Bipartite Graphs and Problem Solving

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Abstract

This paper will begin with a brief introduction to the theory of graphs and will focus primarily on the properties of bipartite graphs. The final section will demonstrate how to use bipartite graphs to solve problems.

1 Graphs

A *Graph* G is defined to be an ordered triple $(V(G), E(G), \phi(G))$, where $V(G)$ is the nonempty set of *vertices* of G, $E(G)$ is the set of *edges* of G, and $\phi(G)$ associates to each edge in $E(G)$ two unordered vertices in $V(G)$.

If $\phi(\text{e}) = uv$, for $e \in E(G)$ and $v, u \in V(G)$, then we say that e is *incident on* v and u, and that v and u are *adjacent* vertices.

A *simple graph* is a graph where $\phi(\text{e}) \neq vv$ for any $v \in V(G)$, and that for $e_0, e_1 \in E(G)$, we have that $e_0 \neq e_1 \iff \phi(e_0) \neq \phi(e_1)$; in other words, a simple graph is a graph where no edge forces a vertex to be adjacent to itself and in which no edges join the same unordered pair of vertices.

For the purpose of this paper, all graphs will be considered to be simple graphs where $V(G)$ and $E(G)$ are finite sets.

Graphs can be visualized in many different ways, but the way that we will follow in this paper is the following: we imagine each element $v \in V(G)$ to be a point floating in space. To represent an edge between two vertices, we simply connect the two vertices with a line segment – but note that the line segment need not be straight and that there is no notion of distance between any two vertices of G, so they may be as close or as far apart as we please. In this paper, we may now talk about edges and vertices as if they were physical objects such as the ones just described.

2 Properties of General Graphs and Introduction to Bipartite Graphs

Every graph has certain properties that can be used to describe it. An important property of graphs that is used frequently in graph theory is the degree of each vertex. The *degree* of a vertex in G is the number of vertices adjacent to it, or, equivalently, the number of edges incident on it. We represent the degree of a vertex by $\deg(v) = r$, where r is the number of vertices adjacent to v. Two easy theorems to prove about degrees are:

Theorem 2.1 (Sum of Degrees) $\sum_{v \in V(G)} \deg(v) = 2e.$

Proof. Each edge in $E(G)$ will contribute to the degree of two different vertices – therefore, the sum of the degrees should be exactly two times the number of edges. Also note that since e is an integer, we have that $2e$ is even. Therefore, the sum of the degrees of every vertex in a graph G is always even.

Theorem 2.2 (Number of Odd Degree Vertices) *In any simple graph, G, the number of vertices with odd degree is even.*

Proof. Partition $V(G)$ into two sets, V_1 and V_2, where V_1 contains every even degree vertex and V_2 contains every odd degree vertex. Note that $\sum_{v \in V(G)} \deg(v)$ is even, and because the sum of any finite number of even numbers is also even, we have that $\sum_{v \in V_1} \deg(v)$ is even. But if we observe the equation $\sum_{v \in V(G)} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v)$ we note that $\sum_{v \in V_2} \deg(v)$ must be even, since if it were odd, then $\sum_{v \in V(G)} \deg(v)$ would also be odd, which contradicts the previous theorem.

Up until now we've looked at general finite graphs, but in order to prove more specific theorems, we must impose restrictions on the kinds of graphs that we consider. Because of their simplicity and their usefulness in solving certain types of problems, we now consider *bipartite graphs.* A bipartite graph is a simple graph in which $V(G)$ can be partitioned into two sets, V_1 and V_2 with the following properties:

1. If $v \in V_1$ then it may only be adjacent to vertices in V_2.
2. If $v \in V_2$ then it may only be adjacent to vertices in V_1.
3. $V_1 \cap V_2 = \emptyset$
4. $V_1 \cup V_2 = V(G)$

We can imagine bipartite graphs to look like two parallel lines of vertices such that a vertex in one line can only connect to vertices in the other line, and **not** to vertices in its own line.

Now that we know what a bipartite graph is, we can begin to prove some theorems about them that will help us in using the properties of bipartite graphs to solve certain problems. We begin by proving two theorems regarding the degrees of vertices of bipartite graphs.

