THE DIRAC AND WEYL SPINOR REPRESENTATIONS

MIKE THVEDT

This paper is concerned with representations of covers of subgroups of the orthogonal group of relativistic spacetime. Specifically, I look at the group $\mathrm{Spin}^+(3,1)$, the universal cover of the allowed isometries of relativistic spacetime, and its representations. These representations have implications in the field of particle physics. It builds off an earlier paper of mine, [10], which gave an elegant connection between paths in $SO(3)$ up to homotopy and the behavior of the electron under the action of spatial rotation.

1. Lie Groups and Universal Covering Groups

Recall that a Lie group is a group and a differentiable manifold such that the group operations are smooth. Also recall that the a cover of a topological space is a topological space together with a locally homeomorphic surjective map called the covering map.

The following information on universal covers and covering groups was covered in [10]. I go over it again here because this particular construction of the universal cover is of direct importance.

Definition 1.1. The *universal cover* of a topological space is a connected cover with the universal property; i.e., the universal cover of X covers all connected covers of X.

It is a property of the universal cover that it is simply connected; furthermore, any simply connected cover is homeomorphic to the universal cover with homeomorphism the covering map. We require an important theorem about universal covers: that they can be constructed as a space of paths.

Theorem 1.2. *Let X be a connected manifold with base point x_0. For any path γ starting at x_0, let $[\gamma]$ be the equivalence class of paths homotopic to γ with endpoints x_0 and $\gamma(1)$. Let U be the set of all such equivalence classes. Then U is the universal cover of X where the covering map $p: U \to X$ is defined by $[\gamma] \mapsto \gamma(1)$.*

We can construct a base on this topology as follows. For a homotopy class of paths $[\gamma] \in U$ and an open neighborhood $D \in X$ where $\gamma(1) \in D$, define $[\gamma, D]$ to be the set of all paths homotopic to any path formed by γ concatenated to some path contained entirely in D. The sets $[\gamma, D]$ give a base for U. In rougher words, two paths α and β are close to each other if α is "almost" homotopic with fixed endpoints to β and $\alpha(1)$ is close to $\beta(1)$.

To show that U is the universal cover, it needs to be shown that U is simply connected, and that this implies the universal property. For the sake of brevity I will not go into this; but the interested reader might see, for example, [8].

Definition 1.3. The *double cover* of a topological space is a cover such that the covering map is a $2:1$ map.

All connected Lie groups have a universal covering group with the covering map being a map of Lie groups. We can construct this as follows: For a Lie group G with identity e, let the group C be the universal cover where the fixed point $x_0 = e$. For $[\gamma], [\delta] \in C$, define $[\gamma\delta]$ as the equivalence class of $\gamma\delta$, where $(\gamma\delta)t = \gamma(t)\delta(t)$. Then C is the universal covering group of G. The proof that this construction works and is a Lie group this rests on the fact that operators in Lie groups act smoothly on paths in Lie groups.

This construction is the easiest construction that I have found, but it misses the universal covers for non-connected Lie groups. A more general construction as quotients of path spaces can be found in [14] pp. 62-68.

It should be pointed out that if a Lie group is not connected then its universal cover will *not* be unique as a Lie group [16]. A specific example of this will occur at the end of this paper.

2. Minkowski Space and the Lorentz Group

Physical spacetime is (we think) a manifold, but not one with the usual Riemannian metric. The mathematician Minkowski showed in 1909 that Einstein's theory of special relativity can be elegantly described if we imagine physical spacetime as what is now called Minkowski space. To do this we not only allow but require that the distance between two points can be nonnegative.

