BASIC RESULTS CONCERNING BROWNIAN MOTION

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Abstract. We give an overview of standard one-dimensional Brownian motion on the dyadic rationals. We then offer a proof that such a process is uniformly continuous on closed intervals and, hence, that the original definition can be sensibly extended to all nonnegative reals. We conclude by investigating some immediate consequences of this new definition such as sample path nondifferentiability and zero set properties.

Contents

1. Introduction

Although it is now a formal mathematical object, the concept of Brownian motion is the product of natural world observation. Specifically, it was first noted by botanist Robert Brown in 1828 during his study of the movement of pollen particles through water. Later mathematical investigations of Brownian motion were conducted by Bachelier (1900) and Einstein (1905) who were attempting to quantify fluctuations in the stock market and heat flow, respectively; however, these derivations and descriptions were not rigorous. In fact, the first rigorous treatment of Brownian motion was not given until the 1920's work of Norbert Wiener. Today, Brownian motion is still actively explored, especially in topics at the intersection of mathematics with physics, biology, and economics, and work related to Brownian motion has even garnered a recent Fields Medal.

It is the purpose of this paper to introduce the reader to some basic *known* results concerning Brownian motion in one dimension. We will only assume familiarity with elementary probability principles, i.e. those not requiring measure theory, and in fact, with the exception of Theorem 5.10., the concept of measure is absent from this paper entirely. For this reason, the careful reader might be slightly wary of the results proven here. We note though that all arguments can be made rigorous with more advanced machinery. It should also be noted that all definitions, facts, and historical comments were synthesized from the references listed below, and

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consequently, no proofs will be offered for results stated in these contexts. On the other hand, all proofs that we do give were independently generated based upon conversations with Professor Lawler.

2. Brownian Motion on the Dyadic Rationals

It is now time to introduce one of the more simple classes of Brownian motion: Brownian motion on the dyadic rationals.

Remark 2.1. Strictly speaking, in the remainder of this section, we only work with $\mathcal{D}$, the nonnegative dyadic rational numbers. Nevertheless, we omit the term "nonnegative" henceforth.

Definition 2.2. A *standard Brownian motion on the dyadic rationals* is a collection of random variables $\{W_t : t \in \mathcal{D}\}$ satisfying the following conditions:

- For each $n \in \mathbb{N}$, the random variables
$$W_{\frac{k}{2^n}} - W_{\frac{(k-1)}{2^n}}, \quad k \in \mathbb{N},$$
are independent normal random variables with mean zero and variance 2^{-n}.
- $W_0 = 0$.

Straightforward though this characterization may seem, the fact that such a process exists is not obvious. In fact, the obstacle posed by the much needed demonstration of existence is the motivation behind restricting our attention, at first, to the dyadic rationals. However, upon making this restriction, existence can be easily demonstrated, and the results below immediately follow.

Fact 2.3. Let $\{W_t : t \in \mathcal{D}\}$ be a standard Brownian motion on the dyadic rationals. Then for all $s, t \in \mathcal{D}$ with $0 \leq s \leq t$, the random variable $W_t - W_s$ has a normal distribution with mean zero and variance $t - s$ and is independent of the collection of random variables $\{W_r : r \leq s\}$.

Fact 2.4. Let $\{W_t : t \in \mathcal{D}\}$ be a standard Brownian motion on the dyadic rationals. Then for any $j \in \mathbb{Z}$, the collection of random variables $\left\{\widetilde{W}_q : q \in \mathcal{D}\right\}$ defined by $\widetilde{W}_q = 2^{\frac{-j}{2}} W_{2^j q}$ for all $q \in \mathcal{D}$ is a standard Brownian motion on the dyadic rationals.

Fact 2.5. Let $\{W_t : t \in \mathcal{D}\}$ be a standard Brownian motion on the dyadic rationals. Then for every $a > 0$,
$$\mathbb{P}\left\{\sup\left\{|W_q| : q \in \mathcal{D} \cap [0,1]\right\} \geq a\right\} \leq 4\mathbb{P}\left\{W_1 \geq a\right\},$$
and in particular,
$$\mathbb{P}\left\{\sup\left\{|W_q| : q \in \mathcal{D} \cap [0,1]\right\} \geq a\right\} \leq \frac{8}{a} \exp\left(\frac{-a^2}{2}\right).$$

The reader should note that analogous and even stronger results hold for standard Brownian motion (Definition 3.3.). In fact, these analogs will be of critical importance in later sections.

We now proceed to our first set of results accompanied by proofs. The goal of this segment is Theorem 2.12., which states that with probability 1, the sample paths of a standard Brownian motion on the dyadic rationals are uniformly continuous on $\mathcal{D} \cap [0,1]$. Four lemmas will precede the statement and proof of this theorem for the sake of clarity.

Notation 2.6. For the remainder of this section, the collection of random variables $\{W_q : q \in \mathcal{D}\}$ is taken to be a standard Brownian motion on the dyadic rationals.

Notation 2.7. Although we abstain from true measure theory in our approach, we still employ the notation $\mathbb{P}\{\omega \in \Omega : X(\omega) \ldots \text{ holds}\}$ wherever it is not too cumbersome. Here, Ω is taken to be the "sample space" and ω an "element" of that sample space. It is hoped that such usage reinforces the notion of a Brownian sample path, i.e. for some fixed $\omega \in \Omega$, the set $\{(t, W_t(\omega)) : t \in \mathcal{D}\,(t \geq 0 \text{ later})\}$.

Lemma 2.8. *For all $\omega \in \Omega$ and $n \in \mathbb{N}$, let*

$$Z_n(\omega) = \sup\left\{|W_s(\omega) - W_t(\omega)| \ : |s-t| \leq 2^{-n};\, s,t \in \mathcal{D} \cap [0,1]\right\}.$$

Then the mapping $q \mapsto W_q(\omega)$ is uniformly continous on $\mathcal{D} \cap [0,1]$ iff $\lim_{n\to\infty} Z_n(\omega) = 0$.

Proof. "$\Rightarrow$ direction." Let $\omega \in \Omega$. Suppose that the mapping $q \mapsto W_q(\omega)$ is uniformly continuous on $\mathcal{D} \cap [0,1]$. Let $\epsilon > 0$. Then, by definition, there exists $\delta > 0$ such that for all $s, t \in \mathcal{D} \cap [0,1]$,

$$|s-t| < \delta \Rightarrow |W_s(\omega) - W_t(\omega)| < \frac{\epsilon}{2}.$$

Now, let $N \in \mathbb{N}$ be such that for all $n \geq \text{N}$, $2^{-n} < \delta$. It follows that for all $n \geq N$ and $s, t \in \mathcal{D} \cap [0,1]$,

$$|s-t| \leq 2^{-n} \Rightarrow |W_s(\omega) - W_t(\omega)| < \frac{\epsilon}{2}.$$

Therefore, for all $n \geq N$,

$$\sup\left\{|W_s(\omega) - W_t(\omega)| : |s-t| \leq 2^{-n};\, s,t \in \mathcal{D} \cap [0,1]\right\} < \epsilon,$$

i.e. for all $n \geq N$, $|Z_n(\omega)| < \epsilon$, as required.

"$\Leftarrow$ direction." Let $\omega \in \Omega$. Suppose that $\lim_{n\to\infty} Z_n(\omega) = 0$. Let $\epsilon > 0$. By definition, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|Z_n(\omega)| < \epsilon$. Let $0 < \delta < 2^{-N}$. It follows that

$$\sup\{|W_s(\omega) - W_t(\omega)| : |s-t| \leq \delta;\ s,t \in \mathcal{D} \cap [0,1]\} < \epsilon.$$

Therefore, we also know that for all $s, t \in \mathcal{D} \cap [0,1]$,

$$|s-t| < \delta \Rightarrow |W_s(\omega) - W_t(\omega)| < \epsilon,$$

i.e. the mapping $q \mapsto W_q(\omega)$ is uniformly continuous on $\mathcal{D} \cap [0,1]$, as required. $\square$

Lemma 2.9. *For all $\omega \in \Omega$, $n \in \mathbb{N}$, and $1 \leq k \leq 2^n$, let*

$$M(k,n,\omega) = \sup\left\{\left|W_q(\omega) - W_{\frac{k-1}{2^n}}(\omega)\right| : q \in \mathcal{D} \cap \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]\right\},$$

$$M_n(\omega) = \max\{M(1,n,\omega), M(2,n,\omega), \ldots, M(2^n,n,\omega)\}.$$

Then the mapping $q \mapsto W_q(\omega)$ is uniformly continuous on $\mathcal{D} \cap [0,1]$ iff $\lim_{n\to\infty} M_n(\omega) = 0$.

