Unconventional Space-filling Curves

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A 2-dimensional space-filling curve, here, is a surjective continuous map from an interval to a 2-dimensional space.

To construct a new space-filling curve distinct and much different from Hilbert's, Peano's, and the Z-order, we first observe that all three of these curves are based on self-similar fractals–in particular, the recursive definition of the sequences of functions that converge to these curves systematically replaces smaller units of the whole with the entire map.

For instance, the space-filling curve given in Munkres, [1], which is there referred to as the "Peano space-filling curve" (271) but which follows more closely Hilbert's design [2], begins with $\mathcal{H}_0$, a parametrized curve from $t = 0$ to $t = 1$. The function increases linearly from $(0, 0)$ to $(\frac{1}{2}, \frac{1}{2})$ at $t = \frac{1}{2}$ and then back to $(1, 0)$ at $t = 0$. $\mathcal{H}_1$, which replaces $\mathcal{H}_0$, is a slightly more complicated function.

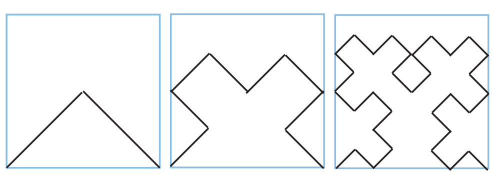

Figure 1: The Hilbert Curve, up to two iterations.

The important observation to make about it is that

first, it still begins at $(0, 0)$ and ends at $(1, 0)$;

[1] Munkres. Topology: A First Course

[2] "Hilbert Curve." Wikipedia, the Free Encyclopedia, acc. August 10, 2007

second, it can be divided into exactly *four* equal-sized segments (that correspond to the four squares obtained by dividing the interval in half vertically and horizontally) that are scaled versions of $\mathcal{H}_0$, which on higher values of m for $\mathcal{H}_m$ will be replaced by scaled versions of $\mathcal{H}_1$;

third, these miniatures of $\mathcal{H}_0$ *can only be replaced* in this way because the scaled copies of $\mathcal{H}_1$ begin and end in exactly the same places as $\mathcal{H}_0$.

Likewise, the interval covered by the Z-order curve is a square containing square numbers of squares. For a given $\mathcal{Z}_m$ the curve begins at $(0,1)$ and ends at $(1,0)$. Each $m+1$ element in the series of functions the converges to the Z-order curve is formed by subdividing further the given squares on which the m^{th} element was identical to scaled versions of $\mathcal{Z}_0$ and replacing each of these with scaled versions of $\mathcal{Z}_1$. The Peano curve, too, follows this pattern; each progression of the sequence of functions further divides the unit interval into smaller squares. Now we can clearly see a formula at work behind these classic space-filling curves:

1. Begin with a polygon that can be divided into scaled copies of itself.
2. Define an initial map from the interval to the polygon and a $\mathcal{F}_1$ which
   (a) starts and ends in the same places as the initial function
   (b) can be subdivided into a finite number of smaller functions such that
      i. the borders between them correspond exactly to their intersections with a further division of the chosen polygon,
      ii. each of them is a scaled, reflected, or rotated copy of $\mathcal{F}_0$
      iii. the union of the images of these smaller functions contains every point in the image of the larger $\mathcal{F}_1$–i.e. that there is no leftover area in $\mathcal{F}_1$ which cannot be replaced.
3. Define the sequence of functions by replacing each smaller instance of the initial function with $\mathcal{F}_1$

However, the condition (1) drastically limits the possibilities of new space-filling curves following this model. Pentagons, for example, do not satisfy this condition. Squares are the obvious choice, especially since the function should map to a square interval I. However, an equilateral triangle can be divided into four triangles by constructing segments between the midpoints of each side. We will construct a space-filling curve based on the Sierpinski triangle, a self-similar fractal obtained from this division.

Fix an arbitrary closed interval $[a, b] \subset \mathbb{R}$ and an arbitrary equilateral triangle T in $\mathbb{R}^2$ using $[a, b]$ as a side. Define the continuous map $\gamma : [a, b] \to T$ by the curve pictured in Figure 1. We will construct a sequence of functions based on an operation that replaces γ with γ', which is pictured in Figure 2. We can apply this operation which transforms all γ to γ' to each γ included within γ'. Some of them are flipped horizontally or rotated, but the only important point for our curve is *where the endpoints are*, since aligning these will guarantee continuity of the map. Figure 3 shows γ". It should now be obvious that simply by giving $a = 0$ and $b = 1$, we obtain a sequence of functions $\{\gamma_n\} : [0, 1] \to T$. Each part of a curve γ_n lies in an equilateral triangle with sides length $(1/2)^n$. The operation defining our sequence reshapes that part (from $t_1 = a$ to $t_2 = b$) to a new path corresponding to γ' which nonetheless lies within the same triangle T, with sides of length $(1/2)^n$. Thus, the distance between any $\gamma_n(t)$ and $\gamma_{n+1}(t)$ is at most $(1/2)^n$. $\{\gamma_n\}$, therefore, is a cauchy sequence of functions (if we choose the Euclidean metric) which converges to a continuous function $\alpha : [0, 1] \to T$.

