REGULAR GRAPHS OF GIVEN GIRTH

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Contents

1. Introduction

This paper gives an introduction to the area of graph theory dealing with properties of regular graphs of given girth. A large portion of the paper is based on exercises and questions proposed by *László* Babai in his lectures and in his Discrete Math lecture notes.

2. Graphs

Graph theory is the study of mathematical structures called graphs. We define a **graph** as a pair (V, E), where V is a nonempty set, and E is a set of unordered pairs of elements of V. V is called the set of **vertices** of G, and E is the set of **edges**. Two vertices a and b are **adjacent** provided $(a, b) \in E$. If a pair of vertices is adjacent, the vertices are referred to as **neighbors**. We can represent a graph by representing the vertices as points and the edges as line segments connecting two vertices, where vertices $a, b \in V$ are connected by a line segment if and only if $(a, b) \in E$. Figure 1 is an example of a graph with vertices $V = \{x, y, z, w\}$ and edges $E = \{(x, w), (z, w), (y, z)\}$.

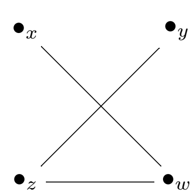

Figure 1

Now we define some relevant properties of graphs.

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Definition 2.1. A **walk** of length k is a sequence of vertices $v_0, v_1, \ldots, v_k$, such that for all $i > 0, v_i$ is adjacent to v_{i-1}.

Definition 2.2. A **connected graph** is a graph such that for each pair of vertices v_1 and v_2 there exists a walk beginning at v_1 and ending at v_2.

Example 2.3. *The graph in Figure 2 is not connected because there is no walk beginning at z and ending at w.*

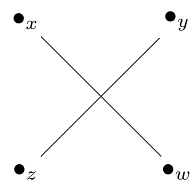

Figure 2

From this point on, we will assume that every graph discussed is connected.

Definition 2.4. A **cycle** of length $k > 2$ is a walk such that each vertex is unique except that $v_0 = v_k$.

Definition 2.5. A **tree** is a graph with no cycles.

Definition 2.6. The **girth** of a graph is the length of its shortest cycle.

Since a tree has no cycles, we define its girth as $\inf \emptyset = \infty$

Example 2.7. *The graph in figure 3 has girth 3.*

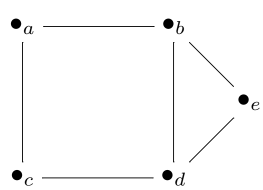

Figure 3

Definition 2.8. The **degree** of a vertex is the number of vertices adjacent to it.

Definition 2.9. A graph is r-**regular** if every vertex has degree r.

Definition 2.10. A **complete graph** is a graph such that every pair of vertices is connected by an edge.

We observe that a complete graph with n vertices is $n-1$-regular, and has

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

edges.

Definition 2.11. A **complete bipartite graph** is a graph whose vertices can be divided into two disjoint sets X and Y, such that every vertex in X is adjacent to every vertex in Y and no pair of vertices within the same set is adjacent.

The complete bipartite graph with $\mid X \mid= x$ and $\mid Y \mid= y$ is denoted $K_{x,y}$.

Definition 2.12. The complement of a graph G, denoted $\bar{G}$, is a graph on the same vertices as G, only the vertices adjacent in G are not adjacent in $\bar{G}$ and the vertices not adjacent in G are adjacent in $\bar{G}$.

Example 2.13. *In Figure 4, $\bar{G}$ is the complement of G.*

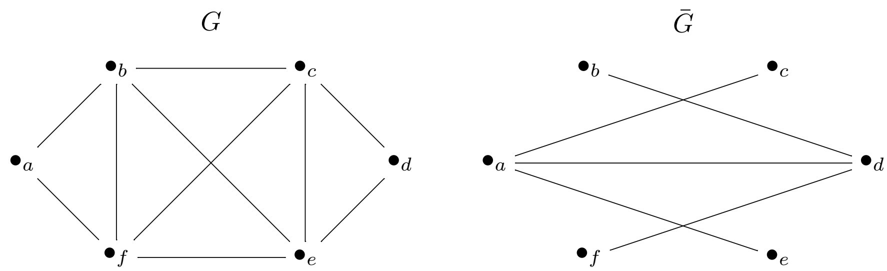

Figure 4

3. Girths of Regular Graphs

Using only the definitions of the previous section and some elementary linear algebra, we are able to prove some interesting results concerning r-regular graphs of a given girth. We begin with two lemmas upon which the rest of the paper will depend.