Definition 2.1. The metric space $\mathbb{R}^{p,q}$ is the vector space $\mathbb{R}^{p+q}$ with basis $\{e_1 \dots e_{p+q}\}$ together with the metric $\|\vec{x}\| = (x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2)$. The metric space $\mathbb{R}^{3,1}$ is called *Minkowski space.*

This is not a metric space in the usual sense. The condition that the metric be positive definite has been replaced with the weaker condition of nondegeneracy. Note that $\mathbb{R}^{p,q}$ and $\mathbb{R}^{q,p}$ are similar as metric spaces (i.e. their metrics differ by a constant), so they share almost all properties; they can be made the same with a simple sign flip.[1]

Conventions. For Minkowski space we usually use write the basis (e_0, e_1, e_2, e_3) such that $\|e_0\| = -1$ and $\|e_1\| = \|e_2\| = \|e_3\| = 1$. Vectors in $\mathbb{R}^{3,1}$ are written (x_0, x_1, x_2, x_3) or (t, x_1, x_2, x_3). The x components with positive index are called the *spatial components*, and x_0 or t is the *timelike component*. Likewise, the metric subspace $\mathbb{R}^3$ with the usual Euclidean metric is *space*. The subspace $\mathbb{R}$ with the negative of the Euclidean metric is *time*. Vectors with a positive norm are called spacelike; vectors with a negative norm timelike, and vectors with zero norm (excluding $\vec{0}$) are called lightlike.

Now that we have a new sort of metric, it follows that we will have new sorts of isometries.

Definition 2.2. The *orthogonal group* $O(p, q)$ is the group of isometries of $\mathbb{R}^{p,q}$. Elements of $O(p, q)$ are called rotations. $O(p, q)$ is Lie under the matrix topology. The *special orthogonal group* $SO(p, q)$ is the subgroup of $O(p, q)$ of rotations that preserve orientation. The *orthochronous special orthogonal group* $SO^+(p, q)$ is the

[1] They are essentially the same as metric spaces, but the usual Clifford algebra is different in these spaces. It turns out that all I cover in this paper, and tons more, can be constructed with Clifford algebrae, in which case one must be careful whether one is in $\mathbb{R}^{3,1}$ or $\mathbb{R}^{1,3}$.

identity component of $SO(p,q)$ as a Lie group. The group $O(3,1)$ is called the *Lorentz group.*

Like $O(4)$, $O(3,1)$ is six-dimensional. It follows from the similarity of $\mathbb{R}^{p,q}$ and $\mathbb{R}^{q,p}$ that $O(p,q) \cong O(q,p)$.

As $\mathbb{R}^3$ as a metric space is a subspace of $\mathbb{R}^{3,1}$, and $SO(3)$ is the identity component of $O(3)$, it follows that $SO(3)$ will be a subgroup of $SO^+(3,1)$. It remains to examine the rotations that alter the path of time.

Definition 2.3. A *Lorentz boost* (or simply *boost*) is a rotation $X \in SO^+(3,1)$ that fixes a plane in the spacelike subspace $\mathbb{R}^3$.

For a vector $\vec{x} = (t, x_1, x_2, x_3)$, we can write a Lorentz boost L_3 in the plane (t, x_3) through the hyperbolic angle φ as follows:

$$L_3 = \begin{pmatrix} \cosh\varphi & 0 & 0 & \sinh\varphi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\varphi & 0 & 0 & \cosh\varphi \end{pmatrix}$$

Definition 2.4. The orthochronous Lorentz group $SO^+(3,1)$ is generated by the spacelike rotations in $SO(3)$ and the Lorentz boosts. We have seen that no rotation, be it boost, spatial or both, is orientation reversing. Furthermore, it is plain from the description of Lorentz boosts that no boost can reverse the direction of time in a timelike vector. For a vector with timelike component t and spacelike components $\vec{x}$, written $(t, \vec{x})$, call T the rotation with $t \mapsto -t$ and fixing $\vec{x}$, and S the spacelike rotation in $O(3)$ with $\vec{x} \mapsto -\vec{x}$. The rotation ST is orientation preserving, so adding ST to $SO^+(3,1)$ gives $SO(3,1)$. Adding S or T to $SO(3,1)$ gives us $O(3,1)$. It turns out these are the only rotations we can add to $SO^+(3,1)$. Then $O(3,1)$ has four connected components, and $O(3,1)/SO^+(3,1) \cong V$, the Klein four-group.