Let x be a point in T. I claim that $x \in \alpha([0, 1])$. For all n, there exists $t \in [0, 1]$, $min(d(\gamma_n(t), n)) \leq (1/2)^n$. Fix $\epsilon > 0$. Choose N large enough that

$$sup\{d(\gamma_N(t), \alpha(t))|t \in [0, 1]\} < \frac{\epsilon}{2}$$

and

$$(1/2)^N < \frac{\epsilon}{2}.$$

We know that there exists t_0 in $[0, 1]$ such that

$$d(x, \gamma_n(t_0)) \leq (1/2)^N.$$

Therefore

$$d(x, \alpha(t_0)) < \epsilon,$$

. By the compactness of $[0, 1]$ under the Euclidean Metric, we know that $\{\alpha(t)|t \in [0, 1]\}$ is also compact. Thus, x is an element of α. $\square$

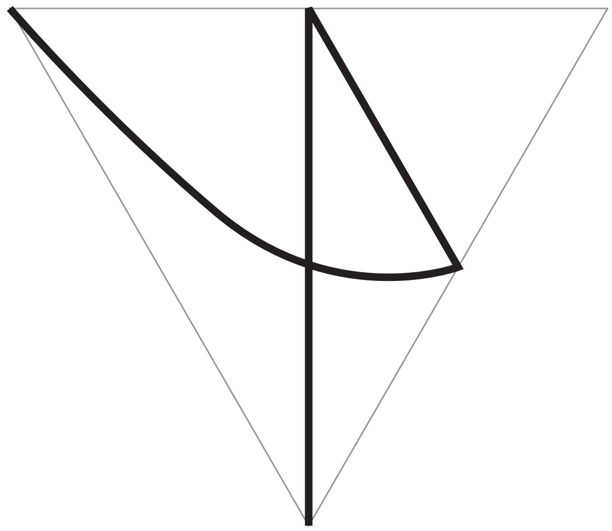

Figure 2: γ

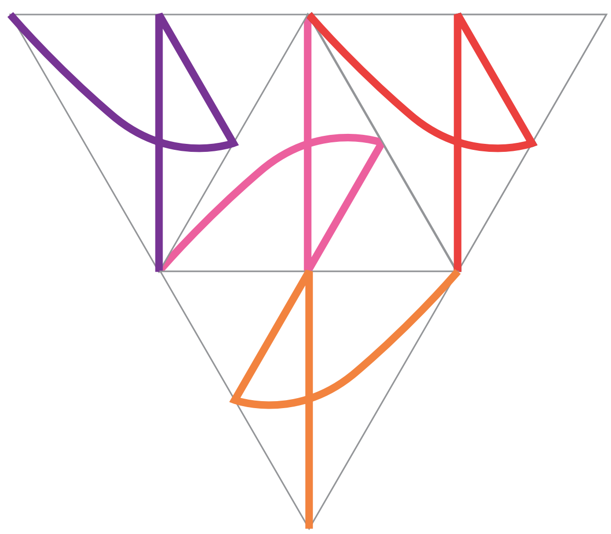

Figure 3: γ'

The interesting thing about this curve is that it can be used to build space-filling curves along many of the polygons resistant to self-similar fractalization. Furthermore, the combination of two of these, one flipped vertically (after multiplying t by a constant λ so that α_λ's domain remains [0,1]), can fill the annulus represented by a unit square with identified sides. Indeed, this curve may be able to fill many triangle-izable surfaces, but the investigation of which exactly it may fill (whether it depends on orient-ability, genus, or anything else) and *how*, must be the subject of another investigation.

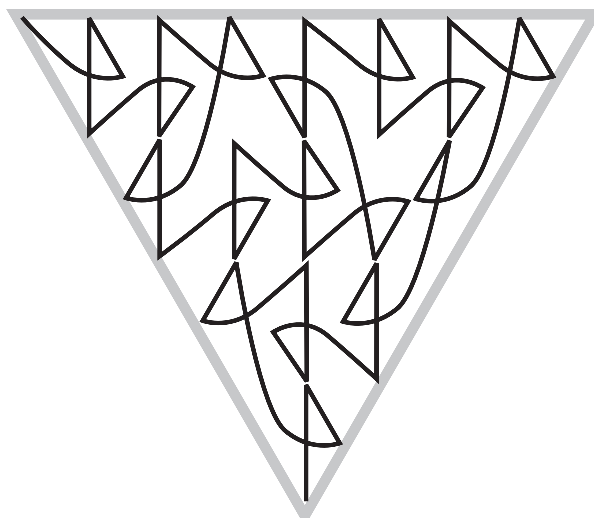

Figure 4: γ"