FOURIER ANALYSIS AND THE DIRICHLET PROBLEM

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Abstract. Fourier analysis began as an attempt to approximate periodic functions with infinite summations of trigonometric polynomials. For certain functions, these sums, known as Fourier series, converge exactly to the original function. Using this property, along with some other properties of trigonometric polynomials, in particular that they are sums of holomorphic and antiholomorphic functions, we are able to solve the Dirichlet problem on the disc. Then, applying the result for the unit disc along with a couple Möbius transformations, we are able to solve the Dirichlet problem on the upper half plane.

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1. Fourier Series

In calculus, we learn that any sufficiently nice function can be approximated by an infinite sum of polynomials called a Taylor series. So, if we consider only periodic functions, it doesn't seem too unreasonable to assume that we could approximate them with infinite sums of sines and cosines (if this apparent guess seems too nonrigorous, rest assured that these summations can be shown to arise quite naturally; see [4]). The existence and convergence of these summations is the basis for Fourier analysis. First, we will determine what form these summations should take, and then we will explore some basic properties of their convergence.

1.1. Suppose we have a Fourier series...

We begin by assuming that we have a 2π periodic function, f, and a summation of trigonometric functions of varying frequencies that converges to it. Thus, we have $f(\theta) = \sum_{k=0}^{\infty} a_k \sin(k\theta) + b_k \cos(k\theta)$.

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However, by Euler's identity, we also have that $e^{ik\theta} = \cos(k\theta) + i\sin(k\theta)$. So, with the correct complex c_ks, we have:

$$f(\theta) = \sum_{k=0}^{\infty} a_k \sin(k\theta) + b_k \cos(k\theta) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}$$

Assuming that this sum converges uniformly, we now must determine the actual values for the c_ks. Before rushing into things, however, we make a slight change to consider functions that are periodic on the interval $[0, 1]$; this prevents us from dividing out constants later. So, we are now solving for the c_ks in $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx}$.

We have

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx}$$

Now, multiply each side by $e^{-2\pi imx}$ and integrate. So,

$$\int_0^1 f(x)e^{-2\pi imx}dx = \sum_{k=-\infty}^{\infty} c_k \int_0^1 e^{2\pi ikx}e^{-2\pi imx}dx$$

Then, we note that the $e^{2\pi ikx}$ are orthogonal with respect to the inner product $\langle g, h\rangle = \int_0^1 g(x)\overline{h(x)}dx$. In other words, it can easily be verified that

$$\int_0^1 e^{2\pi ikx}e^{-2\pi imx}dx = \begin{cases} 0 & (k \neq m) \\ 1 & (k = m) \end{cases}$$

Thus, we determine that

$$c_m = \int_0^1 f(x)e^{-2\pi imx}dx$$

1.2. **Let's define what we just did.** Now that we've explored what these summations should look like when everything works out, it's time to make some definitions out of our discoveries. First, it would be helpful to have a standard name and notation for those c_ks that we just found.

Definition 1.1 (Fourier Coefficients)**.** Let

$$\hat{f}(k) = \int_0^1 f(x)e^{2\pi ikx}dx$$

be the k^{th} Fourier Coefficient of f.

Now, we should formally define what a Fourier Series actually is.

Definition 1.2 (Fourier Series)**.** The Fourier series of f is the trigonometric polynomial with the Fourier coefficients defined above:

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikx}.$$

It is important to keep in mind that these definitions describe how to determine the Fourier series for a given function, but they don't imply anything about the convergence of the series. In fact, even when the Fourier series does converge, it doesn't always converge to the function that it is based on.

1.3. **When does a Fourier series converge?** As it turns out, a certain degree of niceness is required for a Fourier series to converge to the function that it is based on. In fact, there are even continuous functions whose Fourier series diverges at a point (see [1] for a proof of existence).