3. Relativity

It remains to be shown that $\mathbb{R}^{3,1}$ is spacetime in special relativity. Einstein postulated that the laws of physics are invariant under all allowed transformations of space. Importantly, Einstein required that physical laws be the same for both stationary and moving observers. In other words, all motion is to be relative, but the speed of light absolute. How is this accomplished?

Einstein proposed we consider two coordinate systems to be equivalent, and that observers[2] see themselves as stationary in their own coordinate systems. Suppose these two coordinate systems have the same origin, and that the velocity difference between these two observers is $\vec{v}$. He showed that one coordinate system can be transformed to the other by a Lorentz boost in the $\vec{v}$ direction with angle φ where $\|\vec{v}\|/c = \tanh\varphi$. Or you can think about it this way: giving yourself a velocity $\vec{v}$ (with respect to your previous coordinate frame) is the same as giving yourself a Lorentz boost of the same velocity, and then remaining stationary. (Of course, both of these require the same energy and force.)

$c \approx 299{,}700{,}000$ meters per second, but we can set $c = 1$ if we make our units the right size. Now introduce a metric such that the distance covered by any ray of light will be 0. This is precisely the metric of Minkowski space. It follows that

[2]An observer is anything that "observes" the laws of the universe. Just about everything is an observer, including you and me.

any isometry of Minkowski space will preserve the speed of light, which is what we wanted. (Note that the magnitude of a lightlike vector will remain 0 under any Lorentz transformation in the Minkowski metric, but will generally not retain its magnitude in the Euclidean metric.) We have the result that adding velocity to a coordinate system is the same as a rotation in time.

Recall from high school physics or your favorite science fiction television show that we can get closer and closer to the speed of light but not equal or surpass it by normal means of propulsion.[3][4] This implies that a sequence of Lorentz boosts, formed by applying the same boost over and over, will take an observer arbitrarily close to, but never directly at, the speed of light. So we have the following:

Theorem 3.1. *$SO^+(3,1)$ is not compact.*

As we shall soon see, the Lorentz group is not enough to describe all the ways to rotate a particle. We also need to study the covers of $SO^+(3,1)$.

Definition 3.2. $\mathrm{Spin}(n)$ is the double covering group of $SO(n)$. $\mathrm{Spin}(p,q)$ is a double covering group of $SO(p,q)$. A group $\mathrm{Spin}(p,q)$ might not be unique.

For $n > 2$, $\mathrm{Spin}(n)$ is a universal covering group, and is simple and compact. In the theory of so-called semisimple Lie groups, these and the other simple, compact, simply connected Lie groups are the building blocks of compact Lie groups; analogous to how simple finite groups are the building blocks of finite groups.

The reason spin is important to us in this paper is because of the connection between particle physics and spin. Isometries of spacetime do not actually describe all the possible ways to rotate a particle. In particular, a rotation of 2π is not homotopic to a rotation of 0, and somehow particles know this. An electron rotated 2π in $\mathbb{R}^3$ will end up with the opposite phase [9]. Particles with this property are called *spinning*, and a spinning particle said to have spin $n/2$ will be invariant under a $4\pi/n$ rotation.

4. Representations and Spinors

Definition 4.1. A *representation* of a group G is a vector space m together with a homomorphism $\rho : G \to Aut(V)$, the automorphism group of V. When ρ is obvious (and often even when it is not) it is simply said that V is a representation of G, or V represents G. Confusingly, it is also common to refer to the map ρ itself as a representation. Also, the image $\rho(G)$ is sometimes called a representation.