Proof. "$\Rightarrow$ direction." Let $\omega \in \Omega$. Suppose that the mapping $q \mapsto W_q(\omega)$ is uniformly continuous on $\mathcal{D} \cap [0,1]$. By Lemma 2.8., it suffices to show that for all $n \in \mathbb{N}$, $M_n(\omega) \leq Z_n(\omega)$. Let $n \in \mathbb{N}$. Then by definition of $M_n(\omega)$ there exists $1 \leq k_0 \leq 2^n$ such that $M_n(\omega) = M(k_0, n, \omega)$. Using properties of supremum, it follows that

$$\begin{aligned} M_n(\omega) &= \sup\left\{ \left| W_q(\omega) - W_{\frac{k_0-1}{2^n}}(\omega) \right| : q \in \mathcal{D} \cap \left[\frac{k_0-1}{2^n}, \frac{k_0}{2^n} \right] \right\} \\ &\leq \sup\left\{ \left| W_q(\omega) - W_{\frac{k_0-1}{2^n}}(\omega) \right| : \left| q - \frac{k_0-1}{2^n} \right| \leq 2^{-n}; q \in \mathcal{D} \cap [0,1] \right\} \\ &\leq \sup\left\{ |W_s(\omega) - W_t(\omega)| : |s-t| \leq 2^{-n};\ s,t \in \mathcal{D} \cap [0,1] \right\} \\ &= Z_n(\omega), \end{aligned}$$

as required.

"$\Leftarrow$ direction." Let $\omega \in \Omega$. Suppose that $\lim_{n\to\infty} M_n(\omega) = 0$. By Lemma 2.8., it suffices to show that for all $n \in \mathbb{N}$, $Z_n(\omega) \leq 3M_n(\omega)$. Let $n \in \mathbb{N}$. Construct the following sets:

$$\mathcal{A}_n = \left\{ (s,t) : |s-t| \leq 2^{-n};\ s,t \in \mathcal{D} \cap [0,1]\,; \exists\, k \in \{1, 2, \ldots, 2^n\} \text{ such that } t \leq \frac{k}{2^n} \leq s \right\},$$

$$\mathcal{B}_n = \left\{ (s,t) : s,t \in \mathcal{D} \cap [0,1]\,; \exists\, k \in \{1, 2, \ldots, 2^n\} \text{ such that } \frac{k-1}{2^n} < t \leq s < \frac{k}{2^n} \right\}.$$

Note that $\mathcal{A}_n$ and $\mathcal{B}_n$ are disjoint and

$$\mathcal{A}_n \cup \mathcal{B}_n = \left\{ (s,t) : |s-t| \leq 2^{-n};\ s,t \in \mathcal{D} \cap [0,1]\,;\ t \leq s \right\}.$$

Therefore, by properties of supremum,

$$\begin{aligned} &\sup\left\{ |W_s(\omega) - W_t(\omega)| : |s-t| \leq 2^{-n};\ s,t \in \mathcal{D} \cap [0,1] \right\} \\ &\quad = \max\left\{ (\sup\left\{ |W_s(\omega) - W_t(\omega)| : (s,t) \in \mathcal{A}_n \right\}), (\sup\left\{ |W_s(\omega) - W_t(\omega)| : (s,t) \in \mathcal{B}_n \right\}) \right\}. \end{aligned}$$

Now, for a given $1 \leq k \leq 2^n - 1$,

$$\begin{aligned} &\sup\left\{ |W_s(\omega) - W_t(\omega)| : |s-t| \leq 2^{-n};\ t \leq \frac{k}{2^n} \leq s;\ s,t \in \mathcal{D} \cap [0,1] \right\} \\ &\quad \leq \sup\Bigg\{ \left| W_s(\omega) - W_{\frac{k}{2^n}}(\omega) \right| + \left| W_{\frac{k}{2^n}}(\omega) - W_{\frac{k-1}{2^n}}(\omega) \right| + \left| W_t(\omega) - W_{\frac{k-1}{2^n}}(\omega) \right| : \\ &\qquad\qquad |s-t| \leq 2^{-n};\ t \leq \frac{k}{2^n} \leq s;\ s,t \in \mathcal{D} \cap [0,1] \Bigg\} \\ &\quad \leq M(k+1, n, \omega) + M(k, n, \omega) + M(k, n, \omega) \\ &\quad \leq 3M_n(\omega). \end{aligned}$$

Also, if $k = 2^n$, then

$$\begin{aligned} &\sup\left\{ |W_s(\omega) - W_t(\omega)| : |s-t| \leq 2^{-n};\ t \leq \frac{k}{2^n} \leq s;\ s,t \in \mathcal{D} \cap [0,1] \right\} \\ &\quad = \sup\left\{ |W_1(\omega) - W_t(\omega)| : |1-t| \leq 2^{-n};\ t \in \mathcal{D} \cap [0,1] \right\} \\ &\quad \leq \sup\left\{ \left| W_1(\omega) - W_{\frac{2^n-1}{2^n}}(\omega) \right| + \left| W_t(\omega) - W_{\frac{2^n-1}{2^n}}(\omega) \right| : |1-t| \leq 2^{-n};\ t \in \mathcal{D} \cap [0,1] \right\} \\ &\quad \leq M(k, n, \omega) + M(k, n, \omega) \\ &\quad \leq 2M_n(\omega). \end{aligned}$$

It follows that

$$\sup\{|W_s(\omega)-W_t(\omega)| : (s,t)\in\mathcal{A}_n\} \le 3M_n(\omega).$$

In addition, for $1 \le k \le 2^n$, let

$$\mathcal{B}_{k,n} = \left\{(s,t)\in\mathcal{B}_n : \frac{k-1}{2^n} < t \le s < \frac{k}{2^n}\right\}.$$

Note that $\mathcal{B}_{1,n},\ \mathcal{B}_{2,n},\ \ldots,\mathcal{B}_{2^n,n}$ are disjoint and $\bigcup_{k=1}^{2^n}\mathcal{B}_{k,n} = \mathcal{B}_n$. Since

$$\begin{aligned}
&\sup\{|W_s(\omega)-W_t(\omega)| : (s,t)\in\mathcal{B}_n\}\\
&\quad\le \max\left\{\left(\sup\left\{\left|W_s(\omega)-W_{\frac{1-1}{2^n}}(\omega)\right| + \left|W_t(\omega)-W_{\frac{1-1}{2^n}}(\omega)\right| : (s,t)\in\mathcal{B}_{1,n}\right\}\right),\right.\\
&\qquad\qquad\left(\sup\left\{\left|W_s(\omega)-W_{\frac{2-1}{2^n}}(\omega)\right| + \left|W_t(\omega)-W_{\frac{2-1}{2^n}}(\omega)\right| : (s,t)\in\mathcal{B}_{2,n}\right\}\right),\ldots\\
&\qquad\qquad\left(\sup\left\{\left|W_s(\omega)-W_{\frac{2^n-1}{2^n}}(\omega)\right| + \left|W_t(\omega)-W_{\frac{2^n-1}{2^n}}(\omega)\right| : (s,t)\in\mathcal{B}_{2^n,n}\right\}\right)\\
&\quad\le \max\{2M(1,n,\omega),2M(2,n,\omega),\ldots,2M(2^n,n,\omega)\}\\
&\quad= 2M_n(\omega),
\end{aligned}$$

we have, by our previous work, that $Z_n(\omega)\le 3M_n(\omega)$, as required. □

Lemma 2.10. *For all $\epsilon>0$ and $n\in\mathbb{N}$,*

I. $\mathbb{P}\{\omega\in\Omega : M_n(\omega)\ge\epsilon\} \le 2^n\,\mathbb{P}\{\omega\in\Omega : M(1,n,\omega)\ge\epsilon\}$

II. $\mathbb{P}\{\omega\in\Omega : M(1,n,\omega)\ge\epsilon\} = \mathbb{P}\left\{\omega\in\Omega : M(1,0,\omega)\ge 2^{\frac{n}{2}}\epsilon\right\}$.

Proof. "I." Let $\epsilon>0$ and $n\in\mathbb{N}$. Then

$$\begin{aligned}
&\mathbb{P}\{\omega\in\Omega : M_n(\omega)\ge\epsilon\}\\
&\quad=\mathbb{P}\{\omega\in\Omega : \max\{M(1,n,\omega),M(2,n,\omega),\ldots,M(2^n,n,\omega)\}\ge\epsilon\}\\
&\quad\le\sum_{k=1}^{2^n}\mathbb{P}\{\omega\in\Omega : M(k,n,\omega)\ge\epsilon\}\\
&\quad=\sum_{k=1}^{2^n}\mathbb{P}\{\omega\in\Omega : M(1,n,\omega)\ge\epsilon\}\\
&\quad=2^n\,\mathbb{P}\{\omega\in\Omega : M(1,n,\omega)\ge\epsilon\},
\end{aligned}$$

as required.

