CLOSED CURVES ON HYPERBOLIC MANIFOLDS

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ABSTRACT. In this paper I will define a hyperbolic manifold, characterize its universal cover $\mathbb{H}^n$, describe its deck transformations, and prove that a closed curve on a hyperbolic manifold is homotopic to a closed geodesic on the surface.

1. Hyperbolic Manifolds

We begin by discussing a few properties of the universal cover and deck transformations of a compact hyperbolic manifold without boundary. We show that its universal cover is the hyperbolic space $\mathbb{H}^n$, and that its group deck transformations acts by isometries. But first, we define a hyperbolic manifold.

Definition 1.1. A *hyperbolic manifold* is a Riemannian manifold (a differentiable manifold with a smoothly varying inner-product on the tangent space at every point in the manifold) with constant sectional curvature -1.

Proposition 1.2. *If X is a hyperbolic manifold, and $p : \widetilde{X} \to X$ is a covering space of X, then $\widetilde{X}$ can be given the structure of a hyperbolic manifold.*

Proof. Recall first that, as defined in *Riemannian Geometry* by M. do Carmo, a differentiable manifold is the set X and a collection of injective mappings $\mathbf{x}_\alpha : U_\alpha \to X$ of open balls $U_\alpha \subset \mathbb{R}^n$ into X such that

- **(1):** $\bigcup_\alpha \mathbf{x}_\alpha(U_\alpha) = X$
- **(2):** for any pair α, β such that $\mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta) = W \neq \emptyset$, the sets $\mathbf{x}_\alpha^{-1}(U_\alpha)$ and $\mathbf{x}_\beta^{-1}(U_\beta)$ are open sets in $\mathbb{R}^n$ and the mappings $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$ are differentiable.

Let $C = \mathbf{x}_\alpha(U_\alpha)$ for some α be a chart in X. Also, let B with $B \cap C \neq \emptyset$ be a base map, a member of a collection of open sets covering X, whose inverse images under the covering map p are distinct open sets. Note that such a collection exists because p is a covering map. Now consider any lift $D \subset p^{-1}(B \cap C)$. Denote the restriction of p to D by p'. p' is a homeomorphism onto $B \cap C$, so that $p'^{-1} \circ \mathbf{x}_\alpha$ from $E = x_\alpha^{-1}(B \cap C) \subset \mathbb{R}^n$ onto D is an injective map into $\widetilde{X}$. Thus we can construct a family of injective mappings from open sets in $\mathbb{R}^n$ into $\widetilde{X}$.

Now, we must check that (1) and (2) hold. First, let $y \in \widetilde{X}$. Since $p(y) \in X$, $p(y) \in B \cap C$ for some B and C because the collections of base sets and charts both cover X. Since y is contained in exactly one of the the sets $D \subset p^{-1}(B \cap C)$, letting $p' = p|_D$ we see that $y \in p'^{-1} \circ \mathbf{x}_\alpha(E)$, where E is as above. Now suppose that for some D_1 and D_2, $D_1 \cap D_2 \neq \emptyset$. Then $p(D_1 \cap D_2) = p(D_1) \cap p(D_2) = (B_1 \cap C_1) \cap (B_2 \cap C_2)$ where C_1 and C_2 are distinct charts in X with maps $\mathbf{x}_\alpha$ and $\mathbf{x}_\beta$. $(B_1 \cap C_1) \cap (B_2 \cap C_2)$ is non-empty, for otherwise, $D_1 \cap D_2$ would be empty. The transition function between the two charts C_1 and C_2, namely $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$, is differentiable, hence the transition function between D_1 and D_2, $\mathbf{x}_\beta^{-1} \circ p'' \circ p'^{-1} \circ \mathbf{x}_\alpha = \mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$, where $p' = p|_{D_1}$ and $p'' = p|_{D_2}$, is also differentiable.

