INTRODUCTION TO BROWNIAN MOTIONS

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ABSTRACT. This paper aims to present some basic facts about Brownian Motions. It will assume a basic familiarity with probability and random variables. It will begin by defining Brownian Motion in one dimension on the dyadic rationals using a countable number of random variables and then proceed to generalize this to the real line using a continuity argument. Some other consequences of continuity will be proved and then differentiability will be examined.

CONTENTS

1. Introduction

The notion of a Brownian Motion in one dimension may be illustrated by an ant on a long track, walking back and forth. The ant can only move in one direction, but it is free to change directions as often as it likes and at whatever speed it likes, although the ant's speed is controlled enough that there is some sense of "average" speed. We also assume that the ant is completely without memory and the track is entirely featureless, so that the ant's movement is entirely independent of its previous decisions and its current location. Assuming the ant starts at 0, we can consider the probability distribution of the ant's location at time t, denoted by W_t. A few characteristics suggest themselves right away:

(1) The probability distribution $f_{W_t}(x)$ should be continuous in x.
(2) The distribution should be symmetric.
(3) The variance of W_t should be increasing with t.
(4) The distribution should be additive, i.e. $f_{W_{t-s}} = f_{W_t - W_s}$.
(5) The distribution should be self-similar, i.e. $f_{W_t} = f_{W_{t \cdot s}} \cdot g(s)$ for some function g.

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1 is based on assuming that if the ant can travel a certain distance x in time t, that for some ϵ the ant can travel $x + \epsilon$ almost as easily. This suggests that we need to use continuous random variables as opposed to discrete random variables, but it is important to note that this is *not* the same as saying W_t is a continuous function of t. (As illustrated, we should also expect this to be true, because it is equivalent to saying the ant never teleports, but it will need to be proved, whereas we will base our definition on the use of continuous random variables.) 2 & 3 should be obvious from the illustration: the ant has no preference for positive wandering or negative wandering, so it is as likely to be at x as at -x, but it is more likely to be farther from its starting point as it has wandered more. 4 is a basic feature of the assumption that the ant has no memory. 5 is based on picturing the ant making infinitely many decisions in any time interval, so that as time passes, the only difference in what the ant may have done is that it may have moved farther (i.e. the variance may have increased).

2. Formal Definitions

Definition 2.1. A one-dimensional Brownian Motion on a set $S \subset \mathbb{R}$ is a function W_t that maps S to the space of real-valued random variables such that $W_0 = 0$ and if $r \leq s \leq t$ then $W_t - W_s \sim N(0, \sigma^2 \cdot (t-s))$ and $W_t - W_s$ is independent of W_r. If $\sigma^2 = 1$, we call W a standard Brownian motion.

Note how naturally this definition fits our properties outlined above. Normal random variables are continuous and symmetric, all that changes as more time elapses is the variance, and since we have defined the distributions for the *difference* of variables, additivity comes right away.

The obvious question is for what S is it possible to construct such a function. For starters, we shall be content to consider S in the non-negative reals, as it saves our ant considerable trouble if we let it start at a definite point in time. Moreover, it is easier to begin with a countable domain, so we will use the dyadic rationals, $\mathcal{D}$, where $\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n$ and $\mathcal{D}_n = \bigcup_{k=0}^{\infty} \frac{k}{2^n}$. This will prove to be a particularly useful choice of domains when we view Brownian Motion as the limit of random walks, which we will now define.

Definition 2.2. A one dimensional random walk is a function X_t that maps $\mathbb{N}$ to the space of real-valued random variables such that $X_t = \Sigma_{i=1}^{t} B_i$, where $\{B_i\}$ is a set of i.i.d. random variables that equal 1 or -1 each with probability 1/2.

The important part of this definition is that both the time and space dimensions are discrete. We will still consider it a random walk if we have as the domain $\frac{\mathbb{N}}{c_1}, c_1 \in \mathbb{N}$. We will also allow space to be scaled, such that $\{B_i\}$ are divided by some constant c_2. Thus, we might have as a random walk $X_t = \Sigma_{i=1}^{c_1 \cdot t} \frac{B_i}{c_2}$. It is reasonable and correct to think of a Brownian Motion as the limit of rescaling a random walk, where c_1 and c_2 go to infinity, i.e. motion in continuous space and time is the limit of making more frequent but smaller decisions in discrete space and time.

3. Construction

Proposition 3.1. *If* $\{X_{t,n} | t \cdot 2^{2n} \in \mathbb{N}\}, X_{t,n} = \Sigma_{i=1}^{c_1(n)\cdot t} \frac{B_{i,n}}{c_2(n)}$ *is a collection of random walks such that* $c_1(n) = 2^{2n}$*,* $c_2(n) = 2^n$*, and the* $B_{i,n}$ *are i.i.d. random variables which equal* ± 1 *with probability 1/2, then* $X_{t,n} \xrightarrow{n} W_t$*, a standard Brownian Motion.*

Remark 3.2. It should be clear from the statement why the dyadics were a good choice for the domain of our Brownian Motion. Space is being scaled in the simplest way that includes all lattice points from the n^{th} random walk in the $n+1^{st}$ and time is being scaled so as to keep variance constant. Also note that $\forall t \in \mathcal{D}\ \exists N$ such that $\forall n \geq N, X_{t,n}$ is defined. Also note that the $B_i's$ have gained an n index, which is essentially meaningless except to emphasize that after each rescaling, they are redrawn rather than recycled.