However, despite this unfortunate property, there are several conditions that are sufficient for the convergence of a Fourier series (assuming we've decided what sort of convergence we are talking about). For pointwise convergence, differentiability or local bounded variation are both sufficient. On the other hand, convergence in L^p norms and almost everywhere convergence can also result if $f \in L^p$ for most values of p.

Formal theorems regarding these various types of convergence as well as their proofs can be found in [1]. For our purposes, it is only necessary for us to know that the Fourier series of a nice enough function does converge to the function. Thus, it makes sense to use them for the Dirichlet problem on the disc.

2. The Dirichlet Problem on the Disc

Given a connected open set, Ω, and a function, f defined on the boundary of Ω, $\partial\Omega$, the solution to the Dirichlet problem is a function, u, such that

$$\begin{cases} \triangle u = 0 & x \in \Omega \\ u = f & x \in \partial\Omega \end{cases}.$$

Thus, for the open disc, we are looking for a function that is harmonic on the interior of the disc and periodic on the circle. First we will show that the solution will be unique, and then we will explicitly solve the Dirichlet problem on the disc.

2.1. **Uniqueness.** In order to avoid proving the first fairly simple theorem, I refer the interested reader to [2]. I will then use the result to prove the uniqueness of solutions to a given Dirichlet problem.

Theorem 2.1 (Maximal Principle)**.** *Let Ω be any bounded domain, and let $u(x,y)$ in $C^0(\bar{\Omega}) \cap C^2(\Omega)$ be a harmonic function in Ω. Then u attains its maximum value on $\bar{\Omega}$ somewhere on $\partial\Omega$.*

Corollary 2.2. *Let Ω be any bounded domain, and let $u(x,y)$ in $C^0(\bar{\Omega}) \cap C^2(\Omega)$ be harmonic in Ω and let $u(x,y) = 0\ \forall (x,y) \in \partial\Omega$. Then, $u(x,y) = 0\ \forall (x,y) \in \bar{\Omega}$.*

Proof. By the theorem, $u(x,y) \leq 0\ \forall (x,y) \in \Omega$. Then, since $-u(x,y) = 0\ \forall (x,y) \in \partial\Omega$, we also have that $-u(x,y) \leq 0\ \forall (x,y) \in \Omega$. Thus, $u(x,y) = 0\ \forall (x,y) \in \bar{\Omega}$. □

Now we are ready to prove that the solution to a given Dirichlet problem is unique.

Theorem 2.3 (Uniqueness)**.** *Suppose we have two functions u and v such that*

$$\begin{cases} \triangle u = \triangle v = 0 & x \in \Omega \\ u = v = f & x \in \partial\Omega \end{cases}.$$

Then, $u(x) = v(x)\ \forall x \in \bar{\Omega}$.

Proof. Consider $w(x) = u(x) - v(x)$. The function w is harmonic in Ω, since $\triangle w = \triangle u - \triangle v = 0 - 0 = 0$, and $w(x) = u(x) - v(x) = f(x) - f(x) = 0\ \forall x \in \partial\Omega$. So, by the corollary, $w(x) = 0\ \forall x \in \bar{\Omega}$. Thus, $u(x) = v(x)\ \forall x \in \bar{\Omega}$. □

2.2. **Solving the unit disc.** When solving the Dirichlet problem on the unit disc, we first observe that we are looking for a harmonic function that approximates the function, f, from the interior from the disc. Noting that f is periodic, it would be sufficient to find a harmonic function that is equivalent to the Fourier series of f on the boundary of the disc.

First, we note that the Fourier series of f is equivalent to the limit as $r \to 1^-$ of

$$u(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k + \sum_{k=-\infty}^{-1} \hat{f}(k) \overline{z}^{|k|} \quad \text{where} \quad z = re^{2\pi i\theta}.$$

In order for this expression to be useful in this situation, we need to know that this function is harmonic. For this, we will use the fact that every holomorphic function is harmonic (see [3] for this result). Thus, since u is the sum of a holomorphic function, $(\sum_{k=0}^{\infty} \hat{f}(k) z^k$, and an antiholomorphic function, $\sum_{k=-\infty}^{-1} \hat{f}(k) \overline{z}^{|k|}$, in the unit disc, we know that u is harmonic.