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Furthermore, the covering map p is a local diffeomorphism. Any point y in $\widetilde{X}$ has a neigborhood in some D, so that the image of the neighborhood under p is contained entirely in $p(D) = B \cap C$ for some B and C. Since the chart map $p'^{-1} \circ \mathbf{x}_\alpha$ is injective, there is an open neighborhood $U \subset \mathbb{R}^n$ such that $p'^{-1} \circ \mathbf{x}_\alpha(U) \subset D$ Then $p(p'^{-1} \circ \mathbf{x}_\alpha(U)) = \mathbf{x}_\alpha(U)$. Also, $\mathbf{x}_\alpha^{-1}(p(p'^{-1} \circ \mathbf{x}_\alpha)) = \mathrm{Id}$, hence is differentiable. The same is true of the inverse of p. Thus, p is a local diffeomorphism.

It follows from the chain rule that for $y \in \widetilde{X}$ the differential $dp_y : T_y\widetilde{X} \to T_{p(y)}X$ is an isomorphism. Recall that a Riemannian metric is characterized by an inner product on the tangent space of every point. We define an inner product on $T_y\widetilde{X}$ as follows: let $\langle u, v\rangle_y = \langle \mathrm{d}p_y(u), \mathrm{d}p_y(v)\rangle_{p(y)}$. As defined, p is a *local isometry*, a map which preserves the norm on each tangent space.

It can be shown using more advanced differential geometry that sectional curvature is preserved under local isometries. Hence, because X has constant sectional curvature -1, and because p is a local isometry, $\widetilde{X}$ has an induced constant sectional curvature -1. □

Proposition 1.3. *If X is an n-dimensional closed, compact hyperbolic manifold, then the universal cover of X is the hyperbolic n-space, $\mathbb{H}^n$.*

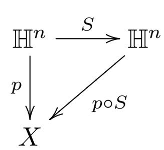
Proof. It follows from the *Cartan-Hadamard Theorem* that every simply-connected, complete hyperbolic n-manifold is isometric to $\mathbb{H}^n$. Any closed, compact manifold is complete. Hence, any covering space of it is also complete. Then, by Proposition 1, given a closed, compact hyperbolic n-manifold X, its universal cover is by definition a simply-connected covering space $\widetilde{X}$, which is complete and can be given a hyperbolic manifold structure, hence $\widetilde{X} \simeq \mathbb{H}^n$. □

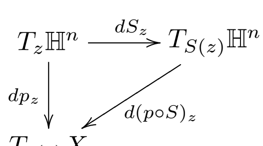
Next we will describe the relationship between the group of deck transformations in $\mathbb{H}^n$ of a hyperbolic surface and isometries of $\mathbb{H}^n$.

Proposition 1.4. *Any hyperbolic n-manifold X is the quotient $\Gamma \setminus \mathbb{H}^n$ of $\mathbb{H}^n$ by a group Γ acting properly discontinuously—in fact freely—on $\mathbb{H}^n$ by orientation-preserving isometries.*

Proof. Let Γ be the group of deck transformations in the universal cover $\mathbb{H}^n$. We will show that the group of deck transformations Γ of a hyperbolic surface X from a discrete group of orientation-preserving isometries of $\mathbb{H}^n$.

By definition, if $S \in \Gamma$, then for any $z \in \mathbb{H}^n$, $p(S(z)) = p(z)$, as in the figure below. Now consider the differentials at a point $z \in \mathbb{H}^n$ $\mathrm{d}p_z : T_z\mathbb{H}^n \to T_{p(z)}X$ and $dS_z : T_z\mathbb{H}^n \to T_{S(z)}\mathbb{H}^n$. By the chain rule, $d(p \circ S)_z = dp_z \circ dS_z$. Also, $d(p \circ S)_z = dp_z$. Hence $dp_z = dp_z \circ dS_z$.