Proof. We have by the Central Limit Theorem that

$$c_2(n) \cdot X_{t,n} = \sum_{i=1}^{c_1(n)\cdot t} B_{i,n} \overset{a}{\sim} N(0, c_1(n) \cdot t) \tag{3.3}$$

$$X_{t,n} = \sum_{i=1}^{c_1(n)\cdot t} \frac{B_{i,n}}{c_2(n)} \overset{a}{\sim} N(0, \frac{c_1(n)}{c_2(n)^2} \cdot t) \tag{3.4}$$

Since $c_1(n) = 2^{2n} = (2^n)^2 = c_2(n)^2$, we have

$$X_{t,n} \overset{a}{\sim} N(0,t) \tag{3.5}$$

So let $W_t = \lim_{n\to\infty} X_{t,n}$ and $W_0 = 0$. Then for $t > s$ we have:

$$W_t - W_s = \lim_{n\to\infty} X_{t,n} - \lim_{n\to\infty} X_{s,n} \tag{3.6}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{2^{2n}\cdot t} \frac{B_{i,n}}{2^n} - \lim_{n\to\infty} \sum_{i=1}^{2^{2n}\cdot s} \frac{B_{i,n}}{2^n} \tag{3.7}$$

$$= \lim_{n\to\infty} \sum_{i=1+2^{2n}\cdot s}^{2^{2n}\cdot t} \frac{B_{i,n}}{2^n} \tag{3.8}$$

$$\sim \lim_{n\to\infty} \sum_{i=1}^{2^{2n}\cdot (t-s)} \frac{B_{i,n}}{2^n} \sim N(0, t-s) \tag{3.9}$$

It should also be intuitively clear by looking at the indices of summation in (3.8) that if $r \leq s \leq t$, $X_{r,n} \perp (X_{t,n} - X_{s,n})$ because for all n large enough that $X_{r,n}$, $X_{s,n}$, $X_{t,n}$ are defined, $(X_{t,n} - X_{s,n})$ and $X_{r,n}$ sum over disjoint $B_{i,n}$. We formally check whether this holds in the limiting case by checking whether the joint probability density function, $f_{W_t - W_s, W_r}$, factors into the product of the marginal densities $f_{W_t - W_s} \cdot f_{W_r}$.

$$f_{W_t-W_s}(x) \cdot f_{W_r}(y) = \lim_{n\to\infty} f_{X_{t,n}-X_{s,n}}(x) \cdot \lim_{m\to\infty} f_{X_{r,m}}(y) \tag{3.10}$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} f_{X_{t,n}-X_{s,n}}(x) \cdot f_{X_{r,m}}(y) \tag{3.11}$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} f_{X_{t,n}-X_{s,n}, X_{r,m}}(x,y) \tag{3.12}$$

To see that $f_{X_{t,n}-X_{s,n},X_{r,m}}(x,y) \xrightarrow{m} f_{X_{t,n}-X_{s,n},W_r}(x,y)$ in measure, we check that $(X_{t,n} - X_{s,n}) \times X_{r,m} \xrightarrow{m} (X_{t,n} - X_{s,n}) \times W_r$ in probability for arbitrary n. Let $\epsilon > 0$.

$$\lim_{m\to\infty} P(|(X_{t,n} - X_{s,n}) \times W_r - (X_{t,n} - X_{s,n}) \times X_{r,m}| > \epsilon) \tag{3.13}$$

$$= \lim_{m\to\infty} P(|W_r - X_{r,m}| > \epsilon) = 0 \tag{3.14}$$

This could be iterated again for n to give $(X_{t,n} - X_{s,n}) \times X_{r,m} \xrightarrow{m,n} (W_t - W_s) \times W_r$. Plugging this into (3.12), we have

$$f_{W_t-W_s}(x) \cdot f_{W_r}(y) = f_{W_t-W_s,W_r}(x,y) \tag{3.15}$$

which is our independence condition. □

4. Continuity

Having neatly defined and constructed a Brownian Motion on $\mathcal{D}$, we would like to have it continuous, so as to be able to extend $\mathcal{D}$ to $\mathbb{R}$.

Theorem 4.1. *With probability 1, if $R \in \mathbb{N}$ the map $t \mapsto W_t$ is uniformly continuous on $\mathcal{D} \cap [0, R]$.*

Remark 4.2. A statement that something like uniform continuity, which sounds rather universal, is true with a certain probability, may sound a little bizarre, perhaps even superfluous since that probability is 1. It makes sense, however, in the same way that a continuous random variable is equal to 17 with probability zero: there exist events within the sample space such that the variable equals 17, but those events have probability measure zero. Similarly, we show that the events which lead to a discontinuous map $t \mapsto W_t$ have measure zero (or that their complement has measure 1, which is equivalent). The only hitch is that it is not obvious how to make statements about the probability that

$$\forall \epsilon > 0\ \exists \delta > 0 \ni \forall t, s \in \mathcal{D} \cap [0, 1], |t - s| < \delta \Rightarrow |W_t - W_s| < \epsilon.$$

To begin, we translate it to set theoretic notation:

$$\bigcap_{\epsilon>0} \bigcup_{\delta>0} |t - s| < \delta \Rightarrow |W_t - W_s| < \epsilon,$$

which is a fine mess, because we have an uncountable intersection and an uncountable union, neither of which is well handled by measure operators. Handling the implication could also get a little sloppy. Hence, we employ some lemmas to reduce uniform continuity to some more tractable statements that involve at most countable unions and intersections.

Lemma 4.3. *If*

$$Z_n = \sup\{|W_s - W_t| : |s - t| \leq 2^{-n}, s, t \in \mathcal{D} \cap [0, 1]\},$$

then $Z_n \to 0 \iff W$ is uniformly continuous.