Lemma 2.3 *If G is a bipartite graph and the bipartition of G is X and Y, then* $\sum_{v\in X} \deg(v) = \sum_{v\in Y} \deg(v)$.

Proof. By induction on the number of edges. We denote the number of elements of X as $|X|$. Suppose $|X| = r$ and $|Y| = s$ for some integers $r, s > 1$. Note that the case where X or Y has one vertex is trivial, as only one edge can be drawn. Take the subgraph of G which consists of only the vertices of G. Now we begin inducting: add one edge from any vertex in X to any vertex in Y. Then, $\sum_{v\in X} \deg(v) = \sum_{v\in Y} \deg(v) = 1$. Now, suppose this is true for n-1 edges and add one more edge. Since this edge adds exactly 1 to both $\sum_{v\in X} \deg(v)$ and $\sum_{v\in Y} \deg(v)$, we have that this is true for all n$\in N$.

A *k-regular* graph G is one such that $\deg(v) = k$ for all $v \in G$.

Theorem 2.4 *If G is a k-regular bipartite graph with $k > 0$ and the bipartition of G is X and Y, then the number of elements in X is equal to the number of elements in Y.*

Proof. We observe $\sum_{v\in X} \deg(v) = k\,|X|$ and similarly, $\sum_{v\in Y} \deg(v) = k\,|Y|$. By the previous lemma, this means that $k\,|X| = k\,|Y| \implies |X| = |Y|$.

In any graph, the ϕ function tells us if a vertex is adjacent to another vertex, but in many cases we need a stronger idea of connectedness. For example, what if we want to know if there are several edges and vertices which we can "link together" to get from one vertex to another? Visually, we are asking if we can start at vertex v and follow an edge which is incident on v to another vertex, follow an edge connected to that vertex and get to a new vertex, and so on until we reach the desired final vertex u? In order to be able to have such a notion of connectedness, we introduce the concept of a path.

A *path* is a finite alternating sequence of vertices and edges such that every edge and vertex in the sequence is distinct. In addition, for the (u, v)-path $= \{ue_1v_1e_2 \ldots e_nv\}$, we must have e_1 incident on u, and, in general, v_n adjacent to

v_{n-1} with e_n being the edge that connects the two. We may think of a path of a graph G as picking a vertex then "walking" along an edge adjacent to it to another vertex and continuing until we get to the last vertex. The *length* of a path is the number of edges contained in the path.

We now use the concept of a path to define a stronger idea of connectedness. Two vertices, u and v in a graph G are *connected* if there exists a (v, u)-path in G. Notice that connection is an equivalence relation: a (v, v) path is just the path $\mathrm{P} = v$; a (u, v)-path is also a (v, u)-path, since the graph is undirected; and if there exists a (u, v)-path and a (v, w) path, for $u, v, w \in V(G)$, then we may simply exclude the last v in the (u, v)-path and join this path to the (v, w)-path to make a (u, w)-path. Because of this, G may be split up into equivalence classes of connectedness. If there is only one equivalence class, then G is *connected*.

A *cycle* is a (v, u)-path where $v = u$. In other words, a cycle is a path with the same first and last vertex. The *length* of the cycle is the number of edges that it contains, and a cycle is *odd* if it contains an odd number of edges.

Theorem 2.5 *A bipartite graph contains no odd cycles.*

Proof. If G is bipartite, let the vertex partitions be X and Y. Suppose that G did contain an odd cycle – then $\mathrm{C} = v_0 e_1 \ldots e_{2k+1} v_0$. Without loss of generality, let v_0 be a vertex in X. Then v_1 must be a vertex in Y, and it is connected to v_0 by e_1. Similarly, e_{2n+1} is preceded by a vertex in X and proceeded by a vertex in Y for all $n \in N$. But e_{2k+1} is proceeded by v_0, which is a vertex in X and therefore cannot also be a vertex in Y.

In fact, any graph that contains no odd cycles is necessarily bipartite, as well. This we will not prove, but this theorem gives us a nice way of checking to see if a given graph G is bipartite – we look at all of the cycles, and if we find an odd cycle we know it is not a bipartite graph.

A *subgraph* H of G is a graph such that $V(H) \subseteq V(G)$, and $E(H) \subseteq E(G)$ and $\phi(H)$ is defined to be $\phi(G)$ restricted to $E(H)$.