Recall that a group action of G on a set X can be defined as a map $\rho : G \to S_X$, the symmetric group of X, and that S_X is precisely the group of automorphisms of X in the category of sets. Then a representation is simply a group action in the category of vector spaces. So for a representation V of G we can speak of G acting on V or vectors in V, and gV can be shorthand for $\rho(g)(V)$.

Definition 4.2. A representation V of G is *reducible* if there exists some subspace $W \subset V$ such that $gW \subset W$ for $\forall g \in G$. An *irreducible representation*, or *irrep*, is one that is not reducible.

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[3] Unless you live in the Star Wars universe.

[4] Recently there have been reports in mainstream media of the speed of light "being broken". This is merely a consequence of mainstream media being bad at science reporting. The speed of light has never been observed to have been broken.

Some representations are more or less the same:

Definition 4.3. Two representations $\rho_1 : G \to V$ and $\rho_2 : G \to W$ (over the same field) are said to be *equivalent* if there exists an isomorphism $\alpha : V \to W$ such that G acts the same on both; i.e., $\alpha \circ \rho_1(g) \circ \alpha^{-1} = \rho_2(g)$.

It turns out that any finite-dimensional representation of a so-called compact semisimple Lie group can be decomposed into the direct sum of irreps [5]. Unfortunately for us, we are interested in representations of covers of $SO^+(3,1)$, which is not compact. Nevertheless, there exist clever tricks for finding the finite-dimensional irreps that we need. The representations we need have a name:

Definition. A representation of a spin group is called a *spinor representation.* Elements of this representation are called *spinors*.

In [10] I touched upon spinor representations of Spin(3). There is an isomorphism $\text{Spin}(3) \cong SU(2)$, and $SU(2)$ acting on $\mathbb{C}^2$ is of course a representation of Spin(3). This representation consists of two-dimensional column spinors with complex entries. These spinors are used to describe the states of electrons in three dimensions. To an electron, we assign an *angular momentum vector* that describes the orientation of a particle in space. In particular, I showed the following:

Theorem 4.4. *Action on spinors of electrons in 3-space by rotation is described completely by* Spin(3)*, the group of rotation paths up to fixed-endpoint homotopy in* $SO(3)$*. Furthermore, each rotation path in* Spin(3) *has precisely the same effect on a particle's angular momentum vector as it does on the angular momentum vector of an ordinary spinning object in classical physics.*

This theorem follows from the two-valued action of $SO(3)$ on spinors by factoring through to Spin(3), and that a rotation of 2π is not the same as the identity.

According to [7], the group $\text{Spin}(3,1)$ acts on particles in Minkowski space in the expected way, which is all we need to infer that this theorem extends to particles in spacetime.

5. The Weyl spinor representations of $\text{Spin}^+(3,1)$

First we need a mathematically convenient description of $\text{Spin}^+(3,1)$, the universal cover of $SO^+(3,1)$.

Claim. $\text{Spin}^+(3,1) \cong SL(2,\mathbb{C})$.

Proof. Proof by exhibition.

Let A be the set of 2x2 Hermitian matrices. A is a metric space over $\mathbb{R}$ (but not $\mathbb{C}$) with norm the determinant. Define an isomorphism $\sigma : \mathbb{R}^{3,1} \cong A$ as follows:

$$\begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{bmatrix} t - x_3 & x_2 - ix_1 \\ x_2 + ix_1 & t + x_3 \end{bmatrix}$$

This is an isomorphism of vector spaces. Furthermore, a simple calculation shows that, for $\vec{v} \in \mathbb{R}^{3,1}$, $\det(\sigma(\vec{v})) = -\|\vec{v}\|$, so σ is a similarity of metric spaces.

Let $SL(2,\mathbb{C})$ act on A not in the usual way but in the following way:

$$a \mapsto sas^*$$

where $s \in SL(2,\mathbb{C})$. Call such a transformation g_s. $(sas^*)^* = sa^*s^* = sas^*$, so this action preserves hermitianivity. It is clear that $g_{st} = g_s g_t$, and that the action of all these g's is smooth. So $SL(2,\mathbb{C})$ acts smoothly on A, and on $\mathbb{R}^{3,1}$ through the similarity σ. This gives a map of Lie groups $\rho : SL(2,\mathbb{C}) \to O(3,1)$.