Proof. " $\Rightarrow$ " Suppose $Z_n \to 0$. Let $\epsilon > 0$. Then $\exists N$ such that $\forall n > N$, $Z_n < \epsilon$. Let $\delta < 2^{-n}$. Then $|s - t| < \delta \Rightarrow |W_s - W_t| \leq Z_n < \epsilon$.
" $\Leftarrow$ " Suppose the map is uniformly continuous. Then given $\epsilon > 0\, \exists \delta > 0$ such that $|t - s| < \delta \Rightarrow |W_t - W_s| < \epsilon$. Then $\exists N$ such that $n > N \Rightarrow 2^{-n} < \delta$. Then, $\forall n > N, Z_n < \epsilon$. □

Lemma 4.4. *Let*

$$M(k,n) = \sup\left\{\left|W_q - W_{\frac{k-1}{2^n}}\right| : q \in \mathcal{D} \cap \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]\right\},$$
$$M_n = \max\{M(1,n), M(2,n), \ldots, M(2^n,n)\}.$$

Then $M_n \to 0 \iff Z_n \to 0$.

Proof. " $\Leftarrow$ " Rewriting M_n as

$$M_n = \sup\{|W_t - W_s| : |s-t| \leq 2^{-n}, s > t, t \in \mathcal{D}_n \cap [0,1], s \in \mathcal{D} \cap [0,1]\}$$

we see that it is the same as Z_n except with more parameters for the supremum. By monotonicity of the supremum ($A \subset B \Rightarrow \sup A \leq \sup B$), we have $M_n \leq Z_n$. " $\Rightarrow$ " Suppose $|s-t| \leq 2^{-n}$ & $s, t \in \mathcal{D} \cap [0,1]$. WLOG, let $s < t$. Let k be such that $\frac{k}{2^n} \leq t < \frac{k+1}{2^n}$. Then

$$\frac{k-1}{2^n} = \frac{k}{2^n} - \frac{1}{2^n} \leq t - 2{-}n \leq s < t < \frac{k+1}{2^n}$$
$$\frac{k-1}{2^n} \leq s < \frac{k+1}{2^n}$$

If $s \in [\frac{k}{2^n}, \frac{k+1}{2^n})$, then

$$|W_s - W_t| = |W_s - W_{\frac{k}{2^n}} + W_{\frac{k}{2^n}} - W_t| \leq |W_s - W_{\frac{k}{2^n}}| + |W_{\frac{k}{2^n}} - W_t| \leq 2M_n$$

If $s \in [\frac{k-1}{2^n}, \frac{k}{2^n})$, then

$$\begin{aligned}|W_s - W_t| &= |W_s - W_{\frac{k-1}{2^n}} + W_{\frac{k-1}{2^n}} - W_{\frac{k}{2^n}} + W_{\frac{k}{2^n}} - W_t| \\ &\leq |W_s - W_{\frac{k-1}{2^n}}| + |W_{\frac{k-1}{2^n}} - W_{\frac{k}{2^n}}| + |W_{\frac{k}{2^n}} - W_t| \leq 3M_n\end{aligned}$$

Thus $Z_n \leq 3M_n$. □

Proof of Theorem 4.1 *Claim 1:*

$$\mathbb{P}\{M_n \geq \epsilon\} = 2^n \mathbb{P}\{M(1,n) \geq \epsilon\} = 2^n \mathbb{P}\{M(1,0) \geq 2^{n/2}\epsilon\}$$

Proof of Claim 1: Index $q \in \mathcal{D} \cap [0, 2^{-n}]$ by $r_n(m) = q, m \in \mathbb{N}$. Then

$$M(1,n) \geq \epsilon \iff \exists m \in \mathbb{N} \ni W_{r_n(m)} \geq \epsilon \tag{4.5}$$

$$\mathbb{P}\{M(1,n) \geq \epsilon\} = \mathbb{P}\left\{\bigcup_{m=1}^{\infty} W_{r_n(m)} \geq \epsilon\right\} \tag{4.6}$$

Since $2^{-n/2} W_{2^n r_n(m)} \sim N(0, (2^{-n/2})^2 \cdot 2^n \cdot r_n(m)) = N(0, r_n(m))$, we may rewrite (4.6) as:

$$\mathbb{P}\{M(1,n) \geq \epsilon\} = \mathbb{P}\left\{\bigcup_{m=1}^{\infty} W_{2^n r_n(m)} \geq 2^{n/2}\epsilon\right\} \tag{4.7}$$

$$= \mathbb{P}\{M(1,0) \geq 2^{n/2}\epsilon\}, \tag{4.8}$$

since $2^n r_n(m)$ now indexes $\mathcal{D} \cap [0,1]$. Also, we may treat $M(1,n), \ldots, M(2^n,n)$ as i.i.d. random variables, independent by our assumption that for $r \leq s \leq t, (W_t -$

$W_s) \perp W_r$ and identically distributed by repeating (4.5)-(4.8), replacing $W_{r_n(m)}$ with $W_{k/2^n+r_n(m)} - W_{k/2^n}$. Thus,

$$\mathbb{P}\{M_n \geq \epsilon\} = \mathbb{P}\left\{\bigcup_{k=1}^{2^n} M(k,n) \geq \epsilon\right\} \tag{4.9}$$

$$\leq \sum_{k=1}^{2^n} \mathbb{P}\{M(k,n) \geq \epsilon\} \tag{4.10}$$

$$= 2^n \mathbb{P}\{M(1,n) \geq \epsilon\} \tag{4.11}$$

$$= 2^n \mathbb{P}\{M(1,0) \geq 2^{n/2}\epsilon\} \tag{4.12}$$

Claim 2:

$$\forall a > 0,\ \mathbb{P}\{M(0,1) \geq a\} \leq 4\mathbb{P}\{W_1 \geq a\}$$

We first simplify by symmetry. Let $M(0,1)^* = \sup\{W_t | t \in \mathcal{D} \cap [0,1]\}$. Clearly, $M(0,1) \geq M(0,1)^*$ with equality holding half the time. We now use conditional probabilities:

$$\mathbb{P}\{W_1 > a | M(0,1)^* > a\} = \mathbb{P}\{W_1 > a | W_t > a \text{ for some } t \in (0,1]\} \tag{4.13}$$

$$\geq \mathbb{P}\{W_1 - W_t > 0 | W_t > a, t \in (0,1]\} \tag{4.14}$$

We have proved that $(W_1 - W_t) \perp W_t$, thus:

$$\mathbb{P}\{W_1 - W_t > 0 | W_t > a, t \in (0,1]\} = \mathbb{P}\{W_1 - W_t > 0\} = 1/2 \tag{4.15}$$