Theorem 2.6 (Subgraph of a Bipartite Graph) *Every subgraph H of a bipartite graph G is, itself, bipartite.*

Proof. If G is bipartite, let the partitions of the vertices be X and Y. Then let $X' = X \cap H$ and $Y' = Y \cap H$. Suppose that this was not a valid bipartition of H – then we have that there exists v and u in X' (without loss of generality) such that v and u are adjacent. But then by the definition of a subgraph, they are also adjacent in G. But then X and Y is not a valid bipartition of G. Therefore, H is a bipartite graph.

This theorem is almost obvious, but we state it for completeness – it is enough to note that the graph G is bipartite to be able to use any and all theorems relating to bipartite graphs for any subgraphs we take of G.

The concept of coloring vertices and edges comes up in graph theory quite a bit. A *k-coloring* is a partition of V(G) into k sets such that each of the k sets are disjoint and no two vertices in the same set are adjacent to each other. A graph which has a k-coloring but no (k-1)-coloring is called *k-colorable.* Normally, it is somewhat tedious and difficult to check to see if a given graph G is k-colorable, but with bipartite graphs it is extremely easy. In fact, we show with the theorem below that k can only be one of two values!

Theorem 2.7 (Bipartite Colorings) *If G is a bipartite graph with a positive number of edges, then G is 2-colorable. If G is bipartite with no edges, it is 1-colorable.*

Proof. Let G be bipartite graph with a positive number of edges, then let the bipartition of G be X and Y. Let the vertices in X represent color 1 and the vertices in Y represent color 2. Then this satisfies the criteria for a valid coloring. Therefore, G has a 2-coloring.

Now we show it has no 1-coloring. If this were true, then every vertex in G represents color 1. But because there is a positive number of edges, one vertex is adjacent to at least one other, which contradicts the fact that this is a coloring. Therefore, G is 2-colorable.

Let G be a bipartite graph with no edges. It has at least a 1-coloring, since no vertex is adjacent to any other vertex. Since we cannot color it with zero colors, it must be the case that G is 1-colorable.

A *k-partite graph* is a graph that may be partitioned into k sets such that no vertex in any of the k sets connects to another element of that same set. We can generalize the previous theorem by saying that every k-partite graph is k-colorable and the proof is similar to the proof for two.

Similar to the idea of coloring, we have that a *matching M* in G is a set of edges such that no two edges share a common vertex. Another way to say this is that the set of edges must be pairwise non-adjacent. A *maximal matching* in G is a matching in G which contains the largest possible number of edges. A *perfect matching* is a matching where the edges in M are incident on each vertex in V(G). Notice that a perfect matching is also a maximal matching, but a maximal matching need not be a perfect matching.

The *neighborhood N(X)* of a subset X of G is a collection of all vertices $u \in Y$ such that u is adjacent to at least one $v \in X$. We will show that there is a beautiful relationship between subgraphs of a bipartite graph and the perfect matchings, but first we need to short lemmas.

Lemma 2.8 *Given a bipartite graph G with bipartition X and Y, if $|X| \neq |Y|$ then there does not exist a perfect matching for G.*

Proof. Suppose we have a perfect matching M. We form a subgraph H which is defined to be the vertices of G with only the edges of the perfect matching and the incidence function ϕ restricted to the edges of the perfect matching. Note that H is still a bipartite graph by Theorem 2.6. Since M is a perfect matching, each vertex has degree exactly 1. If this is the case, then H is a 1-regular graph and by Theorem 2.4 we have that $|X| = |Y|$. This contradicts the fact that $|X| \neq |Y|$, and so there does not exist a perfect matching in G.

Before proving Theorem 2.10, we need some tools to help us prove the second half of the proof. We say that a path P is an *M-alternating path in G* if the edges in P are alternately in $E \setminus M$ and M, where M is a matching in G. An *M-augmenting path in G* is an M-alternating path where the first and last vertex are not incident on by any edge in M.

We will now state a lemma relating matchings and M-augmenting paths. We exclude the proof of the lemma in this paper as it is somewhat tedious, but it is available in Bondy and Murty's Graph Theory text [1].