With this in mind, we have essentially solved the Dirichlet problem on the unit disc, and now we just have to clean up our solution. First, we combine the two sums in the definition of u to get

$$u(re^{2\pi i\theta}) = \sum_{k=-\infty}^{\infty} \hat{f}(k) r^{|k|} e^{2\pi i k\theta}.$$

Then, if this converges nicely enough,

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) r^{|k|} e^{2\pi i k\theta} = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) \sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi i k(\theta - t)} dx.$$

Now, as is often done in Fourier analysis, we try to find a closed form for this convolution operator. So, here's how it's done:

$$\sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi i k(t)} = \sum_{k=0}^{\infty} r^{|k|} e^{2\pi i k(t)} + \sum_{k=1}^{\infty} r^{|k|} e^{-2\pi i k(t)}$$

Noting that each of these is now a geometric series, we have

$$\sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi i k(t)} = \frac{1}{1 - re^{2\pi i t}} + \frac{re^{-2\pi i t}}{1 - re^{-2\pi i t}} = \frac{1 - re^{-2\pi i t} + re^{-2\pi i t} - (re^{-2\pi i t})(re^{2\pi i t})}{1 - (re^{2\pi i t} + re^{-2\pi i t}) + (re^{2\pi i t})(re^{-2\pi i t})}$$

So, we finally have

$$\sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi i k(t)} = \frac{1 - r^2}{1 - 2r\cos(2\pi t) + r^2}.$$

This convolution operator is known as the Poisson kernel for the unit disc, and is denoted $P_r(t) = \frac{1-r^2}{1-2r\cos(2\pi t)+r^2}$. Thus, our cleaned up solution to the Dirichlet problem on the unit disc is

$$u(re^{2\pi i\theta}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) P_r(\theta - t) dt.$$

For specific theorems about the convergence of $P_r * f(x)$ see [1].

3. Möbius Transformations and the Upper Half Plane

A Möbius transformation, for our purposes, is a function that transforms the complex plane in a specific way. Basically, the transformation can be viewed as a composition of rotations, inversions, dilations, and translations on the complex projective space. Formally, this group of transformations consists of functions of the form $m(z) = \frac{az+b}{cz+d}$, where $a, b, c,$ and d are complex constants. It can be shown that generalized circles (circles on the projective space) are mapped to generalized circles under Möbius transformations, so our goal will be to find a transformation that will map the unit circle to the x-axis (making sure that the interior of the circle corresponds to the upper half of the plane). Then, using this transformation, we will convert a Dirichlet problem on the upper half plane to one on the disc, solve it on the disc, and finally transform it back to the upper half plane for the final solution.

3.1. **Which transformations to use?** Our first step is to change a function on the upper half plane to a function on the disc, so we want a transformation, m, that will take points in the disc to points on the upper half plane in a specified manner. The idea is that we compose $f : U \to \mathbb{C}$ with $m : D \to U$ to get a function $f(m(z)) : D \to \mathbb{C}$, where D is the unit disc and U is the upper half plane. So, in order to find a Möbius transformation that will take the unit disc to the upper half plane, we will use what we know about the general form of Möbius transformations and information about its action on specific points. Since there are four unknowns, it makes sense to use four point to define our transformation.

