* $M \in \mathbb{R}$ *such that* $P\{|X_i| < M\} = 1$, *then* $X := \sum_{i=1}^{n} X_i$ *converges in distribution to* $N(n\mu, n\sigma^2)$ *(Normal Distribution with mean* $n\mu$ *and variance* $n\sigma^2$*)*

Note that $|a_{ij}| \leq 1$ with probability 1, so Theorem 4.2 applies. We have that the column sum of the j-th column, $B_j = \sum_{i=1}^{n} a_{ij}$, is converging in distribution to $N(0, n)$, with probability density function (p.d.f.): $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{(-\frac{x}{\sqrt{2}\sigma})^2}$. Noticing that the above p.d.f. is symmetric about 0, we have $E[|B_j|] \to \int_0^\infty 2xf(x)dx$ as $n \to \infty$. The reader should check that the above integral is equal to $\sqrt{\frac{2}{\pi}}\sqrt{n}$.

Lastly, we need to recall the fact that the expected value operator is linear.

Proposition 4.3. *Let $X_1, X_2, ..., X_n$ be random variables and $a_1, a_2, ..., a_n$ be real numbers. Then we have $E[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i E[X_i]$.*

For any two columns i and j, there is no reason why $|B_i|$ and $|B_j|$ are independent. Nevertheless, the above proposition enables us to sum up their expected values and have $E[\sum_{j=1}^{n} |B_j|] = \sum_{j=1}^{n} E[|B_j|] = n\sqrt{\frac{2}{\pi}}\sqrt{n} = \sqrt{\frac{2}{\pi}}n^{\frac{3}{2}}$. We have got the "right number", but how does this calculation give a lower bound for the payout of Player 2's optimal strategy? We need Proposition 4.1.

Let X be the random variable which gives the payout of the game if Player 2 follows the "coin-tossing strategy" outlined above. It is clear that $X = \sum_{j=1}^{n} |B_j|$. Since X can only take on an integer value between $-n^2$ and n^2, it is then a discrete random variable. Then Lemma 4.1 would imply that there exists positive probability that $X \geq E[X] = \sqrt{\frac{2}{\pi}}n^{\frac{3}{2}}$. Since X is completely determined by the outcome of the n coin-tosses, this implies there exists an outcome of the coin-tosses which gives instructions for Player 2 to achieve a payout at least as much as E[X]. By definition, the optimal strategy for Player 2 should achieve a payout at least as much as that achievable using any other strategy, so $\sqrt{\frac{2}{\pi}}n^{\frac{3}{2}}$ is a lower bound for the optimal payout for Player 2.

5. Conclusion

We have demonstrated that if both Player 1 and Player 2 play optimally, we then have $\sqrt{\frac{2}{\pi}}n^{\frac{3}{2}} \leq S(n) \leq \sqrt{2}n^{\frac{3}{2}}$ as $n \to \infty$. So as $n \to \infty$, $|S(n)/n^{\frac{3}{2}}| \leq \sqrt{2}$ and $|n^{\frac{3}{2}}/S(n)| \leq \sqrt{\frac{\pi}{2}}$, which means $S(n) = \Theta(n^{3/2})$.