Let $k \in ker(\rho)$. Then $a = kak^*$ for all $a \in A$. Then $I = kk^*$ implying $k^* = k^{-1}$, so $a = kak^{-1}$. This means that k is in the center of $SL(2,\mathbb{C})$, so $k = \pm I$. So ρ is a $2:1$ map.

Now $SL(2,\mathbb{C})$ is simply connected. Then the image of ρ is also connected, and can only be the identity component of $O(3,1)$. Hence $SL(2,\mathbb{C}) \cong Spin^+(3,1)$, the universal (and double) cover of $SO^+(3,1)$. □

Then $SL(2,\mathbb{C})$ acting on $\mathbb{C}^2$ is naturally a representation of $\mathrm{Spin}^+(3,1)$. We also have the conjugate representation $\bar{\rho} : \mathrm{Spin}^+(3,1) \to SL(2,\mathbb{C})$ acting on $\mathbb{C}^2$ defined by $\bar{\rho}(s) = \overline{\rho(s)}$. Let $\bar{\rho}(\mathrm{Spin}^+(3,1))$ be called $\overline{SL(2,\mathbb{C})}$.

The representations of $SU(2)$ and $SL(2,\mathbb{C})$ are closely related. In fact, it turns out there is a complete classification of the finite-dimensional representations of the former that leads to a convenient classification on the latter, but that is beyond the scope of this paper. This is covered in many texts on the representation theory of Lie groups; for example, [2]. The text [6] provides a physics-based approach.

The representation $\mathbb{C}^2$ acted upon by its special linear group is irreducible since $SL(2,\mathbb{C})$ clearly fixes no subspace of $\mathbb{C}^2$. Its conjugate representation (sometimes called the dotted or right-handed representation) is also irreducible. These are called the *Weyl spinor representations*, and they describe the states of neutrinos [6].

6. The Dirac spinor representation

For electrons the picture is more complicated. We require that the spinors of electrons be representations of the entire spin group, $\mathrm{Spin}(3,1)$ [6]. We can achieve this with *Dirac spinors.*

To do this, we simply take the direct sum of the two Weyl representations. Let χ_1, χ_2 be two Weyl spinors, and $\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$ a Dirac spinor. An element $s \in \mathrm{Spin}^+(3,1)$ acts on Dirac spinors as the matrix $\begin{pmatrix} s & 0 \\ 0 & \bar{s} \end{pmatrix}$. To get a representation of the entire spin group we add the matrix j:

$$j = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

This matrix represents the total reflection sending a vector $\vec{u}$ to $-\vec{u}$. j commutes with every $\begin{pmatrix} s & 0 \\ 0 & \bar{s} \end{pmatrix}$; also, $j^2 = -1$ which represents a 2π rotation. So j has all the necessary properties we need for it to doubly cover total reflection.

It would also be possible to pick a properly commuting j such that $j^2 = 1$, representing the identity rotation. Such a representation would also give a double covering of $SO(3,1)$. It follows that there are two choices of the nonorthochronous group $\mathrm{Spin}(3,1)$.

In fact, we can do even better. We can add the matrices

$$s = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$t = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

in order to get a double covering representation of $O(3,1)$ [11]. I won't prove it here. This is called a pin representation. This is only one of several ways to have a pin representation consisting of Dirac spinors. There are generally at least two homeomorphic but nonisomorphic Pin groups, due to the nonconnectedness of the orthogonal group; the one in which two reflections are 2π are called Pin_- and the one in which two reflections are the identity Pin_+.

In literature, one often reads about "the" spin and pin groups. These usually refer to the groups constructed as subgroups of Clifford algebrae; which are unique.

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