Lemma 3.1. *The number of vertices of an r-regular graph with an odd girth of $g = 2d + 1$ (where $d \in \mathbb{Z}^+$) is n, such that*

$$n \geq 1 + \sum_{i=0}^{d-1} r(r-1)^i.$$

We will first show this for $g = 5$:

Proof of Lemma 3.1 for $g = 5$. Let G be an r-regular graph with girth 5. Take a vertex v_0 of G. Let $V_0 = \{v_0\}$. v_0 must be adjacent to r vertices. Let V_1 be the set containing those r vertices. None of the elements of V_1 can be adjacent to one another, because that would create a 3-cycle. Thus, each of those vertices must be adjacent to an additional $r - 1$ vertices. Each of these vertices must be unique in order to avoid creating a 4-cycle. Let V_2 be the set of all vertices adjacent to elements of V_1 except v_0. Elements of V_2 can be adjacent to one another, creating 5-cycles. Thus, adding up the vertices in Figure 5, we have

$$\begin{aligned} & n = 1 + r + r(r-1) \\ \Rightarrow \quad & n = 1 + \sum_{i=0}^{1} r(r-1)^i. \end{aligned}$$

$\square$

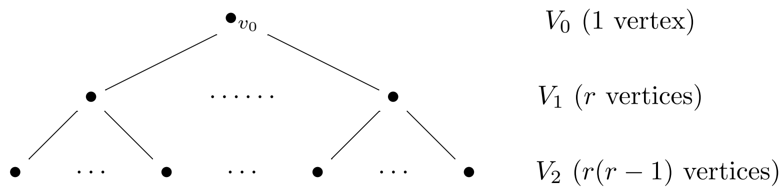

Figure 5

Now we extend this to any $g = 2d + 1$.

Proof of Lemma 3.1. Let G be an r-regular graph with girth $g = 2d + 1$. Take a vertex v_0 of G. Let $V_0 = \{v_0\}$. v_0 must be adjacent to r vertices. Let V_1 be the set consisting of those r vertices. None of the elements of V_1 can be adjacent to one another without creating a 3-cycle. Now, we will create more sets of vertices, as in Figure 5. The vertices adjacent to those in V_1 (not including v_0) will be in the set V_2. Similarly, if v is a vertex, we define $V_k = \{v \mid v \text{ is adjacent to an element of } V_{k-1} \text{ and } v \notin V_n \text{ where } n < k-1\}$. There can be no cycles within vertices in $\bigcup_{i=0}^{d-1} V_i$. Suppose there were such a cycle. Then, without loss of generality, we can assume that the cycle is a result of an adjacency between two elements within the same set V_n. Then, there would be a $2n+1$-cycle, and since $n < d$, $2n+1$ must be less than g. Thus, there can be no such cycle, which implies that each vertex of V_d is a unique vertex of G. Each set V_n ($< n \leq d$) has a cardinality of $(r-1)$ times the number of vertices of V_{n-1}, and V_1 has r vertices. Thus, V_k has $r(r-1)^{k-1}$ vertices. So, summing the cardinalities of V_0 through V_d we have,

$$n \geq 1 + \sum_{i=0}^{d-1} r(r-1)^i$$

□

We now give a similar lemma for graphs of even girth.

Lemma 3.2. *The number of vertices of an r-regular graph with a girth of $g = 2d$ (where $d \geq 2$) is n, where*

$$n \geq 1 + (r-1)^{d-1} + \sum_{i=0}^{d-2} r(r-1)^i = 2\sum_{i=0}^{d-1}(r-1)^i.$$

We will prove this in two different ways. In the first proof, we will begin with a vertex of the graph, as in the proof of Lemma 3.1, and in the second one we will begin with two adjacent vertices.