Sending the points $z = -i, 1, i, -1$ on the unit circle to the points $z = -1, 0, 1, \infty$ on the x-axis with the transformation $m(z) = \frac{az+b}{cz+d}$, we get

$$a + b = 0,\ ai + b = ci + d,\ -ai + b = ci - d,\ \text{and}\ -c + d = 0.$$

Then, we solve this system of equations to discover that

$$a = -b = -ic = -id.$$

Noting that the ratio in the transformation would cancel out any change in the magnitude of the constants, we can choose any value for a and determine the other variables from that. For $a = 1$, we have

$$a = 1, b = -1, c = i, d = i.$$

Thus, $m(z) = \frac{z-1}{iz+i}$. We will also need to invert this later to take the upper half plane back to the disc. The interested reader can verify that $m^{-1}(z) = \frac{iz+1}{-iz+1}$.

3.2. **Transforming the Dirichlet problem.** Now that we have our transformations, we can go ahead and solve the Dirichlet problem on the upper half plane. So, we seek a function $u(z)$ such that $\begin{cases} \triangle u = 0 & z \in U \\ u = f & z \in \partial U \end{cases}$. Composing the function f with our transformation, m, we have $f(m(z))$, which is now a boundary condition on the unit disc. Next, we use our solution for the Dirichlet problem on the unit disc, to find

$$u'(re^{2\pi i\theta}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(m(e^{2\pi it}))P_r(\theta - t)dt.$$

So, we have solved the Dirichlet problem for $f(m(z))$ on the unit circle, and we just have to use the inverse transformation we found earlier to transform it back to the upper half plane. Composing our function $u' : D \to \mathbb{C}$ with $m^{-1} : U \to D$, we should get our function $u : U \to \mathbb{C}$. Now we need to determine the form of this function. After composing these functions we have

$$u(z) = u'(m^{-1}(z)) = \int_{?}^{?} f(m(m^{-1}(t)))P_?(?)dt.$$

In order to complete this solution, we now need to determine how the bounds of the integral and the Poisson kernel were affected by composing with our inverse transformation. First, we examine the bounds. The basic observation here is that the bounds on the solution to the Dirichlet problem on the disc spans the entire circle, so the bounds on the solution to the Dirichlet problem on the upper half plane should span the entire circle that the x-axis makes up on the projective space. So, we have an integral from negative infinity to infinity.

Now, we must consider how the Poisson kernel is affected by the composition. The problem here is that the Poisson kernel was given as a function of the angle for each radius rather than as a function of the complex number itself. So, we need to find a way of expressing the Poisson kernel as a function of z. To achieve this, we use the following.

$$P_r(t) = \frac{1-r^2}{1-2r\cos(2\pi t)+r^2} = P(z) = \frac{1-z\overline{z}}{1-z-\overline{z}+z\overline{z}} \text{ where } z = re^{2\pi i\theta}.$$

Now that we have the Poisson kernel as an actual function of z, we can just compose it with m^{-1}. So, going through a very messy calculation that the reader can verify if he/she has a lot of free time, we have

$$P(m^{-1}(x+iy)) = \frac{y}{x^2+y^2}.$$

This function is called the Poisson kernel for the upper half plane, and is denoted $P_y(x) = \frac{y}{x^2+y^2}$.

So, we finally have a solution to the Dirichlet problem on the upper half plane. Filling in the question marks from our last steps, we have

$$u(x+iy) = \int_{-\infty}^{\infty} f(t)P_y(x-t)dt = P_y * f(x)$$

as our solution. Now, since it would be nice to have our solution on the real plane rather than the complex plane, we make one last (almost insignificant) change to

$$u(x,y) = \int_{-\infty}^{\infty} f(t)P_y(x-t)dt = P_y * f(x)$$

The reader can verify that this is equivalent to the solution determined in a very different way in [1].

References

[1] Javier Duoandikoetxea. Fourier Analysis. American Mathematical Society. 2001.

[2] Karl E. Gustafson. Introduction to Partial Differential Equations and Hilbert Space Methods. Dover Publications, Inc. 1999.

[3] Walter Rudin. Real and Complex Analysis. McGraw-Hill, Inc. 1966.

[4] Elias Stein and Rami Shakarchi. Fourier Analysis, An Introduction. Princeton University Press. 2003.