Employing the definition of conditional probability and (4.14), we have:

$$1/2 \leq \mathbb{P}\{W_1 > a | M(0,1)^* > a\} = \frac{\mathbb{P}\{W_1 > a \cap M(0,1)^* > a\}}{\mathbb{P}\{M(0,1)^* > a\}} \tag{4.16}$$

Noting that $W_1 > a \Rightarrow M(0,1)^* > a$, we rewrite the numerator as $\mathbb{P}\{W_1 > a\}$ and rearrange the inequality, yielding:

$$\mathbb{P}\{M(0,1)^* > a\} \leq 2\mathbb{P}\{W_1 > a\} \tag{4.17}$$

If we defined $M(0,1)^{-*} = \inf\{W_t | t \in \mathcal{D} \cap [0,1]\}$, we would have:

$$\mathbb{P}\{M(0,1) > a\} = \mathbb{P}\{M(0,1)^* > a \cup M(0,1)^{-*} < -a\} \tag{4.18}$$

$$\leq \mathbb{P}\{M(0,1)^* > a\} + \mathbb{P}\{M(0,1)^{-*} < -a\} \tag{4.19}$$

$$= 2\mathbb{P}\{M(0,1)^* > a\} \tag{4.20}$$

$$\leq 4\mathbb{P}\{W_1 > a\} \tag{4.21}$$

We now incorporate our claims and lemmas to conclude:

$$\mathbb{P}\{t \mapsto W_t \text{ is unif. cont.}\} = \mathbb{P}\{M_n \to 0\} \tag{4.22}$$

Given some sequence $\{\epsilon_n\}_n \to 0$, we see that if $\exists N \in \mathbb{N}\ \forall n > N,\ M_n < \epsilon_n \Rightarrow M_n \to 0$. Thus,

$$\mathbb{P}\{M_n \to 0\} \geq \mathbb{P}\left\{\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} M_n < \epsilon_n\right\} \tag{4.23}$$

$$\text{by De Morgan's law} = 1 - \mathbb{P}\left\{\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} M_n \geq \epsilon_n\right\} \tag{4.24}$$

We recall a basic proposition of measure theory: for a finite measure space (such as a probability space), if $\forall i \in \mathbb{N} A_N \supset A_{N+1}$, then

$$\mu\left\{\bigcap_{i=1}^{\infty} A_N\right\} = \lim_{i\to\infty} \mu\{A_N\} \tag{4.25}$$

Taking our A_N's to be $\bigcup_{n=N}^{\infty} M_n \geq \epsilon_n$, it is obvious that $A_N \supset A_{N+1}$ just by looking at the indices of the union. Thus,

$$\mathbb{P}\left\{\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} M_n \geq \epsilon_n\right\} = \lim_{N\to\infty} \mathbb{P}\left\{\bigcup_{n=N}^{\infty} M_n \geq \epsilon_n\right\} \tag{4.26}$$

$$\leq \lim_{N\to\infty} \sum_{n=N}^{\infty} \mathbb{P}\{M_n \geq \epsilon_n\} \tag{4.27}$$

$$\text{by claims 1 \& 2:} \quad \leq \lim_{N\to\infty} \sum_{n=N}^{\infty} 2^{n+2}\mathbb{P}\{W_1 \geq 2^{n/2}\epsilon_n\} \tag{4.28}$$

Note that we are now essentially done, because all we have to do is choose a sequence $\epsilon_n \to 0$ in such a way that the series above converges, because

$$\sum_{n=N}^{\infty} a_n < \infty \Rightarrow \lim_{N\to\infty} \sum_{n=N}^{\infty} a_n = \lim_{N\to\infty}\left[\left(\sum_{n=1}^{\infty} a_n\right) - \left(\sum_{n=1}^{N} a_n\right)\right] \tag{4.29}$$

$$= \left(\sum_{n=1}^{\infty} a_n\right) - \lim_{N\to\infty}\left(\sum_{n=1}^{N} a_n\right) \tag{4.30}$$

$$= \left(\sum_{n=1}^{\infty} a_n\right) - \left(\sum_{n=1}^{\infty} a_n\right) = 0. \tag{4.31}$$

Since $W_1 \sim N(0,1)$, we observe that $0 < a \leq x \Rightarrow -x^2 \leq -ax$, so

$$\Phi(a) = \int_a^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{4.32}$$

$$\leq \int_a^{\infty} \frac{1}{\sqrt{2\pi}} e^{-ax/2} \tag{4.33}$$

$$= -\sqrt{\frac{2}{\pi}}\frac{1}{a}e^{-ax/2}\Big|_a^{\infty} \tag{4.34}$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{a}e^{-a^2/2} \tag{4.35}$$

$$\therefore \Phi(2^{n/2}\epsilon_n) \leq \sqrt{\frac{2}{\pi}}\frac{2^{-n/2}}{\epsilon_n}e^{-2^n\epsilon_n^2/2}. \tag{4.36}$$