Proof #1 of Lemma 3.2. Let G be an r-regular graph with girth $g = 2d$, where $d \geq 2$. Let v_0 be a vertex in G. Let $V_0 = \{v_0\}$. v_0 must be adjacent to r vertices. Let V_1 be the set consisting of those r vertices. None of the elements of V_1 can be adjacent to one another without creating a 3-cycle. The vertices adjacent to those in V_1 (not including v_0) will be in the set V_2. Similarly, if v is a vertex, we define $V_k = \{v \mid v \text{ is adjacent to an element of } V_{k-1} \text{ and } v \notin V_n \text{ where } n < k-1\}$. There can be no cycles within vertices in $\bigcup_{i=0}^{d-1} V_i$. Suppose there were such a cycle. Then, without loss of generality, we can assume that the cycle is a result of an adjacency between two elements within the same set V_n. Then, there would be a $2n+1$-cycle, and since $n < d$, $2n+1$ must be less than g. Thus, there can be no such cycle, which implies that there must be more vertices than just those in sets V_1 through V_{d-1}. Each set V_n $(n > 0)$ has a cardinality of $(r-1)$ times the number of vertices of V_{n-1}, and V_1 has r vertices. Thus, V_k has $r(r-1)^{k-1}$ vertices. Thus, we have counted a total of

$$n \geq 1 + \sum_{i=0}^{d-2} r(r-1)^i$$

vertices so far. However, each vertex in V_{d-1} must have $r-1$ additional neighbors which are not in $\bigcup_{i=0}^{d-1} V_i$ in order to satisfy the girth requirement of $2d$ and r-regularity. That gives $r(r-1)^{d-1}$ additional vertices. However, these vertices need not be unique in order to create a $2d$-cycle. Since each of the vertices can have at most r neighbors within V_{d-1}, we can divide by r so that we only actually need an additional $(r-1)^{d-1}$ unique vertices. So, summing the cardinalities of V_0 through V_{d-1}, and the additional vertices that are adjacent to those in V_{d-1}, we have,

$$n \geq 1 + (r-1)^{d-1} + \sum_{i=0}^{d-2} r(r-1)^i.$$

□

Proof #2 of Lemma 3.2. Let G be an r-regular graph with girth $g = 2d$, where $d \geq 2$. Let a_0 and b_0 be adjacent vertices in G. Let $V_1 = \{a_0, b_0\}$. Each vertex in V_1 must be adjacent to an additional $r-1$-vertices. These vertices must be unique in order to prevent a 3-cycle from being created. Define V_2 as the set of the $2(r-1)$ vertices adjacent to the vertices in V_1 (not including elements of V_1. Similar to the first proof, we define $V_k = \{v \mid v \text{ is adjacent to an element of } V_{k-1} \text{ and } v \notin V_n$ where $n \leq k-1\}$. There can be no cycles within vertices in $\bigcup_{i=0}^{d-1} V_i$. Suppose there were such a cycle. Then, without loss of generality, we can assume that the cycle is a result of an adjacency between two elements within the same set V_n. Then, there would be a $2n$-cycle, and since $n < d$, $2n$ must be less than $g = 2d$. Thus, there can be no such cycle, which implies (since g is even) that each vertex of V_d is a unique vertex of G. Each set V_n has a cardinality of $(r-1)$ times the number of vertices of V_{n-1} and V_1 has 2 vertices. Thus, V_k has $2(r-1)^{k-1}$ vertices. So, summing the cardinalities of V_1 through V_d, we obtain

$$n \geq 2 \sum_{i=0}^{d-1} (r-1)^i.$$

Now, simple algebra shows that $1+(r-1)^{d-1}+\sum_{i=0}^{d-2} r(r-1)^i = 2\sum_{i=0}^{d-1}(r-1)^i$: Let $x = r-1$. Then the left hand side of the equation becomes:

$$\begin{aligned} LHS &= (x+1)\sum_{i=0}^{d-2} x^i + x^{d-1} + 1 \\ &= \frac{(x+1)(x^{d-1}-1)}{x-1} + x^{d-1} + 1 \\ &= \frac{1}{x-1}\left[(x+1)(x^{d-1}-1) + (x^{d-1}+1)(x-1)\right] \\ &= \frac{1}{x-1}(x^d - x + x^{d-1} - 1 + x^d - x^{d-1} + x - 1) \\ &= \frac{2x^d-2}{x-1} = 2\left(\frac{x^d-1}{x-1}\right) = 2\sum_{i=0}^{d-1} x^i = 2\sum_{i=0}^{d-1}(r-1)^i = RHS \end{aligned}$$

□

4. Minimal Regular Graphs of Given Girth

A natural question that arises from the lemmas in the previous section is when do the equalities hold? That is, for which pairs of $g = 2d+1$ and r can we find a graph such that the number of vertices is

$$n = 1 + \sum_{i=0}^{d-1} r(r-1)^i, \tag{1}$$

and for which pairs of $g = 2d$ and r can we find a graph such that the number of vertices is

$$n = 2\sum_{i=0}^{d-1}(r-1)^i? \tag{2}$$

For small values of g there are many values of r for which the equality holds. We begin with $g = 3$. In this case, we must find r-regular graphs with girth 3 and $1 + r(r-1)^0 = 1 + r$ vertices.

Observation 4.1. *For every integer $r > 1$, there exists an r-regular graph with girth 3 and exactly $r+1$ vertices.*

Proof. Given an integer $r > 1$, the complete graph with $r+1$ vertices is r-regular and has a girth of 3. □

In order to further understand regular graphs of given girth, we must introduce some linear algebra relevant to graph theory.

Definition 4.2. The **adjacency matrix** A of a graph G is the square matrix whose entries are a_{ij}, where

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the adjacency matrix of a (non-directed) graph is symmetric, that is $A^T = A$, since adjacency is a symmetric relation. That is, i is adjacent to $j \iff j$ is adjacent to i. We also observe that the sum of the entries of the ith row or of the ith column is equal to the degree of the ith vertex. Therefore, the sum of any row or any column of the adjacency matrix of an r-regular graph is r.

A result of the spectral theorem is that every symmetric matrix has an orthonormal eigenbasis, and real eigenvalues. Thus, the same holds for every adjacency matrix.

Observation 4.3. *The square of an adjacency matrix has entries b_{ij}, where b_{ij} is equal to the number of vertices which are adjacent to both vertex i and vertex j for $i \neq j$, and b_{ii} is equal to the degree of vertex i.*

Proof. Suppose A is an n by n adjacency matrix of a labeled graph. Then, by matrix multiplication, A^2 will have entries b_{ij}, where

$$b_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}.$$

We notice that the product $a_{ik}a_{kj}$ will be nonzero if and only if k is adjacent to both i and j, in which case $a_{ik} = 1$ and $a_{kj} = 1$ so $a_{ik}a_{kj} = 1$. Thus,

$$b_{ij} = \sum_{k=1}^{l} 1 = l,$$

where l is the number of vertices which are adjacent to both vertex i and vertex j. b_{ii} will thus simply be the number of vertices adjacent to i, or the degree of i, since i shares all adjacencies with itself. □

Since each entry of A^2 has a geometric interpretation within the graph, it is reasonable to infer that the same holds for A^k. In fact, we can state the following lemma.