"II." Let $\epsilon>0$ and $n\in\mathbb{N}$. By properties of Brownian motion,

$$\begin{aligned}
&\mathbb{P}\left\{\omega\in\Omega : M(1,0,\omega)\ge 2^{\frac{n}{2}}\epsilon\right\}\\
&\quad=\mathbb{P}\left\{\omega\in\Omega : \frac{M(1,0,\omega)}{2^{\frac{n}{2}}}\ge\epsilon\right\}\\
&\quad=\mathbb{P}\left\{\omega\in\Omega : \sup\left\{\frac{|W_q(\omega)-W_0(\omega)|}{2^{\frac{n}{2}}} : q\in\mathcal{D}\cap[0,1]\right\}\ge\epsilon\right\}\\
&\quad=\mathbb{P}\left\{\omega\in\Omega : \sup\left\{\left|W_{2^{-n}q}(\omega)-W_0(\omega)\right| : q\in\mathcal{D}\cap[0,1]\right\}\ge\epsilon\right\}\\
&\quad=\mathbb{P}\left\{\omega\in\Omega : \sup\left\{|W_t(\omega)-W_0(\omega)| : t\in\mathcal{D}\cap\left[0,\frac{1}{2^n}\right]\right\}\ge\epsilon\right\}\\
&\quad=\mathbb{P}\{\omega\in\Omega : M(1,n,\omega)\ge\epsilon\},
\end{aligned}$$

as required. $\square$

Lemma 2.11. *Define a sequence* $(\epsilon_m)_{m\in\mathbb{N}}$ *by*

$$\epsilon_m = 2^{\frac{-m}{4}}.$$

Then

I. $\displaystyle\sum_{m=1}^{\infty} 2^m \mathbb{P}\left\{\omega \in \Omega \,:\, M(1,m,\omega) \geq \epsilon_m\right\} < \infty$

II. $\displaystyle\lim_{n\to\infty} \mathbb{P}\left\{\omega \in \Omega : M_m(\omega) \geq \epsilon_m \textit{ for some } m \geq n\right\} = 0.$

Proof. "I." By Lemma 2.10., for any strictly positive sequence $(\delta_m)_{m\in\mathbb{N}}$,

$$\begin{aligned}
&\sum_{m=1}^{\infty} 2^m\, \mathbb{P}\left\{\omega \in \Omega \,:\, M(1,m,\omega) \geq \delta_m\right\}\\
&\quad= \sum_{m=1}^{\infty} 2^m\, \mathbb{P}\left\{\omega \in \Omega \,:\, M(1,0,\omega) \geq 2^{\frac{m}{2}}\delta_m\right\}\\
&\quad\leq \sum_{m=1}^{\infty} 2^m\, \frac{8}{2^{\frac{m}{2}}\delta_m} \exp\left(\frac{-(2^{\frac{m}{2}}\delta_m)^2}{2}\right)\\
&\quad= 8\sum_{m=1}^{\infty} 2^{\frac{m}{2}}\, \frac{1}{\delta_m} \exp\left(\frac{-2^m\delta_m^2}{2}\right).
\end{aligned}$$

Therefore, it suffices to show that

$$\sum_{m=1}^{\infty} 2^{\frac{m}{2}}\, \frac{1}{\epsilon_m} \exp\left(\frac{-2^m\epsilon_m^2}{2}\right) < \infty.$$

Now, for all $m \geq 10$,

$$\frac{7m}{2}\log(2) \leq 2^{\frac{m}{2}}.$$

Hence, for all $m \geq 10$,

$$-2^m 2^{\frac{-m}{2}} = -2^{\frac{m}{2}} \leq -\frac{7m}{2}\log(2) = -\frac{m}{2}\log(2) - 3m\log(2) = \log(2^{\frac{-m}{2}}) - 3m\log(2).$$

Therefore, for all $m \geq 10$,

$$-2^m\epsilon_m^2 \leq \log(\epsilon_m^2) + \log(2^{-3m}) = 2\log\left(\epsilon_m 2^{\frac{-3m}{2}}\right)$$

and

$$\exp\left(\frac{-2^m\epsilon_m^2}{2}\right) \leq \epsilon_m 2^{\frac{-3m}{2}}.$$

Consequently, for all $m \geq 10$,

$$\exp\left(\frac{-2^m\epsilon_m^2}{2}\right)\frac{2^{\frac{m}{2}}}{\epsilon_m} \leq 2^{-m},$$

and so the series above converges, as required.

"II." Let $\epsilon > 0$. By 2.11.I., we can find an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\sum_{m=n}^{\infty} 2^m\, \mathbb{P}\left\{\omega \in \Omega \,:\, M(1,m,\omega) \geq \epsilon_m\right\} < \epsilon.$$

But for all $n \geq \mathrm{N}$,

$$\begin{aligned} &\mathbb{P}\left\{\omega \in \Omega \,:\, M_m(\omega) \geq \epsilon_m \text{ for some } m \geq n\right\} \\ &\quad \leq \sum_{m=n}^{\infty} \mathbb{P}\left\{\omega \in \Omega :\, M_m(\omega) \geq \epsilon_m\right\} \\ &\quad \leq \sum_{m=n}^{\infty} 2^m\, \mathbb{P}\left\{\omega \in \Omega \,:\, M(1,m,\omega) \geq \epsilon_m\right\} < \epsilon, \end{aligned}$$

i.e. $\lim_{n\to\infty} \mathbb{P}\left\{\omega \in \Omega : M_m(\omega) \geq \epsilon_m \text{ for some } m \geq n\right\} = 0$, as required. □

Theorem 2.12.

$$\mathbb{P}\left\{\omega \in \Omega : q \mapsto W_q(\omega) \textit{ is uniformly continuous on } \mathcal{D} \cap [0,1]\right\} = 1.$$

Proof. By Lemma 2.9., it suffices to show that

$$\mathbb{P}\left\{\omega \in \Omega \,:\, \lim_{n\to\infty} M_n(\omega) = 0\right\} = 1.$$

But Lemma 2.11.II. states that

$$\begin{aligned} &\lim_{n\to\infty} \left(1 - \mathbb{P}\left\{\omega \in \Omega :\, M_m(\omega) < \epsilon_m \text{ for all } m \geq n\right\}\right) \\ &\quad = \lim_{n\to\infty} \mathbb{P}\left\{\omega \in \Omega :\, M_m(\omega) \geq \epsilon_m \text{ for some } m \geq n\right\} \\ &\quad = 0, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \mathbb{P}\left\{\omega \in \Omega \,:\, M_m(\omega) < \epsilon_m \text{ for all } m \geq n\right\} = 1.$$

Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$1 - \epsilon \,<\, \mathbb{P}\left\{\omega \in \Omega \,:\, M_m(\omega) < \epsilon_m \text{ for all } m \geq n\right\} \,\leq\, 1.$$

But by the Squeeze Theorem, for all $n \in \mathbb{N}$,

$$\mathbb{P}\left\{\omega \in \Omega \,:\, M_m(\omega) < \epsilon_m \text{ for all } m \geq n\right\} \,\leq\, \mathbb{P}\left\{\omega \in \Omega : \lim_{m\to\infty} M_m(\omega) = 0\right\},$$

so

$$1 - \epsilon \,<\, \mathbb{P}\left\{\omega \in \Omega \,:\, \lim_{m\to\infty} M_m(\omega) = 0\right\} \,\leq\, 1.$$

Since $\epsilon > 0$ was arbitrary, it follows that

$$\mathbb{P}\left\{\omega \in \Omega : \lim_{m\to\infty} M_m(\omega) = 0\right\} = 1,$$

as required. □

3. Standard Brownian Motion

The previous section restricted its attention to $\mathcal{D} \cap [0,1]$. However, the following fact can be demonstrated in a similar manner.

Fact 3.1. Let $\{W_t \,:\, t \in \mathcal{D}\}$ be a standard Brownian motion on the dyadic rationals. Then Theorem 2.12. can be generalized so that for all $a, b \in \mathbb{R}$ with $b \geq a \geq 0$,

$$\mathbb{P}\left\{\omega \in \Omega : q \mapsto W_q(\omega) \text{ is uniformly continuous on } \mathcal{D} \cap [a,b]\right\} = 1.$$

This observation provides the key step in extending the definition of Brownian motion to all nonnegative reals. The technique used for this extension employs Cauchy sequences and is the standard one from analysis. After this extension has been conducted, the next fact can be immediately proven.