Suppose, for $y \in \mathbb{H}^n$, that $\langle u, v\rangle_z$ is the inner product on the tangent space $T_z\mathbb{H}^n$. Recall from Proposition 1.2 that the inner product on the tangent space $T_{S(z)}\mathbb{H}^n$ is defined to be

$$\langle dS_z(u), dS_z(v)\rangle_{S(z)} = \langle dp_z(dS_z(u)), dp_z(dS_z(v))\rangle_{p(S(z)} = \langle dp_z(u), dp_z(v)\rangle_{p(z)} = \langle u, v\rangle_z$$

Thus, S preserves the inner product on every tangent space, and is therefore an isometry.

Now we shall show that Γ acts properly discontinuously by showing that every Γ-orbit is discrete and the the order of the stabilizer of each point is finite. The latter condition is automatically satisfied because deck transformations act freely. Consider any point $x \in X$. x has a contractible neigborhood in which any loop is homotopic to the trivial loop. Hence, there is a $d_x > 0$ such that every non-trivial loop through x has length greater or equal to d_x. Every point in the fiber over x in $\mathbb{H}^n$ must be at least distance d_x from any other point in the fiber, and since elements of Γ map lifts of x to lifts of x, the Γ orbit of any lift $\widetilde{x}$ of x is discrete. Since every point in $\mathbb{H}^n$ is a lift of some point in X, every Γ-orbit in $\mathbb{H}^n$ is discrete.

By *Proposition 1.40* from Hatcher, since Γ acts freely on $\mathbb{H}^n$, for any $x \in X$ and any two lifts $\widetilde{x}$ and $\widetilde{x}'$ of x, there exists $S \in \Gamma$ such that $S(\widetilde{x}) = \widetilde{x}'$. Hence, $\Gamma \setminus \mathbb{H}^n$ is X. □

Next, we shall build a proof of the theorem that any closed curve on a hyperbolic surface is homotopic to a closed geodesic on the manifold, drawing from *Fuchsian Groups* by Svetlana Katok and adapting her arguments to higher dimension.

2. Closed Curves on Hyperbolic Surfaces

We will begin this section with a brief classification of isometries of hyperbolic n-space. The following results come from hyperbolic geometry. By the translation distance of an isometry T, we mean $\inf_{x\in\mathbb{H}^n} \rho(x, T(x))$, where $\rho(\cdot,\cdot)$ is the distance function on $\mathbb{H}^n$ induced by the Riemannian metric.

Definition 2.1. A *hyperbolic transformation* is an isometry such that the translation distance is greater than 0. A hyperbolic transformation fixes two points on the boundary of $\mathbb{H}^n$ and none in $\mathbb{H}^n$ itself. It preserves its *axis*, the unique geodesic segment connecting the two fixed points, and moves all points in $\mathbb{H}^n$ along the axis.

Definition 2.2. A *parabolic transformation* is an isometry such that the translation distance is equal to 0, but is not realized by any point in $\mathbb{H}^n$. A parabolic transformation fixes a single point on the boundary of $\mathbb{H}^n$, and no points in $\mathbb{H}^n$. All points in $\mathbb{H}^n$ rotate around the fixed point. In fact, any Euclidean $(n-1)$-sphere tangent to $\mathbb{R}^{n-1} \cup \{\infty\}$ at the fixed point is preserved under a parabolic transformation.

Definition 2.3. An *elliptic transformation* is an isometry such that the translation distance is equal to 0 and is realized by a single point in $\mathbb{H}^n$. An elliptic transformation fixes one point in $\mathbb{H}^n$, while all other points rotate about the fixed point.

Definition 2.4. For a group of isometries Γ acting properly discontinuously, and a point p not fixed by any element of Γ except Id, a *Dirichlet region* centered at p is

$$D_p(\Gamma) = \{z \in \mathbb{H}^n | \rho(p, z) \le \rho(z, T(p)) \forall T \in \Gamma\}.$$

$D_p(\Gamma)$ is a fundamental region for the action of Γ.