Observation 4.4. *Let A be an adjacency matrix of a labeled graph G, with n vertices. Then the entry a_{ij} of A^k, where $k \geq 1$, is the number of walks of length k from vertex j to vertex i.*

Proof by Induction. The statement holds for $k = 1$ by the definition of an adjacency matrix. Assume the statement holds for $k = l$. Then, we multiply A^l and A to obtain A^{l+1}. It does not matter in which order we multiply them since matrix multiplication is associative. Let the entries of A be denoted b_{ij} and let the entries of A^l be denoted c_{ij}. The entry a_{ij} of A^{l+1} is the dot product of the ith row of A and the jth column of A^l. That is,

$$a_{ij} = \sum_{m=1}^{n} b_{im}c_{mj}.$$

We notice that the product $b_{im}c_{mj}$ will be nonzero if and only if i is adjacent to m and there is at least one walk of length l from j to m. this is equivalent to saying there is at least one walk of length $l+1$ from j to i whose second to last vertex is m. Thus, if $b_{im}c_{mj}$ is nonzero, it is equal to the number of walks of length $l+1$ from j to i whose second to last vertex is m. Thus, summing over n, we find that $a_{ij} = \sum_{m=1}^{n} b_{im}c_{mj}$ is the total number of walks of length l from j to i. □

Now that we have studied adjacency matrices, let us briefly return to r-regular graphs of girth 3 with $r+1$ vertices (complete graphs) in order to give an example.

Example 4.5. *Let G be a complete graph of degree $r > 1$, and let A be its adjacency matrix. Since G is complete, the entries of A are*

$$a_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j. \end{cases}$$

Also, since G has $r + 1$ vertices, every pair of unique vertices has exactly $r - 1$ neighbors in common, so the entries of A^2 are

$$a_{ij} = \begin{cases} r & \text{if } i = j \\ r - 1 & \text{if } i \neq j. \end{cases}$$

Thus, for the adjacency matrix of the complete graph, the following equality holds:

$$A + A^2 = rJ,$$

where J is the matrix of all ones.

Now, let us look at r-regular graphs of girth $g = 4$ ($d = 2$). According to Lemma 3.2, at least n vertices are required to make such a graph, where

$$2\sum_{i=0}^{1}(r-1)^i = 2(r-1)^0 + 2(r-1)^1 = 2 + 2r - 2 = 2r.$$

In fact, only $2r$ vertices are *required*, as stated in the following lemma.

Lemma 4.6. *For every integer $r > 1$, the only r-regular graph G that has girth $g = 4$ and exactly $2r$ vertices is the complete bipartite graph $K_{r,r}$.*

Proof. Let x be a vertex in G. x is adjacent to r vertices. Let Y be the set of vertices adjacent to x. Let X be the set of r vertices not adjacent to x (including x itself). No two vertices in Y can be adjacent without creating a 3-cycle. Thus, every vertex of Y must be adjacent to every vertex in X in order for G to be r-regular. Thus, no two vertices in X can be adjacent without creating a 3-cycle. Therefore, G is the complete bipartite graph $K_{r,r}$. Suppose $v \in X$ and $z \in Y$. Then, the vertices x, y, v, and z create a four cycle. Thus, G has girth 4. □

By using Lemma 4.6, we can make the following observation about the adjacency matrix of an r-regular graph of girth four with $2r$ vertices:

Observation 4.7. *Let G be an r-regular graph of girth four with $2r$ vertices, and let A be an adjacency matrix of G. Then the following equality holds:*

$$A^2 + rA = rJ.$$

Proof. Let i and j be vertices in G. Since G is a complete bipartite graph, i and j have r neighbors in common if i is not adjacent to j and no neighbors in common if i is adjacent to j. Thus, A^2 has zeros where A has ones, and r's where A has zeros. Thus,

$$A^2 + rA = rJ.$$

□

Now let us approach the problem of when the equality in equation (1) holds for regular graphs of girth $g = 5$ ($d = 2$). That is, we must find for which values of r we can find a graph with girth 5 and n vertices, where

$$n = 1 + r(r-1)^0 + r(r-1)^1 = 1 + r + r^2 - r = 1 + r^2$$

It turns out that there are only a finite number of values of r for which the equality holds. In fact, there are at most 5 values: $\{1,2,3,7,57\}$. We say "at most" because such a graph has not yet been discovered for $r = 57$. To give an idea of how complex the graphs become as r increases, let's look at the simplest of these graphs:

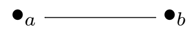

Figure 6: $r = 1$, $n = 2$

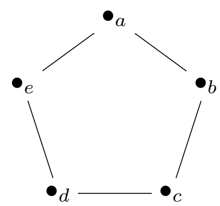

Figure 7: $r = 2$, $n = 5$

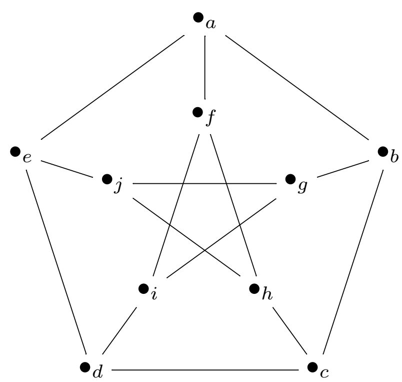

Figure 8 (Petersen Graph): $r = 3$, $n = 10$

The graph for $r = 7$ is called the Hoffman-Singleton graph, and has 50 vertices and 175 edges. In fact, the theorem in this section regarding graphs of girth five is known as the Hoffman-Singleton Theorem. Before we state the Hoffman-Singleton Theorem, we must discuss some features of regular graphs with girth ≥ 5 for which the equality in equation (1) holds.

Lemma 4.8. *If G is an r-regular graph, has girth at least 5, and $n = r^2 + 1$ vertices, the number of common neighbors of a pair x, y of vertices is zero if x, y are adjacent and 1 if x, y are not adjacent.*

Proof. For this proof, we will be directly referring to the proof of Lemma 3.1 for $g = 5$ and to Figure 5. Without loss of generality, let $x = v_0$.

Case 1: $y \in V_1$. In this case, y is adjacent to x and x and y have no neighbors in common, so the lemma holds.

Case 2: $y \in V_2$. In this case, y is not adjacent to x, and x and y are both adjacent to exactly one vertex in V_1. Thus the lemma holds. $\square$

Theorem 4.9 (Hoffman-Singleton Theorem)**.** *If there exists an r-regular graph G of girth greater than or equal to 5 such that the number of vertices n satisfies the equality $n = r^2 + 1$, then $r \in \{1, 2, 3, 7, 57\}$.*

See appendix for proof.

5. Further Studies

It is possible to study the minimal regular graphs of girth greater than 5, however, such studies require more complex mathematical tools than those used in this paper, and some have even been studied with no success yet.

6. Appendix

The following proof was given in a lecture by *László* Babai. I added it for completeness of the paper.

Proof of Theorem 4.9. Let A be the adjacency matrix of G, and let $\bar{A}$ be the adjacency matrix of $\bar{G}$. By Observation 4.3 and Lemma 4.8, we have

$$A^2 = \bar{A} + rI, \tag{3}$$

where I is the identity matrix. Also, by the definition of the complement of a graph, we have

$$A + \bar{A} + I = J, \tag{4}$$

where J is the matrix of all ones. Thus, substituting equation (3) into equation (4), we have

$$\begin{aligned} & A + (A^2 - rI) + I = J \\ \Rightarrow \quad & A + A^2 - (r-1)I = J \end{aligned} \tag{5}$$

As observed earlier, the sum of the elements of each row of A is r. Thus,

$$A \cdot \mathbf{1} = \begin{bmatrix} r \\ r \\ \vdots \end{bmatrix},$$

where $\mathbf{1}$ is the vector of all ones. So we have $A \cdot \mathbf{1} = r \cdot \mathbf{1}$, which implies that $\mathbf{1}$ is an eigenvector of A with eigenvalue r. By the Spectral Theorem (as described earlier), A has an orthogonal eigenbasis, $\mathbf{e}_1 = \mathbf{1}, \mathbf{e}_2, \ldots, \mathbf{e}_n$. That is, $\mathbf{e}_i \perp \mathbf{e}_j$ for all pairs of $i, j \in \{1, 2, \ldots, n\}$ such that $i \neq j$. Thus, $\mathbf{1} \perp \mathbf{e}_i$ for all $i \geq 2$. So, for $i \geq 2$, we have

$$J\mathbf{e}_i = \begin{bmatrix} \mathbf{e}_i \cdot \mathbf{1} \\ \vdots \\ \mathbf{e}_i \cdot \mathbf{1} \end{bmatrix} = \mathbf{0}.$$