Thus, let $\epsilon_n = 2^{-n/4}$. Then

$$\Phi(2^{n/2}\epsilon_n) \leq \sqrt{\frac{2}{\pi}} 2^{-n/4} e^{-(2^{n/2})/2} \tag{4.37}$$

$$\sum_{n=N}^{\infty} 2^{n+2}\mathbb{P}\{W_1 \geq 2^{n/2}\epsilon_n\} \leq 4\sqrt{\frac{2}{\pi}} \sum_{n=N}^{\infty} 2^{3n/4} e^{-(2^{n/2})/2} \tag{4.38}$$

$$2 < e \Rightarrow \quad \leq 4\sqrt{\frac{2}{\pi}} \sum_{n=N}^{\infty} e^{\frac{3n-2\cdot 2^{n/2}}{4}} \tag{4.39}$$

$$= 4\sqrt{\frac{2}{\pi}} \sum_{n=N}^{\infty} \underbrace{e^{\frac{3n-2^{n/2}}{4}}}_{\to 0} \underbrace{e^{-\frac{2^{n/2}}{4}}}_{<e^{-n},\text{ large } n} \tag{4.40}$$

$$< \infty \tag{4.41}$$

Combining (4.22),(4.24),(4.28), and (4.41), we get:

$$\mathbb{P}\{t \mapsto W_t \text{ is uniformly continuous on [0,1]}\} = 1 - 0 = 1 \tag{4.42}$$

Note that we really only needed to assume a bounded interval. The proof could be repeated with some additional constants to prove that the map is uniformly continuous on $[0, N]$ with probability 1.

5. Extension

We now look to extend our Brownian Motion from $\mathcal{D}$ to $\mathbb{R}$. Given $x_0 \in \mathbb{R} \cap [0, 1]$. Let $x_n = \max_{k=0}^{2^n}\{k/2^n \leq x_0\}$. Then

$$x_n + 1/2^n > x_0 \Rightarrow 0 \leq x_0 - x_n < 1/2^n$$
$$\Rightarrow |x_0 - x_n| < 1/2^n \Rightarrow x_n \to x_0$$

Proposition 5.1. $\exists \lim_{n\to\infty} W_{x_n}$

Proof. Let $\epsilon > 0$. Uniform continuity lets us assign $\delta_\epsilon > 0$ such that $|t - s| < \delta_\epsilon \Rightarrow |W_t - W_s| < \epsilon/4\ \forall t, s \in \mathcal{D} \cap [0, 1]$. Let $N_\epsilon \in \mathbb{N}$ be such that $1/2^{N_\epsilon} < \delta_\epsilon$. Let $n, m > N_\epsilon$. Because $\{x_n\}$ is non-decreasing, we have $x_{N_\epsilon} \leq x_n$. We also know from the definition that $\forall n, x_n \leq x_0$ and that $x_{N_\epsilon} + 1/2^{N_\epsilon} > x_0$. Thus,

$$x_{N_\epsilon} \leq x_n \leq x_0 < x_{N_\epsilon} + 1/2^{N_\epsilon} \tag{5.2}$$

$$0 \leq x_n - x_{N_\epsilon} < 1/2^{N_\epsilon} \tag{5.3}$$

$$|x_n - x_{N_\epsilon}| < 1/2^{N_\epsilon} \tag{5.4}$$

We could follow the same steps for m and conclude

$$|x_m - x_{N_\epsilon}| < 1/2^{N_\epsilon}$$

Thus we get:

$$|W_{x_n} - W_{x_m}| \leq |W_{x_n} - W_{x_{N_\epsilon}}| + |W_{x_m} - W_{x_{N_\epsilon}}| < \epsilon/2 \tag{5.5}$$

which makes $\{x_n\}_n$ a Cauchy sequence in $\mathbb{R}$, which is a complete space, hence it is a convergent sequence. □

Definition 5.6.

$$\forall x_0 \in \mathbb{R} \cap [0, 1], W_{x_0} = \lim_{n\to\infty} W_{x_n}.$$

Remarks 5.7. This definition agrees with the former definition on $\mathcal{D}$ since $\{W_{x_n}\}_n$ is eventually constant if $x_0 \in \mathcal{D}$. We can also easily verify that $t \mapsto W_t$ is uniformly continuous on $\mathbb{R}$. Let $\epsilon < 0, |t-s| < \delta_\epsilon, n > N_\epsilon$. Then:

$$|W_t - W_s| \leq \underbrace{|W_t - W_{t_n}|}_{<\epsilon/4} + \underbrace{|W_s - W_{s_n}|}_{<\epsilon/4} + |W_{t_n} - W_{\frac{t_n+s_n}{2}}| + |W_{\frac{t_n+s_n}{2}} - W_{s_n}| \tag{5.8}$$

WLOG, let $s < t$. If we choose n high enough that $t_n \in [s,t]$, we have

$$|t_n - \frac{t_n+s_n}{2}| = |\frac{t_n+s_n}{2} - s_n| = |\frac{t_n-s_n}{2}| \leq \left(\underbrace{|t_n - s|}_{<|t-s|<\delta_\epsilon} + \underbrace{|s - s_n|}_{\delta_\epsilon}\right)/2 \tag{5.9}$$

Thus (5.8) becomes

$$|W_t - W_s| \leq \frac{4\epsilon}{4} = \epsilon \tag{5.10}$$

Note that, just as before, we could have proved that the map $t \mapsto W_t$ is continuous on $\mathbb{R} \cap [0,n]$ for any $n \in \mathbb{N}$, but it would have required using more constants. We also note that if we did this $\forall n \in \mathbb{N}$, we would be able to prove by taking $n \to \infty$ that $t \mapsto W_t$ is continuous on $\mathbb{R}$ with probability one, but uniform continuity would not hold. This is a product of the fact that continuity is a feature of points individually, so a function will be continuous on a union of sets on which it is contiuous (ignoring the possible messiness of one-sided continuity, which does not show up here). Uniform continuity, however, is a feature of sets, and as such does not necessarily survive infinite union.