Lemma 2.9 *A matching M in G is a maximum matching if and only if G contains no M-augmenting path.*

Theorem 2.10 (Perfect Matching) *A perfect matching exists on a bipartite graph G with bipartition X and Y if and only if for every subset S of X we have $|S| \leq |N(S)|$ and $|X| = |Y|$. That is, if for every subset S of X, the number of elements in S is less than or equal to the number of elements in the neighborhood of S.*

Proof.

Assume we have a perfect matching, $M \subseteq E(G)$. By lemma 2.8, it must be the case that $|X| = |Y|$. Then we have that each vertex in S has at least one adjacent vertex in N(S) and possibly more since each vertex in S may, in addition, be adjacent to other vertices besides the one that it is adjacent to in the matching. It follows that $|S| \leq |N(S)|$.

Assume that we have $|S| \leq |N(S)|$ for every $S \subseteq X$. Now, let M be a maximum matching in G, but suppose that it does not saturate one vertex in X (i.e., M is not a perfect matching). Let $x \in X$ be a vertex which is not M-saturated and construct a set Z of all vertices which are connected to x by M-alternating paths. Because M is a maximum matching, we have from the previous lemma that x is the only M-unsaturated vertex in Z. Now we create two sets, $T = Z \cap X$ and $R = Z \cap Y$. We have that the vertices in $T \setminus \{x\}$ are matched under M with the vertices in R, and so we have that $|R| = |T| - 1$. Since every vertex in N(T) is connected to x by an M-alternating path, we have that N(T) $= R$. But now we have that $|N(T)| = |T| - 1 < |T|$, which contradicts our assumptions. Therefore, M is a perfect matching.

3 Problems

In this section, we show that bipartite graphs can be useful in problem solving by presenting two problems that are easily solvable using the theory of bipartite graphs.

Problem 3.1 *Take a standard* 8×8 *chessboard and remove the top left square and the bottom right square. Prove that we cannot cover the board with* 1×2 *dominoes with no dominoes overlapping.*

Solution. We create a bipartite graph G from the chess board. Let each white square be a vertex in X and let each black square be a vertex in Y. Note that each domino must cover exactly one black and one white square and so we will connect a vertex in X to a vertex in Y if a domino covers both of them. Because we have removed the upper left and the lower right square, which are both either white or black, we have that, without loss of generality, $|X| = 30$ and $|Y| = 32$. Now suppose that each of the squares was able to be covered. Then each vertex has degree exactly one, since it is covered by one domino. This means that G is 1-regular, and by Theorem 2.4 we have that $30 = |X| = |Y| = 32$ which is absurd.

It's interesting to note that the choice of squares was somewhat arbitrary – so long as both of the squares are black or both are white, the proof will still work. Notice that if one black and one white square were to be removed, then $|X| = |Y| = 31$. Since the degree of each vertex is exactly one , it follows that each $v \in X$ is adjacent to exactly one vertex $u \in Y$, since if there exists a point $w \in Y$ such that $\deg(w)$ $= 2$ or 0, this means that either a domino overlapped another domino or a domino only covered one white square and not a black square. Neither of these things can happen. Then we have that for each subset $S \subseteq X$, $|S| = |N(S)|$, and so this is perfect matching by Theorem 2.10.

Problem 3.2 *There are a set of boys and a set of girls. Each boy only likes girls and each girl only likes boys. A common friend wants to match each boy with a girl such that the boy and girl are both happy – but they both will only be happy if the boy likes the girl and the girl likes the boy. Is it possible for every situation? What property of the set of boys and girls is necessary to make such a perfect matching possible?*

Solution. The solution is simple once we construct a bipartite graph representing this situation. We first make a bipartite graph G with bipartition X and Y. Let X represent the boys, Y represent the girls. Let there be an edge which connects $v \in X$ and $u \in Y$ if v likes u and u likes v. We notice first that the desired matching would require $|X| = |Y|$ by lemma 2.10. Now we use Theorem 2.10 to note that there exists such a matching if and only if we have that $|S| \leq |N(S)|$ for every S$\subseteq X$. Therefore, for this matching to be possible, we require that for each subset of $|S|$ boys, the total number of different girls that they like is greater than or equal to the number of boys in the subset.

[1] Graph Theory with Applications, Bondy and Murty. North Holland Publishers, 1976. Fifth Edition. The relevant proof is on page 78.