Lemma 2.5. *The vertices of a Dirichlet region F=$D_p(\Gamma)$ centered at p of a group Γ of isometries acting properly discontinously on $\mathbb{H}^n$ are isolated in $\mathbb{H}^n$.*

Proof. Suppose the vertices are not isolated. Then there is a limit point v_0 of the vertices in $\mathbb{H}^n$. In any neighborhood of v_0 with radius ϵ, F has infinitely many vertices v_n. Since each vertex is the intersection of n geodesic hyperplanes, there are also infinitely many geodesic segments contained entirely inside in the same neighborhood of v_0. In fact, all but finitely many of the hyperplane segments connecting the vertices inside the neighborhood are contained entirely inside of it. Furthermore, each geodesic hyperplane segment is the perpendicular bisector of the unique geodesic line-segment connecting the points p and $T(p)$ for some $T \in \Gamma - \mathrm{Id}$. That is, the point $T(p)$ is twice the hyperbolic distance from p as the bisecting geodesic segment lying in the ϵ-neighborhood of v_0. Let r be twice the distance from p to v_0. Then the endpoint of every segment connecting p to each $T(p)$ passing through the neighboorhood of v_0 lies inside the closed ball $\overline{B(p, r+2\epsilon)}$ of radius $r + 2\epsilon$ centered at p. But there are infinitely many such $T(p)$ inside of $\overline{B(p, r+2\epsilon)}$, which is a compact subset of $\mathbb{H}^n$. This contradicts the discreteness of the Γ-orbit of p, which contradicts the properly discontinuous action of Γ. Hence the vertices of $D_p(\Gamma)$ are isolated. □

Proposition 2.6. *For any group Γ acting properly discontinuously on $\mathbb{H}^n$ by isometries, if the hyperbolic manifold $\Gamma \setminus \mathbb{H}^n$ is compact, then its volume $\nu(\Gamma \setminus \mathbb{H}^n) < \infty$ and Γ contains no parabolic elements.*

Proof. First, we will show that if Γ has a non-compact Dirichlet region, then $\Gamma \setminus \mathbb{H}^n$ is non-compact. Consider oriented geodesic rays r coming out of p. Recall that a Dirichlet region is convex, so r either intersect the boundary ∂F of F once or lie entirely inside F. Define $l(r)$ to be the length of the segment of r from p to ∂F. If r lies inside F, $l(r) = \infty$. $l(r)$ is continuous where it is not ∞, so if it is bounded for all r, F is compact. If F is not compact, there is at least one r such that $l(r) = \infty$. Thus, after indentifying congruent sides and vertices of F, the resulting orbifold is non-compact.

Now, we will show that if Γ has a compact dirichlet region $D_p(\Gamma)$, that it contains no parabolic elements. Recall from the proof of Proposition 1.4, for each $x \in \Gamma \setminus \mathbb{H}^n$, there exists $d_x > 0$, the length that each loop must attain if it is to be non-trivial. If $\Gamma \setminus \mathbb{H}^n$ is compact, then $\inf_x d_x = d$ is realized and it is greater than 0 because any non-trivial loop has non-zero length. Then the minimum distance between a point in $D_p(\Gamma)$ and a point in its orbit is d. Suppose Γ contains a parabolic element S. Then $\inf_{z \in D_p(\Gamma)} \rho(z, S(z)) = 0$ because a parabolic element moves points an arbitrarily small distance, and we obtain a contradiction.

Therefore, if $\Gamma \setminus \mathbb{H}^n$ is compact, it has a compact Dirichlet region, and if it has a compact Dirichlet region, it contains no parabolic elements. □

Note that since Γ acts freely on $\mathbb{H}^n$, it cannot contain any elliptic elements either because each one has a fixed point in $\mathbb{H}^n$.