6. Intermediate Value Theorem

One of the biggest advantages to continuity is that it allows us to invoke the intermediate value theorem. This is particularly helpful in considering the probability that W_t takes on a given value for $t \in A \subset \mathbb{R}$. Because W_t has a continuous distribution, the probability distribution function is useless in answering this type of question, since $\forall a \in \mathbb{R}\ \mathbb{P}\{W_t = a\} = 0$ and $\mathbb{P}\left\{\bigcup_{t\in A} W_t = a\right\}$ cannot be reduced to a sum if A is uncountable. However, if A is an interval (s,t), we can easily check

$$\mathbb{P}\{(W_s \leq a \cap W_t \geq a) \cup (W_s \geq a \cap W_t \leq a)\}.$$

Proposition 6.1. *With probability one* $\forall b > 0, \exists t > b$ *such that* $W_t = 0$.

Proof. WLOG, let $W_b = a_0 < 0$. Then let

$$p = \Phi(a_0) = \mathbb{P}\{W_{b+1} - W_b > -a_0\} < 1/2 \tag{6.2}$$

and let $q = 1 - p$. If $W_{b+1} = a_1 < 0$ then $\exists x \in \mathbb{R}$ such that

$$\mathbb{P}\{W_{b+x_2} - W_{b+1} > -a_1\} = \Phi\left(\frac{-a_1}{\sqrt{x_2}}\right) \tag{6.3}$$

Since $\lim_{x_2\to\infty} \Phi(-a_1/\sqrt{x_2}) = 1/2$, we may choose an x_2 such that $\Phi(a_1/\sqrt{x_2}) = p$. Thus, $\mathbb{P}\{W_{b+1} > 0 \text{ or } W_{b+1} < 0 < W_{b+x_2}\} = p + qp$. If we continued picking x_n in this way, the probability that the first $n-1$ variables are less than 0 but the n^{th} variable is greater than 0 will be $q^{n-1}p$. Thus

$$\mathbb{P}\left\{\bigcup_{n=1}^{\infty} W_{b+x_n} > 0\right\} = p\sum_{n=0}^{\infty} q^n = p\frac{1}{1-q} = p/p = 1 \tag{6.4}$$

Thus, by the intermediate value theorem, with probability one $\exists t > b \ni W_t = 0$. $\square$

Note that this same argument could be used to prove that the motion eventually reaches any a with probability one. Interestingly, one need not wait particularly long for the motion to return to its starting position:

Proposition 6.5. *With probability one, $\forall \epsilon > 0, \exists t \in (0, \epsilon)$ such that $W_t = 0$.*

Proof. We first consider $\mathbb{P}\{\exists t \in (0,1)$ such that $W_t = 0\}$. WLOG, assume $W_1 = a_0 < 0$. The goal is to repeat the trick above, but coming progressively closer to zero. To do this, we need to know the conditional cdf

$$t > s, \ F_{W_s|W_t}(0|W_t = a_0 < 0)$$

which we can realize by examining the corresponding pdf. To make things notationally simpler, let $X = W_s, Y = W_t - W_s, Z = Wt$, all of which are normal random variables. By definition, $X \perp Y$ and $X + Y = Z$. Denote $\text{Var}(X) = \sigma_x^2$ and $\text{Var}(Y) = \sigma_y^2$. Clearly $\text{Var}(Z) = \sigma_x^2 + \sigma_y^2$. Consider:

$$f_{X|Z}(x|z) = \frac{f_{X,Z}(x,z)}{f_Z(z)} \tag{6.6}$$

$$= \frac{f_{X,Y}(x, z-x)}{f_Z(z)} \tag{6.7}$$

$$= \frac{f_X(x) f_Y(z-x)}{f_Z(z)} \tag{6.8}$$

$$= \frac{1/(2\pi\sigma_x\sigma_y)\exp(-1/2[x^2/\sigma_x^2 + (z-x)^2/\sigma_y^2])}{1/\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}\exp(-1/2[z^2/(\sigma_x^2+\sigma_y^2)])} \tag{6.9}$$

$$= \frac{\sqrt{\sigma_x^2+\sigma_y^2}}{\sqrt{2\pi}\sigma_x\sigma_y}\exp\left[-\frac{1}{2}\cdot\frac{x^2(1+\frac{\sigma_y^2}{\sigma_x^2}) + (z-x)^2(1+\frac{\sigma_x^2}{\sigma_y^2}) - z^2}{\sigma_x^2+\sigma_y^2}\right] \tag{6.10}$$

We can tell already that the above is in the form of a normal distribution. If we expand the numerator of the argument of the exponential, replacing $R = \sigma_x^2/\sigma_y^2$, we get:

$$\begin{aligned}
&x^2 + x^2R^{-1} + z^2 - 2zx + x^2 + z^2R - 2zxR + x^2R - z^2 \\
=&x^2(2 + R^{-1} + R) - 2zx(1+R) + z^2R \\
=&[x^2 - 2zx\frac{1+R}{2+R^{-1}+R} + z^2\frac{R}{2+R^{-1}+R}](2+R^{-1}+R) \\
=&[x^2 - 2zx\frac{1+R}{2+R^{-1}+R} + z^2(\frac{1+R}{2+R^{-1}+R})^2 \\
&\underbrace{+ z^2\frac{R}{2+R^{-1}+R} - z^2\frac{1+2R+R^2}{(2+R^{-1}+R)^2}}_{=0}](2+R^{-1}+R) \\
=&[(x - z\frac{1+R}{2+R^{-1}+R})^2](2+R^{-1}+R)
\end{aligned}$$