Fact 3.2. Let $\{W_t : t \in \mathcal{D}\}$ be a standard Brownian motion on the dyadic rationals. Then for each $\omega \in \Omega$, the mapping $t \mapsto W_t(\omega)$ can be extended to all nonnegative reals so that the collection $\{W_t : t \geq 0\}$ satisfies the following conditions:

- For all $0 \leq s \leq t$, the random variable $W_t - W_s$ has a normal distribution with mean zero and variance $t - s$ and is independent of the collection of random variables $\{W_r : r \leq s\}$.
- $\mathbb{P}\{\omega \in \Omega : t \mapsto W_t(\omega) \text{ is continuous for all } t \geq 0\} = 1$.
- $W_0 = 0$.

Given the relation to Definition 2.2., a collection of random variables possessing the above properties warrants the expected definition.

Definition 3.3. Let $\{W_t : t \geq 0\}$ be a collection of random variables satisfying the properties given in Fact 3.2. We then say that $\{W_t : t \geq 0\}$ is a *standard Brownian motion.*

In our present study, it will be convenient to have a broader understanding of Brownian motion than that afforded by the above concept alone. The next definition fills this gap, and the fact afterward suggests that the "required" properties are preserved.

Definition 3.4. Let $x \in \mathbb{R}$. A *Brownian motion starting at* x is a collection of random variables $\left\{\widehat{W}_t : t \geq 0\right\}$ such that for all $t \geq 0$, $\widehat{W}_t := W_t + x$ where the collection $\{W_t : t \geq 0\}$ is a standard Brownian motion.

Fact 3.5. Let $x \in \mathbb{R}$ and $\left\{\widehat{W}_t : t \geq 0\right\}$ a Brownian motion starting at x. Then the collection of random variables $\left\{\widehat{W}_t : t \geq 0\right\}$ satisfies the following conditions:

- For all $0 \leq s \leq t$, the random variable $W_t - W_s$ has a normal distribution with mean zero and variance $t - s$ and is independent of the collection of random variables $\{W_r : r \leq s\}$.
- $\mathbb{P}\left\{\omega \in \Omega : t \mapsto \widehat{W}_t(\omega) \text{ is continuous for all } t \geq 0\right\} = 1$.
- $W_0 = x$.

4. Nondifferentiability of Standard Brownian Motion

The goal of this section is Theorem 4.6., which states that with probablity 1, the mapping $t \mapsto W_t(\omega)$ is nowhere differentiable on $(0, 1)$. We note that this interval was chosen for convenience alone. In fact, the same statement can be demonstrated for any open interval, and, hence, with probability 1, the mapping $t \mapsto W_t(\omega)$ is nowhere differentiable for $t > 0$. We also remark that we once again precede the statement and proof of the main theorem with four lemmas for the sake of clarity.

Notation 4.1. For the remainder of this document, the collection of random variables $\{W_t : t \geq 0\}$ is taken to be a standard Brownian motion.

Lemma 4.2. *For all $\omega \in \Omega$, if the mapping $t \mapsto W_t(\omega)$ is differentiable for some $t_0 \in (0,1)$, then*

I. *There exists $\delta > 0$ and $C < \infty$ such that*
$$r, s \in \left[t_0 - \frac{\delta}{2}, t_0 + \frac{\delta}{2}\right] \Rightarrow |W_r(\omega) - W_s(\omega)| \le C\delta$$

II. *For all $\eta \in (0, \delta)$,*
$$r, s \in \left[t_0 - \frac{\eta}{2}, t_0 + \frac{\eta}{2}\right] \Rightarrow |W_r(\omega) - W_s(\omega)| \le C\eta.$$

Proof. "I." Let $\omega \in \Omega$. Suppose that the mapping $t \mapsto W_t(\omega)$ is differentiable for some $t_0 \in (0,1)$. Define the function $\varphi_\omega : [0,1] \to \mathbb{R}$ by $\varphi_\omega(t) = W_t(\omega)$. By definition, given any $\epsilon > 0$, there exists $\delta > 0$ such that for all $h \in \mathbb{R}$,
$$0 < |h| < \delta \Rightarrow \left|\frac{\varphi_\omega(t_0 + h) - \varphi_\omega(t_0)}{h} - \varphi'_\omega(t_0)\right| < \frac{\epsilon}{2}.$$
Without loss of generality, we can further specify that $\delta > 0$ be such that $[t_0 - \delta, t_0 + \delta] \subseteq [0,1]$. It follows that for all $h \in \mathbb{R}$,
$$0 < |h| < \delta \Rightarrow |\varphi_\omega(t_0 + h) - \varphi_\omega(t_0) - h\varphi'_\omega(t_0)| < \frac{\delta\epsilon}{2},$$
which implies that for all $h \in \mathbb{R}$,
$$\begin{aligned} &0 < |h| < \delta \Rightarrow |\varphi_\omega(t_0 + h) - \varphi_\omega(t_0)| < \frac{\delta\epsilon}{2} + |h\varphi'_\omega(t_0)| < \frac{\delta\epsilon}{2} + \delta\,|\varphi'_\omega(t_0)| \\ &\quad < \delta\epsilon + 2\delta\,|\varphi'_\omega(t_0)| = \delta\,(\epsilon + 2\,|\varphi'_\omega(t_0)|\,. \end{aligned}$$
Moreover, for all $h, h' \in \mathbb{R}$, $0 < \max\{|h|, |h'|\} < \delta$ implies that
$$\left|\frac{\varphi_\omega(t_0 + h) - \varphi_\omega(t_0)}{h} - \varphi'_\omega(t_0)\right| + \left|\frac{\varphi_\omega(t_0 + h') - \varphi_\omega(t_0)}{h'} - \varphi'_\omega(t_0)\right| < \epsilon,$$
which means that
$$|\varphi_\omega(t_0 + h) - \varphi_\omega(t_0) - h\varphi'_\omega(t_0)| + |\varphi_\omega(t_0 + h') - \varphi_\omega(t_0) - h'\varphi'_\omega(t_0)| < \delta\epsilon.$$
Therefore, for all $h, h' \in \mathbb{R}$, $0 < \max\{|h|, |h'|\} < \delta$ implies that
$$|\varphi_\omega(t_0 + h) - \varphi_\omega(t_0 + h') - (h - h')\varphi'_\omega(t_0)| < \delta\epsilon$$
and so for these $h \in \mathbb{R}$,
$$\begin{aligned} &|\varphi_\omega(t_0 + h) - \varphi_\omega(t_0 + h')| < \delta\epsilon + |(h - h')\varphi'_\omega(t_0)| \\ &\quad < \delta\epsilon + 2\delta\,|\varphi'_\omega(t_0)| = \delta\,(\epsilon + 2\,|\varphi'_\omega(t_0)|)\,. \end{aligned}$$
Therefore, setting $\epsilon = 1$ and $C = (1 + 2\,|\varphi'_\omega(t_0)|)$, it follows that
$$r, s \in \left[t_0 - \frac{\delta}{2}, t_0 + \frac{\delta}{2}\right] \Rightarrow |W_r(\omega) - W_s(\omega)| \le C\delta,$$
as required.

"II." Since the above argument can also be applied toward any $\eta \in (0, \delta)$, the proof is complete. $\square$

Lemma 4.3. *For all $\omega \in \Omega$, $n \geq 3$, $k \in \{1, 2, \ldots, n-2\}$, let*

$$M(k,n,\omega) = \max\left\{\left|W_{\frac{k}{n}}(\omega) - W_{\frac{k-1}{n}}(\omega)\right|\,,\,\left|W_{\frac{k+1}{n}}(\omega) - W_{\frac{k}{n}}(\omega)\right|\,,\,\left|W_{\frac{k+2}{n}}(\omega) - W_{\frac{k+1}{n}}(\omega)\right|\right\},$$

$$M_n(\omega) = \min\left\{M(1,n,\omega), M(2,n,\omega), \ldots, M(n-2,n,\omega)\right\}.$$