Before stating and proving the final theroem, it will be important to clarify the notion of a *closed geodesic*. A closed geodesic is a closed path which locally minimizes the distance between two points on it at every point on the path. A closed geodesic can therefore be defined as an embedding of S^1 into the manifold $\Gamma \setminus \mathbb{H}^n$ with the characteristic property of minimizing distances. This is distinct from an embedding of $[0, 1]$ into $\Gamma \setminus \mathbb{H}^n$ as a geodesic segment because even if it begins and ends at the same point p, it can fail to satisy the condition of locally minimizing distance in a neighborhood of p. This distinction is crucial to the theorem because, if we did require the geodesic we seek to be closed (allowing merely a closed geodesic segment), we would be able to remove the hypothesis that the manifold be compact.

Theorem 2.7. *Any closed curve on a closed, compact hyperbolic manifold X is homotopic to a closed geodesic on X.*

Proof. Let $\gamma : [0, 1] \to X$ be a closed curve in X. We can lift γ to a path $\widetilde{\gamma}$ in $\mathbb{H}^n$ such that $p(\widetilde{\gamma}(0)) = p(\widetilde{\gamma}(1))$.

Let Γ be the group of deck transformations for X in $\mathbb{H}^n$. By proposition 1.4, Γ is a discrete group of isometries. By proposition 2.7, Γ contains only hyperbolic elements and acts properly discontinuously and freely on $\mathbb{H}^n$. Furthermore, by *Proposition 1.39* from Hatcher, Γ is isomorphic to $\pi_1(X)$.

There is a hyperbolic transformation $T \in \Gamma$ which takes $\widetilde{\gamma}(0)$ to $\widetilde{\gamma}(1)$. Since T is hyperbolic, it fixes exactly two points on the Euclidean boundary $\mathbb{R}^{n-1} \cup \{\infty\}$ of $\mathbb{H}^n$ and fixes set-wise the axis A the unique geodesic that connects the fixed points of T.

There exists a free homotopy f_t of γ in X such that the point $\gamma(0) = \gamma(1)$ of γ is "homotoped" onto the projection $p(A)$ in X of the geodesic fixed by T in $\mathbb{H}^n$. By the homotopy lifting property (*Proposition 1.30* from Hatcher), f_t lifts to a unique homotopy, $\widetilde{f_t}$ in $\mathbb{H}^n$. Denote $\widetilde{f_1}(\widetilde{\gamma}(0)) = z_0$ and $\widetilde{f_1}(\widetilde{\gamma}(1)) = z_1$. Since $p(z_0) = p(z_1)$ and since $f_1(\gamma)$ is homotopic to γ, we have that $T(z_0) = z_1$.

Recall that any loop in X lifts to a path in the universal cover $\mathbb{H}^n$. Because of the bijection between elements of the fundamental group $\pi_1(X)$ and Γ, the deck transformation taking the beginning of the lifted path to its end represents the same element of $\pi_1(X)$ as the loop in X. Hence any other path with the same end points in $\mathbb{H}^n$ represents the same element of $\pi_1(X)$, so the projections of both paths are homotopic loops.

Therefore, $p(\widetilde{f_1}(\widetilde{\gamma})) = f_1(\gamma)$ is homotopic to $p(A)$. Since homotopy equivalence is an equivalence relation, it follows that γ is homotopic to $p(A)$.

It remains to show that the projection $p(A)$ of the axis of T onto X is a closed geodesic in X. Consider the points z_0 and z_1 in A. Since A is fixed by T, any geodesic segment of A ending at z_0 is mapped isometrically under T to a segment of A ending at z_1. Similarly, any geodesic segment of A beginning at z_0 is mapped isometrically under T to a segment of A beginning at z_1. Hence, the projection of the segment of A from z_0 to z_1 is a closed geodesic in X. □

References

[1] M. P. do Carmo. Riemannian Geometry. Birkhäuser Boston. 1992.
[2] A. Hatcher. Algebraic Topology. Cambridge University Press. 2002.
[3] S. Katok. Fuchsian Groups. University of Chicago Press. 1992.