Thus:

$$f_{X|Z}(x|z) = \frac{\sqrt{\sigma_x^2+\sigma_y^2}}{\sqrt{2\pi}\sigma_x\sigma_y}\exp\left(-\frac{1}{2}\frac{(x-z\frac{1+R}{2+R^{-1}+R})^2}{(\sigma_x^2+\sigma_y^2)/(2+R^{-1}+R)}\right) \tag{6.11}$$

Simplifying the denominator within the exponential, we have:

$$(\sigma_x^2+\sigma_y^2)/(2+R^{-1}+R) = \frac{\sigma_x^2}{2+\sigma_y^2/\sigma_x^2+\sigma_x^2/\sigma_y^2}+\frac{\sigma_y^2}{2+\sigma_y^2/\sigma_x^2+\sigma_x^2/\sigma_y^2} \tag{6.12}$$

$$= \frac{\sigma_x^2\sigma_y^2}{2\sigma_y^2+\sigma_y^4/\sigma_x^2+\sigma_x^2}+\frac{\sigma_x^2\sigma_y^2}{2\sigma_x^2+\sigma_x^4/\sigma_y^2+\sigma_y^2} \tag{6.13}$$

$$= \frac{\sigma_x^2\sigma_y^2\cdot\sigma_x^2}{2\sigma_y^2\sigma_x^2+\sigma_y^4+\sigma_x^4}+\frac{\sigma_x^2\sigma_y^2\cdot\sigma_y^2}{2\sigma_x^2\sigma_y^2+\sigma_x^4+\sigma_y^4} \tag{6.14}$$

$$= \sigma_x^2\sigma_y^2\frac{\sigma_x^2+\sigma_y^2}{(\sigma_x^2+\sigma_y^2)^2} \tag{6.15}$$

$$= \frac{\sigma_x^2\sigma_y^2}{\sigma_x^2+\sigma_y^2} \tag{6.16}$$

Thus, we have:

$$X|Z{=}z \sim N\left(z\left[\frac{1+R}{2+R^{-1}+R}\right], \frac{\sigma_x^2\sigma_y^2}{\sigma_x^2+\sigma_y^2}\right) \tag{6.17}$$

$$W_s|W_t{=}a \sim N\left(a\left[\frac{1+s/(t-s)}{2+(t-s)/s+s/(t-s)}\right], \frac{s(t-s)}{t}\right) \tag{6.18}$$

$$\sim N\left(a\left[\frac{1+s/(t-s)}{1+s/(t-s)+t/s}\right], s-\frac{s^2}{t}\right) \tag{6.19}$$

$$\sim N\left(a\left[\frac{t-\not{s}+\not{s}}{\not{t}-\not{s}+\not{s}+t^2/s-\not{t}}\right], s-\frac{s^2}{t}\right) \tag{6.20}$$

$$\sim N\left(a\frac{s}{t}, s-\frac{s^2}{t}\right) \tag{6.21}$$

Thus, if we fix t to be ϵ, we can see:

$$\mathbb{P}\{W_s > 0|W_\epsilon = a\} = \Phi\left(\frac{-a(s/\epsilon)}{\sqrt{s-s^2/\epsilon}}\right) \tag{6.22}$$

$$= \Phi\left(\frac{-a\sqrt{s\epsilon}}{\sqrt{1-s}}\right) \xrightarrow{s\to 0} \Phi(0) = 1/2 \tag{6.23}$$

Thus, for any $p \in (0, 1/2)$, we may choose some $s \in (0, \epsilon)$ such that $\mathbb{P}\{W_s > 0|W_\epsilon = a\} = p$. We may thus repeat the same trick as above and conclude

$$\forall\epsilon > 0\ \mathbb{P}\{\exists s \in (0, \epsilon) \text{ such that } W_s = 0\} = 1 \tag{6.24}$$

Note that there is a difference between the above statement and the goal of the proposition. We need to get the $\forall\epsilon > 0$ inside the probability sign. We can accomplish this by taking a monotonic sequence converging $\epsilon_n \to 0$, taking the intersection, and invoking (4.25) like so:

$$\mathbb{P}\left\{\bigcap_{n\in\mathbb{N}} \exists s \in (0,\epsilon_n) \text{ such that } W_s = 0\right\} \tag{6.25}$$

$$= \lim_{n\to\infty} \mathbb{P}\left\{\exists s \in (0,\epsilon_n) \text{ such that } W_s = 0\right\} = 1 \tag{6.26}$$

□

7. Non-Differentiability

Theorem 7.1. *With probabiility one, the map $t \mapsto W_t$ is differentiable nowhere in $(0,1)$.*

Proof. We begin with some observations about differentiable functions.
Claim 1 Suppose $\exists t \in (0,1)$ such that W_t is differentiable. Then $\exists \delta > 0, C < \infty$ such that $\forall \epsilon \in (0,\delta)$, $s, s' \in [t-\epsilon, t+\epsilon] \Rightarrow |W_s - W_{s'}| \le C\epsilon$.
Proof of Claim 1

$$\frac{|W_s - W_{s'}|}{2\epsilon} \le \frac{|W_s - W_t| + |W_t - W_{s'}|}{2\epsilon} \tag{7.2}$$

Using the Taylor expansion, we get:

$$= \frac{|W_t + (s-t)W_t' + R(s) - W_t| + |W_t + (s'-t)W_t' + R(s') - W_t|}{2\epsilon} \tag{7.3}$$

$$\le \underbrace{\frac{|s-t|}{2\epsilon}}_{\le 1/2}|W_t'| + \underbrace{\frac{|s'-t|}{2\epsilon}}_{\le 1/2}|W_t'| + \frac{2|R(s)|}{2\epsilon}, \tag{7.4}$$

where R is the remainder function. Note that the existence of the derivative guarantees that $|R(s)|/\epsilon \xrightarrow{\delta\to 0} 0$. Thus, we may choose δ such that $|R(s)|/\epsilon \le |W_t'|$, which means $C = 4|W_t'|$ completes the proof of the claim.
Claim 2 Let

$$M(k,n) = \max\left\{\left|W_{\frac{k}{n}} - W_{\frac{k-1}{n}}\right|, \left|W_{\frac{k+1}{n}} - W_{\frac{k}{n}}\right|, \left|W_{\frac{k+2}{n}} - W_{\frac{k+1}{n}}\right|\right\}$$
$$M_n = \min\{M(1,n), \ldots, M(n,n)\}$$