Then, if the mapping $t \mapsto W_t(\omega)$ is differentiable for some $t_0 \in (0,1)$, it follows that there exists $C, N < \infty$ such that for all $n \geq N$,

$$M_n(\omega) \leq \frac{C}{n}.$$

Proof. Let $\omega \in \Omega$. Suppose that the mapping $t \mapsto W_t(\omega)$ is differentiable for some $t_0 \in (0,1)$. Define the function $\varphi_\omega : [0,1] \to \mathbb{R}$ by $\varphi_\omega(t) = W_t(\omega)$. By Lemma 4.2., there exists $\delta > 0$ and $C < \infty$ such that

$$r, s \in \left[t_0 - \frac{\delta}{2}, t_0 + \frac{\delta}{2}\right] \subseteq (0,1) \Rightarrow |W_r(\omega) - W_s(\omega)| \leq C\delta$$

and for all $\eta \in (0, \delta)$,

$$r, s \in \left[t_0 - \frac{\eta}{2}, t_0 + \frac{\eta}{2}\right] \Rightarrow |W_r(\omega) - W_s(\omega)| \leq C\eta.$$

Let $N \in \mathbb{N}$ be such that for all $n \geq N$, $\frac{1}{n} \leq \frac{\delta}{8}$. It follows that for all $n \geq N$, there exists $k' \in \{1, 2, \ldots, n-2\}$ such that

$$\left[\frac{k'-1}{n}, \frac{k'}{n}\right], \left[\frac{k'}{n}, \frac{k'+1}{n}\right], \left[\frac{k'+1}{n}, \frac{k'+2}{n}\right] \subseteq \left[t_0 - \frac{\delta}{2}, t_0 + \frac{\delta}{2}\right]$$

and

$$t_0 \in \text{at least one of } \left[\frac{k'-1}{n}, \frac{k'}{n}\right], \left[\frac{k'}{n}, \frac{k'+1}{n}\right], \text{ or } \left[\frac{k'+1}{n}, \frac{k'+2}{n}\right].$$

By the above statement, this implies that for all $n \geq N$,

$$\max\left\{\left|W_{\frac{k'}{n}}(\omega) - W_{\frac{k'-1}{n}}(\omega)\right|\,,\,\left|W_{\frac{k'+1}{n}}(\omega) - W_{\frac{k'}{n}}(\omega)\right|\,,\,\left|W_{\frac{k'+2}{n}}(\omega) - W_{\frac{k'+1}{n}}(\omega)\right|\right\} \leq C\frac{6}{n},$$

so $M_n(\omega) \leq C\,\frac{6}{n}$ for all $n \geq N$, as required. □

Lemma 4.4. *There exists $\alpha \in \mathbb{R}$ such that for all $C \in \mathbb{R}, n \in \mathbb{N}$, and $k \in \{1, 2, \ldots, n-2\}$,*

I. $\mathbb{P}\left\{\omega \in \Omega : \ M(k,n,\omega) \leq \frac{C}{n}\right\} = \left[\mathbb{P}\left\{\omega \in \Omega : \ \left|W_{\frac{1}{n}}(\omega)\right| \leq \frac{C}{n}\right\}\right]^3$

II. $\mathbb{P}\left\{\omega \in \Omega : \ \left|W_{\frac{1}{n}}(\omega)\right| \leq \frac{C}{n}\right\} \leq \frac{C\alpha}{\sqrt{n}}$.

Proof. "I." Let $C \in \mathbb{R}$, $n \in \mathbb{N}$, and $k \in \{1, 2, \dots, n-2\}$. Then

$$\begin{aligned}
&\mathbb{P}\left\{\omega \in \Omega : M(k,n,\omega) \le \frac{C}{n}\right\} \\
&= \mathbb{P}\left\{\omega \in \Omega : \max\left\{\left|W_{\frac{k}{n}}(\omega) - W_{\frac{k-1}{n}}(\omega)\right|, \left|W_{\frac{k+1}{n}}(\omega) - W_{\frac{k}{n}}(\omega)\right|, \left|W_{\frac{k+2}{n}}(\omega) - W_{\frac{k+1}{n}}(\omega)\right|\right\} \le \frac{C}{n}\right\} \\
&= \mathbb{P}\left\{\omega \in \Omega : \left|W_{\frac{k}{n}}(\omega) - W_{\frac{k-1}{n}}(\omega)\right| \le \frac{C}{n}, \left|W_{\frac{k+1}{n}}(\omega) - W_{\frac{k}{n}}(\omega)\right| \le \frac{C}{n}, \left|W_{\frac{k+2}{n}}(\omega) - W_{\frac{k+1}{n}}(\omega)\right| \le \frac{C}{n}\right\} \\
&= \mathbb{P}\left\{\omega \in \Omega : \left|W_{\frac{k}{n}}(\omega) - W_{\frac{k-1}{n}}(\omega)\right| \le \frac{C}{n}\right\} \mathbb{P}\left\{\omega \in \Omega : \left|W_{\frac{k+1}{n}}(\omega) - W_{\frac{k}{n}}(\omega)\right| \le \frac{C}{n}\right\} \\
&\qquad \cdot \mathbb{P}\left\{\omega \in \Omega : \left|W_{\frac{k+2}{n}}(\omega) - W_{\frac{k+1}{n}}(\omega)\right| \le \frac{C}{n}\right\} \\
&= \left[\mathbb{P}\left\{\omega \in \Omega : \left|W_{\frac{1}{n}}(\omega)\right| \le \frac{C}{n}\right\}\right]^3,
\end{aligned}$$

as required.

"II." Let $C \in \mathbb{R}$ and $n \in \mathbb{N}$. By definition of Brownian motion,

$$\begin{aligned}
&\mathbb{P}\left\{\omega \in \Omega : \left|W_{\frac{1}{n}}(\omega)\right| \le \frac{C}{n}\right\} \\
&= \mathbb{P}\left\{\omega \in \Omega : \frac{-C}{n} \le W_{\frac{1}{n}}(\omega) \le \frac{C}{n}\right\} \\
&= \int_{\frac{-C}{n}}^{\frac{C}{n}} \frac{1}{\sqrt{2\pi\frac{1}{n}}} \exp\left(\frac{-x^2}{2\frac{1}{n}}\right) dx \\
&= \int_{\frac{-C}{n}}^{\frac{C}{n}} \sqrt{\frac{n}{2\pi}} \exp\left(\frac{-nx^2}{2}\right) dx \\
&= 2\sqrt{\frac{n}{2\pi}} \int_0^{\frac{C}{n}} \exp\left(\frac{-nx^2}{2}\right) dx \\
&\le 2\sqrt{\frac{n}{2\pi}} \int_0^{\frac{C}{n}} 1\, dx = 2\sqrt{\frac{n}{2\pi}}\,\frac{C}{n} = \frac{\alpha C}{\sqrt{n}}
\end{aligned}$$

where $\alpha = \frac{2}{\sqrt{2\pi}}$, as required. □

Lemma 4.5. *For all $C \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{P}\left\{\omega \in \Omega : M_n(\omega) \le \frac{C}{n}\right\} = 0.$$

Proof. Let $\epsilon > 0$ and $C \in \mathbb{R}$. Choose $N \ge 3$ so that for all $n \ge N$,

$$\left[\frac{\alpha C}{\sqrt{n}}\right]^3 < \epsilon.$$

Additionally, for all $n \ge N$, let $k_n' \in \{1, 2, \dots, n-2\}$ be such that

$$M(k_n', n, \omega) = \min\left\{M(1,n,\omega), M(2,n,\omega), \dots, M(n-2,n,\omega)\right\}.$$

It follows that for all $n \geq N$,

$$\begin{aligned}
&\mathbb{P}\left\{\omega \in \Omega : M_n(\omega) \leq \frac{C}{n}\right\} \\
&= \mathbb{P}\left\{\omega \in \Omega : M(k_n', n, \omega) \leq \frac{C}{n}\right\} \\
&\leq \left[\frac{\alpha C}{\sqrt{n}}\right]^3 < \epsilon,
\end{aligned}$$

as required. □

Theorem 4.6.

$$\mathbb{P}\left\{\omega \in \Omega : \ q \mapsto W_q(\omega) \ \textit{is nowhere differentiable on} \ (0,1)\right\} = 1.$$

Proof. It suffices to show that

$$\mathbb{P}\left\{\omega \in \Omega : \ q \mapsto W_q(\omega) \text{ is somewhere differentiable on } (0,1)\right\} = 0.$$

Let $\epsilon > 0$. For any $\omega \in \Omega$, if the mapping $q \mapsto W_q(\omega)$ is somewhere differentiable on $(0,1)$, let $C_\omega < \infty$ and $N_\omega < \infty$ be such that for all $n \geq N_\omega$,

$$M_n(\omega) \leq \frac{C_\omega}{n}.$$

Otherwise, if for some $\omega \in \Omega$ the mapping $q \mapsto W_q(\omega)$ is nowhere differentiable on $(0,1)$, let $C_\omega = N_\omega = 1$. Define a random variable $X : \Omega \to \mathbb{R}$ by $X(\omega) = \max(C_\omega, N_\omega)$. By properties of probability,

$$\lim_{n\to\infty} \mathbb{P}\left\{\omega \in \Omega : \ X(\omega) \leq n\right\} = \mathbb{P}\left\{\omega \in \Omega : \ X(\omega) < \infty\right\} = 1.$$