Multiplying equation (5) by $\mathbf{e}_i$ ($i \geq 2$) to the right, we obtain

$$(A^2 + A - (r-1)I)\mathbf{e}_i = J\mathbf{e}_i = \mathbf{0}. \tag{6}$$

We know that $A\mathbf{e}_i = \lambda_i \mathbf{e}_i$, and thus $A^2\mathbf{e}_i = \lambda_i^2 \mathbf{e}_i$, so equation (6) becomes

$$\lambda_i \mathbf{e}_i + \lambda_i \mathbf{e}_i - (r-1)\mathbf{e}_i = \mathbf{0},$$

and thus

$$\lambda_i^2 + \lambda_i - (r-1) = 0. \tag{7}$$

Since this is a quadratic equation, we know that it has at most 2 roots. Thus, A must have at most 3 eigenvalues; let's call them r, μ_1, and μ_2 with multiplicities 1, m_1, and m_2 respectively.

μ_1 and μ_2 must be roots of equation (7). Thus, we have

$$\mu_{1,2} = \frac{-1 \pm \sqrt{1 + 4(r-1)}}{2} = \frac{-1 \pm \sqrt{4r-3}}{2}.$$

Let $s = \sqrt{4r-3}$. Then, $s^2 = 4r - 3$, or , equivalently,

$$r = \frac{s^2 + 3}{4}. \tag{8}$$

Since A is an n by n matrix, we have

$$1 + m_1 + m_2 = n. \tag{9}$$

We also know that the sum of the eigenvalues is equal to the trace of the matrix, so

$$r + m_1\mu_1 + m_2\mu_2 = 0. \tag{10}$$

Since $n = r^2 + 1$, we can rewrite equation (9) as

$$m_1 + m_2 = r^2. \tag{11}$$

We know that

$$\mu_{1,2} = \frac{-1 \pm s}{2}.$$

Without loss of generality, let

$$\mu_1 = \frac{-1+s}{2} \quad \text{and} \quad \mu_2 = \frac{-1-s}{2}.$$

Now we can rewrite equation (10) as

$$r + m_1\frac{-1+s}{2} + m_2\frac{-1-s}{2} = 0,$$

or

$$2r + s(m_1 - m_2) - (m_1 + m_2) = 0. \tag{12}$$

Consider the case where s is irrational. Then $m_1 - m_2 = 0$ (since m_1, m_2, and r are all integers), and we have

$$2r - (m_1 + m_2) = 0,$$

or

$$2r - r^2 = 0,$$

by equation (11). Since $r \neq 0$, this equation implies that $r = 2$. This number does, in fact, occur in the theorem as a possible value of r.

From now on, we will assume that s is rational, and thus s is an integer (since it is the square root of an integer).

Substituting equation (11) into equation (12), we have

$$2r + s(m_1 - m_2) - r^2 = 0.$$

Plugging in equation (8), we have

$$2\frac{s^2+3}{4} + s(m_1 - m_2) - \frac{(s^2+3)^2}{16} = 0.$$

Simplifying, we obtain

$$s^4 - 2s^2 - 16(m_1 - m_2)s - 15 = 0.$$

Thus, s must be a divisor of 15. That is, $s \in \{\pm 1, \pm 3, \pm 5, \pm 15\}$. Thus, by equation (7), we have $r \in \{1, 3, 7, 57\}$. Including the case where s is irrational, we add 2 to the possible values of r, which brings us to the conclusion that $r \in \{1, 2, 3, 7, 57\}$. □

References

[1] Babai, László. Lectures and *Discrete Math Lecture Notes*.

[2] Biggs, Norman. *Algebraic Graph Theory*. 2nd ed. Cambridge: Cambridge UP, 1993.