Then if $\exists t \in [0,1]$ such that W_t is differentiable, then $\exists C < \infty, n_0$ such that $\forall n \ge n_0,\ M_n \le C/n$.
Proof of Claim 2 Suppose W_t is differentiable at $t \in [0,1]$. We apply Claim 1 and get

$$\exists \delta > 0, C < \infty\ \forall \epsilon \in (0,\delta):\ s, s' \in [t-\epsilon, t+\epsilon] \Rightarrow |W_s - W_s'| \le C\epsilon$$

Choose n_0 such that $1/n_0 < \delta$. Then for $n > n_0$, we let $\epsilon = 1/n$. Let k be such that $t \in [\frac{k}{n}, \frac{k+1}{n}]$. It follows that

$$\frac{k-1}{n}, \frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n} \in [t-\epsilon, t+\epsilon]$$

Thus, by Claim 1,

$$|W_{\frac{k}{n}} - W_{\frac{k-1}{n}}|, |W_{\frac{k+1}{n}} - W_{\frac{k}{n}}|, |W_{\frac{k+2}{n}} - W_{\frac{k+1}{n}}| \le C\epsilon = \frac{C}{n}$$

Thus, $M_n \leq M(k,n) \leq \frac{C}{n}$.
Next we use the assumption that W_t is a Brownian Motion and consider

$$\mathbb{P}\left\{M(k,n) \leq \frac{C}{n}\right\} = \mathbb{P}\left\{\bigcap_{j=k}^{k+2} \left|W_{\frac{j}{n}} - W_{\frac{j-1}{n}}\right| \leq \frac{C}{n}\right\} \tag{7.5}$$

$$\text{independence} \Rightarrow \ = \prod_{j=k}^{j=k+2} \mathbb{P}\left\{\left|W_{\frac{j}{n}} - W_{\frac{j-1}{n}}\right| \leq \frac{C}{n}\right\} \tag{7.6}$$

$$\text{identical distributions} \Rightarrow \ = \left[\mathbb{P}\left\{\left|W_{\frac{1}{n}}\right| \leq \frac{C}{n}\right\}\right]^3 \tag{7.7}$$

$$= \left[\Phi\left(\frac{C}{n}\sqrt{n}\right) - \Phi\left(-\frac{C}{n}\sqrt{n}\right)\right]^3 \tag{7.8}$$

$$= \left[\frac{1}{\sqrt{2\pi}}\int_{-\frac{C}{\sqrt{n}}}^{\frac{C}{\sqrt{n}}} e^{-x^2/2} dx\right]^3 \tag{7.9}$$

Since $e^{-x^2/2} \leq 1 \ \forall x$, we may use the bound

$$\mathbb{P}\left\{M(k,n) \leq \frac{C}{n}\right\} \leq \left[\sqrt{\frac{2}{\pi n}} C\right]^3 \tag{7.10}$$

Using Claim 2, we consider the event

$$\exists t \in [0,1] \text{ such that } \exists W_t' \subset \exists C, n_0 \ \forall n > n_0, M_n \leq C/n \tag{7.11}$$

$$\mathbb{P}\{\exists t \in [0,1] \text{ such that } \exists W_t'\} \leq \mathbb{P}\left\{\bigcup_{C=1}^{\infty} \bigcup_{n_0=1}^{\infty} \bigcap_{n=n_0}^{\infty} M_n \leq C/n\right\} \tag{7.12}$$

$$\leq \sum_{C=1}^{\infty} \sum_{n_0=1}^{\infty} \mathbb{P}\left\{\bigcap_{n=n_0}^{\infty} M_n \leq C/n\right\} \tag{7.13}$$

$$\text{by(4.25)} \ = \sum_{C=1}^{\infty} \sum_{n_0=1}^{\infty} \lim_{n\to\infty} \mathbb{P}\left\{M_n \leq C/n\right\} \tag{7.14}$$

$$= \sum_{C=1}^{\infty} \sum_{n_0=1}^{\infty} \lim_{n\to\infty} \mathbb{P}\left\{\bigcup_{k=1}^{n} M(k,n) \leq \frac{C}{n}\right\} \tag{7.15}$$

$$\leq \sum_{C=1}^{\infty} \sum_{n_0=1}^{\infty} \lim_{n\to\infty} n\mathbb{P}\left\{M(k,n) \leq \frac{C}{n}\right\} \tag{7.16}$$

$$\leq \sum_{C=1}^{\infty} \sum_{n_0=1}^{\infty} \lim_{n\to\infty} n \left[\sqrt{\frac{2}{\pi n}} C\right]^3 = 0 \tag{7.17}$$

□

Thus the probability of $t \mapsto W_t$ being differentiable anywhere on $[0,1]$ is zero. As before, we could make the same argument for an arbitrary interval, say $[0,n]$. Since the probability of being differentiable on the union of these intervals would be less than or equal to the sum of the probabilities for each interval, it follows that the probability of being anywhere differentiable is zero.

Note that being continuous everywhere and differentiable nowhere is rather special. There are certainly ways to explicitly define such functions, but they generally seem highly contrived. With Brownian Motion, we have a process that generates infinitely many such functions which is really rather intuitive.