Thus, there exists $K \in \mathbb{N}$ such that for all $n \geq K$

$$1 - \frac{\epsilon}{2} < \mathbb{P}\left\{\omega \in \Omega : \ X(\omega) \leq n\right\} \leq 1.$$

It follows that

$$\mathbb{P}\left\{\omega \in \Omega : \ \max(C_\omega, N_\omega) > K\right\} < \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\left\{\omega \in \Omega : \ q \mapsto W_q(\omega) \text{ is somewhere differentiable on } (0,1)\right\} \\
&\leq \mathbb{P}\left\{\omega \in \Omega : \text{for all } n \geq N_\omega, \ M_n(\omega) \leq \frac{C_\omega}{n}\right\} \\
&\leq \mathbb{P}\left\{\omega \in \Omega : \ \text{for all } n \geq K, \ M_n(\omega) \leq \frac{K}{n}\right\} + \frac{\epsilon}{2}.
\end{aligned}$$

Now, for each $l \in \mathbb{N}$ such that $l \geq K$,

$$\mathbb{P}\left\{\omega \in \Omega : \ \text{for all } n \geq K, \ M_n(\omega) \leq \frac{K}{n}\right\} \leq \mathbb{P}\left\{\omega \in \Omega : \ M_l(\omega) \leq \frac{K}{l}\right\}.$$

Thus, since Lemma 4.5. implies that there exists $L \geq K$ such that for all $l \geq L$,

$$\mathbb{P}\left\{\omega \in \Omega : \ M_l(\omega) \leq \frac{K}{l}\right\} < \frac{\epsilon}{2},$$

we have

$$\mathbb{P}\left\{\omega \in \Omega : \ \text{for all } n \geq K, \ M_n(\omega) \leq \frac{K}{n}\right\} < \frac{\epsilon}{2}.$$

So

$$\mathbb{P}\{\omega \in \Omega : q \mapsto W_q(\omega) \text{ is somewhere differentiable on } (0,1)\} < \epsilon,$$

and because $\epsilon > 0$ was arbitrary,

$$\mathbb{P}\{\omega \in \Omega : q \mapsto W_q(\omega) \text{ is somewhere differentiable on } (0,1)\} = 0,$$

as required. □

5. The Zero Set of Standard Brownian Motion

To facilitate our remaining work, we require the following results characterizing certain transformations on $\{W_t : t \geq 0\}$.

Fact 5.1. Let $s > 0$. For all $r \geq 0$, define a random variable $\widehat{W_r} : \Omega \to \mathbb{R}$ by $\widehat{W_r}(\omega) = \sqrt{s}\, W_{\frac{r}{s}}(\omega)$. Then $\left\{\widehat{W_r} : r \geq 0\right\}$ is a standard Brownian motion.

Fact 5.2. Let $N \in \mathbb{N}$. For all $t \geq 0$, define a random variable $\widetilde{W_t} : \Omega \to \mathbb{R}$ by $\widetilde{W_t}(\omega) = W_{t+N}(\omega) - W_N(\omega)$. Then $\left\{\widetilde{W_t} : t \geq 0\right\}$ is a standard Brownian motion.

We additionally require the use of a result known as the "Reflection Principle," which derives its name from the particular symmetry employed in its proof.

Fact 5.3. Let $b > 0$. Then for any $t > 0$,

$$\mathbb{P}\{\omega \in \Omega : \exists r \in [0,t] \text{ with } W_r(\omega) \geq b\} = 2\mathbb{P}\{\omega \in \Omega : W_t(\omega) \geq b\}.$$

We are now prepared to investigate a final matter concerning standard Brownian motion: the properties of the "zero set" given by $\{t \geq 0 : W_t = 0\}$. In particular, our goals for this section are Theorems 5.6., 5.8., and 5.10., all of which examine the "distribution" of the zero set over the nonnegative reals. For the sake of clarity, we will present a supporting lemma before each main result.

Remark 5.4. The proofs of Theorems 5.6. and 5.8. employ similar techniques. We only include both for completeness. Additionally, we note that the interval $[0,1]$ considered in Theorem 5.10. was chosen for convenience alone. In fact, the same statement can be demonstrated for any closed interval and, hence, with probability one, the Lebesgue measure of the zero set of standard Brownian motion is zero.

Lemma 5.5. *For all $N \in \mathbb{N}$,*

$$\mathbb{P}\{\omega \in \Omega : \exists t > N \text{ with } W_t(\omega) = 0\} = 1.$$

Proof. Let $N \in \mathbb{N}$ and $\epsilon > 0$. Since for all $m \in \mathbb{N}$,

$$\mathbb{P}\{\omega \in \Omega : -m \leq W_N(\omega) \leq m\} = \int_{-m}^{m} \frac{1}{\sqrt{2\pi N}} \exp\left(\frac{-x^2}{2N}\right) dx,$$

there exists $M \in \mathbb{N}$ such that

$$\mathbb{P}\{\omega \in \Omega : -M \leq W_N(\omega) \leq M\} \geq 1 - \epsilon.$$

For all $t \geq 0$, define a random variable $X_t : \Omega \to \mathbb{R}$ by $X_t(\omega) = W_{t+N}(\omega) - W_N(\omega)$. Then $\{X_t : t \geq 0\}$ is a standard Brownian motion, and for all $l > 0$,

$$\begin{aligned}
&\mathbb{P}\{\omega \in \Omega : \exists s' \in [0,l] \text{ with } X_{s'}(\omega) \leq -2M\} \\
&\quad = \mathbb{P}\{\omega \in \Omega : \exists s \in [0,l] \text{ with } X_s(\omega) \geq 2M\} \\
&\quad = 2\mathbb{P}\{\omega \in \Omega : X_l(\omega) \geq 2M\} \\
&\quad = 2\int_{2M}^{\infty} \frac{1}{\sqrt{2\pi l}} \exp\left(\frac{-x^2}{2l}\right) dx.
\end{aligned}$$

Therefore, there exists an $L \in \mathbb{N}$ such that

$$\begin{aligned}
&\mathbb{P}\{\omega \in \Omega : \exists s' \in [0,L] \text{ with } X_{s'}(\omega) \leq -2M\} \\
&\quad = \mathbb{P}\{\omega \in \Omega : \exists s \in [0,L] \text{ with } X_s(\omega) \geq 2M\} \\
&\quad \geq 1 - \epsilon.
\end{aligned}$$

It follows that if

$$\begin{aligned}
\mathcal{A} &= \{\omega \in \Omega : \exists s' \in [0,L] \text{ with } X_{s'}(\omega) \leq -2M\} \text{ and} \\
\mathcal{B} &= \{\omega \in \Omega : \exists s \in [0,L] \text{ with } X_s(\omega) \geq 2M\},
\end{aligned}$$

then

$$\begin{aligned}
&\mathbb{P}\{\omega \in \Omega : \exists s, s' \in [0,L] \text{ with } X_{s'}(\omega) \leq -2M \text{ and } X_s(\omega) \geq 2M\} \\
&\quad = \mathbb{P}(\mathcal{A} \cap \mathcal{B}) = 1 - \mathbb{P}([\mathcal{A} \cap \mathcal{B}]^c) \\
&\quad \geq 1 - \mathbb{P}(\mathcal{A}^c) - \mathbb{P}(\mathcal{B}^c) \geq 1 - 2\epsilon.
\end{aligned}$$

By continuity of Brownian motion, this implies that

$$\begin{aligned}
&\mathbb{P}\{\omega \in \Omega : \exists t > N \text{ with } W_t(\omega) = 0\} \\
&\quad \geq \mathbb{P}\Big\{\exists s, s' \in [0,L] \text{ such that } W_{s'+N} - W_N \leq -2M \text{ and } W_{s+N} - W_N \geq 2M \\
&\qquad\qquad \Big| -M \leq W_N \leq M\Big\}\, \mathbb{P}\{-M \leq W_N \leq M\} \\
&\quad \geq \mathbb{P}\{\exists s, s' \in [0,L] \text{ with } W_{s'+N} - W_N \leq -2M \text{ and } W_{s+N} - W_N \geq 2M\}(1-\epsilon) \\
&\quad = \mathbb{P}\{\exists s, s' \in [0,L] \text{ with } X_{s'} \leq -2M \text{ and } X_s \geq 2M\}(1-\epsilon) \\
&\quad \geq (1-2\epsilon)(1-\epsilon) = 1 - 3\epsilon + 2\epsilon^2 \geq 1 - 3\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this means that

$$\mathbb{P}\{\omega \in \Omega : \exists t > N \text{ with } W_t(\omega) = 0\} = 1,$$

as required. □

Theorem 5.6. *For all $N \in \mathbb{N}$,*

$$\mathbb{P}\{\omega \in \Omega : \textit{there are infinitely many times } t > N \textit{ with } W_t(\omega) = 0\} = 1.$$

Proof. Let $N \in \mathbb{N}$. Suppose that

$$\mathbb{P}\{\omega \in \Omega : \text{ there exist at most finitely many times } t > N \text{ with } W_t(\omega) = 0\} = \delta$$

for some $\delta > 0$. Then

$$\mathbb{P}\{\omega \in \Omega : \exists N_\omega < \infty \text{ such that for all } t > N_\omega,\ W_t(\omega) \neq 0\} \geq \delta.$$

Construct the following sets:

$$\begin{aligned}
\mathcal{A} &= \{\omega \in \Omega : \exists N_\omega < \infty \text{ such that for all } t > N_\omega,\ W_t(\omega) \neq 0\}, \\
\mathcal{A}_n &= \{\omega \in \Omega : \exists N_\omega < n \text{ such that for all } t > N_\omega,\ W_t(\omega) \neq 0\}\ (\forall n \in \mathbb{N}).
\end{aligned}$$

Then

$$\bigcup_{n=1}^{\infty} \mathcal{A}_n = \mathcal{A} \text{ and for all } n \in \mathbb{N},\ \mathcal{A}_n \subseteq \mathcal{A}_{n+1}.$$

Therefore,

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{A}_n) = \mathbb{P}(\mathcal{A}) \geq \delta,$$

which implies that there exists $M \in \mathbb{N}$ such that

$$\mathbb{P}(\mathcal{A}_M) \geq \frac{\delta}{2}.$$

Thus,

$$\mathbb{P}\{\omega \in \Omega : \text{for all } t > M,\, W_t(\omega) \neq 0\} \geq \frac{\delta}{2},$$

which contradicts Lemma 5.5. It follows that

$$\mathbb{P}\{\omega \in \Omega : \text{there are infinitely many times } t > N \text{ with } W_t(\omega) = 0\} = 1,$$

as required. □

Lemma 5.7. *For all* $\epsilon > 0$,

$$\mathbb{P}\{\omega \in \Omega : \exists t \in (0,\epsilon) \text{ with } W_t(\omega) = 0\} = 1.$$

Proof. Let $\epsilon > 0$. Since

$$\begin{aligned}
&\mathbb{P}\{\omega \in \Omega : \text{for all } t \in (0,\epsilon), W_t(\omega) \neq 0\} \\
&\quad = \mathbb{P}\{\omega \in \Omega : \text{for all } t \in (0,\epsilon), W_t(\omega) > 0\} + \mathbb{P}\{\omega \in \Omega : \text{for all } t \in (0,\epsilon), W_t(\omega) < 0\} \\
&\quad = 2\mathbb{P}\{\omega \in \Omega : \text{for all } t \in (0,\epsilon), W_t(\omega) < 0\},
\end{aligned}$$

it suffices to show that

$$\mathbb{P}\{\omega \in \Omega : \text{for all } t \in (0,\epsilon), W_t(\omega) < 0\} = 0.$$

The above is satisified if and only if

$$\mathbb{P}\{\omega \in \Omega : \exists t \in (0,\epsilon) \text{ with } W_t(\omega) \geq 0\} = 1.$$

Let $\eta > 0$. Then $\delta > 0$ can be chosen so that

$$\begin{aligned}
&\mathbb{P}\{\omega \in \Omega : \exists t \in (0,\epsilon) \text{ with } W_t(\omega) \geq \delta\} \\
&\quad = 2\mathbb{P}\{\omega \in \Omega : W_\epsilon(\omega) \geq \delta\} \\
&\quad = 2\int_\delta^\infty \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(\frac{-x^2}{2\epsilon}\right) dx \\
&\quad \geq 1 - \eta.
\end{aligned}$$

This implies that

$$\mathbb{P}\{\omega \in \Omega : \exists t \in (0,\epsilon) \text{ with } W_t(\omega) \geq 0\} \geq 1 - \eta,$$

and since $\eta > 0$ was arbitrary,

$$\mathbb{P}\{\omega \in \Omega : \exists t \in (0,\epsilon) \text{ with } W_t(\omega) \geq 0\} = 1,$$

as required. □

Theorem 5.8. *For all* $\epsilon > 0$,

$$\mathbb{P}\{\omega \in \Omega : \textit{there are infinitely many times } t \in (0,\epsilon) \textit{ with } W_t(\omega) = 0\} = 1.$$

Proof. Let $\epsilon > 0$. Suppose that

$$\mathbb{P}\left\{\omega \in \Omega : \text{there are at most finitely many times } t \in (0,\epsilon) \text{ with } W_t(\omega) = 0\right\} = \delta$$

for some $\delta > 0$. Then

$$\mathbb{P}\left\{\omega \in \Omega : \exists \eta > 0 \text{ such that for all } t < \eta,\ W_t(\omega) \neq 0\right\} \geq \delta.$$

Construct the following sets:

$$\mathcal{A} = \left\{\omega \in \Omega : \exists \eta > 0 \text{ such that for all } t < \eta,\ W_t(\omega) \neq 0\right\},$$
$$\mathcal{A}_n = \left\{\omega \in \Omega : \exists \eta > \frac{1}{n} \text{ such that for all } t < \eta,\ W_t(\omega) \neq 0\right\} (\forall n \in \mathbb{N}).$$

Then

$$\bigcup_{n=1}^{\infty} \mathcal{A}_n = \mathcal{A} \text{ and for all } n \in \mathbb{N},\ \mathcal{A}_n \subseteq \mathcal{A}_{n+1}.$$

Therefore,

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{A}_n) = \mathbb{P}(\mathcal{A}) \geq \delta,$$

which implies that there exists $M \in \mathbb{N}$ such that

$$\mathbb{P}(\mathcal{A}_M) \geq \frac{\delta}{2}.$$

Thus,

$$\mathbb{P}\left\{\omega \in \Omega : \text{for all } t < \frac{1}{M},\ W_t(\omega) \neq 0\right\} \geq \frac{\delta}{2},$$

which contradicts Lemma 5.7. It follows that

$$\mathbb{P}\left\{\omega \in \Omega : \text{there are infinitely many times } t \in (0,\epsilon) \text{ with } W_t(\omega) = 0\right\} = 1,$$

as required. □

Lemma 5.9. *For all $t > s > 0$,*

$$\mathbb{P}\left\{\omega \in \Omega : \exists r \in [s,t] \text{ with } W_r(\omega) = 0\right\} \leq \frac{2}{\pi}\sqrt{\frac{t}{s} - 1}.$$

Proof. Let $t > s > 0$. For all $r \geq 0$, define a random variable $\widehat{W}_r : \Omega \to \mathbb{R}$ by $\widehat{W}_r(\omega) = \sqrt{s}\, W_{\frac{r}{s}}(\omega)$. Then $\left\{\widehat{W}_r : r \geq 0\right\}$ is a standard Brownian motion. Therefore,

$$\begin{aligned}
&\mathbb{P}\left\{\omega \in \Omega : \exists r \in [s,t] \text{ with } W_r(\omega) = 0\right\} \\
&= \mathbb{P}\left\{\omega \in \Omega : \exists r \in [s,t] \text{ with } \widehat{W}_r(\omega) = 0\right\} \\
&= \mathbb{P}\left\{\omega \in \Omega : \exists r \in [s,t] \text{ with } \sqrt{s}W_{\frac{r}{s}}(\omega) = 0\right\} \\
&= \mathbb{P}\left\{\omega \in \Omega : \exists r \in [s,t] \text{ with } W_{\frac{r}{s}}(\omega) = 0\right\} \\
&= \mathbb{P}\left\{\omega \in \Omega : \exists l \in \left[1, \frac{t}{s}\right] \text{ with } W_l(\omega) = 0\right\}.
\end{aligned}$$

By continuity of Brownian motion, it follows that

$$\begin{aligned}
&\mathbb{P}\left\{\exists l \in \left[1, \frac{t}{s}\right] \text{ with } W_l = 0\right\} \\
&= \mathbb{P}\left\{\exists l \in \left[1, \frac{t}{s}\right] \text{ with } W_l = 0 \middle| W_1 > 0\right\} \mathbb{P}\left\{W_1 > 0\right\} + \mathbb{P}\left\{\exists l \in \left[1, \frac{t}{s}\right] \text{ with } W_l = 0 \middle| W_1 < 0\right\} \\
&\qquad \cdot \mathbb{P}\left\{W_1 < 0\right\} \\
&= \frac{1}{2}\left[\mathbb{P}\left\{\exists l \in \left[1, \frac{t}{s}\right] \text{ with } W_l = 0 \middle| W_1 > 0\right\} + \mathbb{P}\left\{\exists l \in \left[1, \frac{t}{s}\right] \text{ with } W_l = 0 \middle| W_1 < 0\right\}\right] \\
&= \frac{1}{2}\left[\mathbb{P}\left\{\exists l \in \left[1, \frac{t}{s}\right] \text{ with } W_l \leq 0 \middle| W_1 > 0\right\} + \mathbb{P}\left\{\exists l \in \left[1, \frac{t}{s}\right] \text{ with } W_l \geq 0 \middle| W_1 < 0\right\}\right].
\end{aligned}$$

Now, given that $W_1 = x > 0$, $\left\{\breve{W}_r : r \geq 0\right\}$ is a standard Brownian motion, and $\left\{\widetilde{W}_r : r \geq 0\right\}$ is a Brownian motion starting at x,

$$\begin{aligned}
&\mathbb{P}\left\{\exists l \in \left[1, \frac{t}{s}\right] \text{ with } W_l \leq 0\right\} \\
&= \mathbb{P}\left\{\exists l \in \left[0, \frac{t}{s} - 1\right] \text{ with } \widetilde{W}_l \leq 0\right\} \\
&= \mathbb{P}\left\{\exists l \in \left[0, \frac{t}{s} - 1\right] \text{ with } \breve{W}_l \leq -x\right\} \\
&= \mathbb{P}\left\{\exists l \in \left[0, \frac{t}{s} - 1\right] \text{ with } \breve{W}_l \geq x\right\} \\
&= 2\mathbb{P}\left\{\breve{W}_{\frac{t}{s}-1} \geq x\right\}.
\end{aligned}$$

If Y and X are independent random variables such that $Y \sim N\left(0, \frac{t}{s} - 1\right)$ and $X \sim N(0,1)$, then by properties of Brownian motion, this implies that

$$\mathbb{P}\left\{\exists l \in \left[1, \frac{t}{s}\right] \text{ with } W_l \leq 0 \middle| W_1 > 0\right\} = 2\mathbb{P}\left\{Y \geq X \middle| X > 0\right\}.$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\left\{\exists l \in \left[1, \frac{t}{s}\right] \text{ with } W_l \leq 0 \middle| W_1 > 0\right\} \\
&= 2 \iint_{y \geq x > 0} \frac{1}{\sqrt{2\pi\left(\frac{t}{s} - 1\right)}} \exp\left(\frac{-y^2}{2\left(\frac{t}{s} - 1\right)}\right) \frac{2}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) dx\, dy \\
&= 2 \frac{1}{\sqrt{2\pi\left(\frac{t}{s} - 1\right)}} \frac{2}{\sqrt{2\pi}} \int_0^\infty \exp\left(\frac{-y^2}{2\left(\frac{t}{s} - 1\right)}\right) \left(\int_0^y \exp\left(\frac{-x^2}{2}\right) dx\right) dy \\
&\leq \frac{4}{2\pi\sqrt{\left(\frac{t}{s} - 1\right)}} \int_0^\infty \exp\left(\frac{-y^2}{2\left(\frac{t}{s} - 1\right)}\right) \left(\int_0^y 1\, dx\right) dy \\
&= \frac{2}{\pi\sqrt{\left(\frac{t}{s} - 1\right)}} \int_0^\infty \exp\left(\frac{-y^2}{2\left(\frac{t}{s} - 1\right)}\right) y\, dy.
\end{aligned}$$

It follows that

$$\mathbb{P}\left\{\exists l \in \left[1, \frac{t}{s}\right] \text{ with } W_l \leq 0 \middle| W_1 > 0\right\}$$
$$\leq \frac{-2}{\pi}\sqrt{\frac{t}{s}-1}\left[\exp\left(\frac{-y^2}{2\left(\frac{t}{s}-1\right)}\right)\Bigg|_0^\infty\right] = \frac{2}{\pi}\sqrt{\frac{t}{s}-1}.$$

Now, using exactly the same technique, it is possible to show that

$$\mathbb{P}\left\{\exists l \in \left[1, \frac{t}{s}\right] \text{ with } W_l \geq 0 \middle| W_1 < 0\right\} \leq \frac{2}{\pi}\sqrt{\frac{t}{s}-1}$$

as well, so we can conclude that

$$\mathbb{P}\left\{\omega \in \Omega : \exists l \in \left[1, \frac{t}{s}\right] \text{ with } W_l(\omega) = 0\right\} \leq \frac{1}{2}\left[\frac{2}{\pi}\sqrt{\frac{t}{s}-1} + \frac{2}{\pi}\sqrt{\frac{t}{s}-1}\right] = \frac{2}{\pi}\sqrt{\frac{t}{s}-1},$$

as required. □

Theorem 5.10. *Let λ denote Lebesgue measure on $\mathbb{R}$. Then*

$$\mathbb{P}\{\omega \in \Omega : \lambda(\{t \in [0,1] : W_t(\omega) = 0\}) = 0\} = 1.$$

Proof. Define a random variable $X : \Omega \to \mathbb{R}$ by

$$X(\omega) = \lambda(\{t \in [0,1] : W_t(\omega) = 0\}).$$

It suffices to show that $\mathbb{E}[X] = 0$. For all $n \in \mathbb{N}$, define a random variable $Y_n : \Omega \to \mathbb{R}$ by

$$Y_n(\omega) = \frac{1}{n}\left(\# \text{ of } k \in \{0,1,\dots,n-1\} \text{ such that } \exists t \in \left[\frac{k}{n}, \frac{k+1}{n}\right] \text{ with } W_t(\omega) = 0\right).$$

Clearly, for all $\omega \in \Omega$ and $n \in \mathbb{N}$, $X(\omega) \leq Y_n(\omega)$, and so

$$\mathbb{E}[X] \leq \mathbb{E}[Y_n]$$

for all $n \in \mathbb{N}$. Thus, since X is nonnegative, it suffices to show that

$$\lim_{n\to\infty} \mathbb{E}[Y_n] = 0.$$

Now, for all $n \in \mathbb{N}$ and $k \in \{0,1,\dots,n-1\}$, define a random variable $Z_{k,n} : \Omega \to \mathbb{R}$ by

$$Z_{k,n}(\omega) = \begin{cases} 1 & \text{if } \exists t \in \left[\frac{k}{n}, \frac{k+1}{n}\right] \text{ with } W_t(\omega) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then for all $n \in \mathbb{N}$,

$$Y_n = \frac{1}{n}\left(\sum_{k=0}^{n-1} Z_{k,n}\right),$$

so

$$\begin{aligned}\mathbb{E}\left[Y_n\right] &= \mathbb{E}\left[\frac{1}{n}\left(\sum_{k=0}^{n-1} Z_{k,n}\right)\right] \\ &= \frac{1}{n}\left(\sum_{k=0}^{n-1} \mathbb{E}\left[Z_{k,n}\right]\right) \\ &= \frac{1}{n}\left(\sum_{k=0}^{n-1} \mathbb{P}\left\{\omega \in \Omega : \exists t \in \left[\frac{k}{n}, \frac{k+1}{n}\right] \text{ with } W_t\left(\omega\right) = 0\right\}\right) \\ &\leq \frac{1}{n}\left[1 + \sum_{k=1}^{n-1} \frac{2}{\pi}\sqrt{\frac{\frac{k+1}{n}}{\frac{k}{n}} - 1}\right] \\ &= \frac{1}{n}\left[1 + \sum_{k=1}^{n-1} \frac{2}{\pi}\sqrt{\frac{1}{k}}\right].\end{aligned}$$

But for all $n \in \mathbb{N}$, this last expression is just the arithmetic average of the first n terms of the sequence $\left(a_m\right)_{m\in\mathbb{N}}$ given by

$$a_m = \begin{cases} 1 & \text{if } m = 1, \\ \frac{2}{\pi}\sqrt{\frac{1}{m-1}} & \text{otherwise.} \end{cases}$$

Therefore, since $\left(a_m\right)_{m\in\mathbb{N}}$ is a nonnegative sequence tending to 0 as m approaches ∞, this implies that

$$\lim_{n\to\infty} \mathbb{E}\left[Y_n\right] = 0,$$

as required. □

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7. References

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