# On the Fundamental Group of a Generalized Lens Space

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**Abstract**

The purpose of this paper is to provide a brief overview of the topological properties of higher-dimensional versions of the 3-dimensional lens space $L(p, q)$. Specifically, this work focuses on determining the fundamental groups of the generalized lens space using general methods in covering spaces.

## 1 Covering Spaces

**Definition 1**
A continuous map between topological spaces, $p : X^c \mapsto X$, is a *covering map* if and only if
**1.** $p$ is surjective
**2.** For all $x \in X$, there exists an open neighborhood $U$ around $x$ such that $p^{-1}(U) = \bigcup_{j \in J} U_j$ with $U_j$ pairwise disjoint and open and $p : U_j \mapsto U$ a homeomorphism for each $U_j$.

**Definition 2**
A group action of a group $G$ acting on a topological space $X$ is *properly discontinuous* if and only if for each $x \in X$ there exists an open neighborhood $U$ around $x$, such that $g_1 \circ U \cap g_2 \circ U = \emptyset$ for all $g_1, g_2 \in G$ with $g_1 \neq g_2$.

**Definition 3**
Suppose $p : X^c \mapsto X$ is a covering map and $f : Y \mapsto X$ is continuous. A *lift* of $f$ is a continuous map $f_l : Y \mapsto X^c$ such that $p \circ f_l = f$.

**Theorem 1.1**
Let $G$ act on $X$. Then if the action is properly discontinuous, $p : X \mapsto X/G$ under the canonical projection is a covering map.

*Proof* : Let $U$ be an open neighborhood of $x \in X$ such that $g_1 \circ U \cap g_2 \circ U = \emptyset$ for all $g_1 \neq g_2$ with $g_1, g_2 \in G$. Since $p$ is open, $p(U)$ is open, and $p^{-1}(p(U))$ are the orbits of $U$, which are open sets of the form $g \circ U$ with $g \in G$. Hence, because $p : g \circ U \mapsto p(U)$ is bijective, open, and continuous, it is a homeomorphism for each $g \circ U$. □

**Theorem 1.2**
Let $p : X^c \mapsto X$ be a covering map. Then $p$ is an open map.

*Proof* : Let $U$ be an open subset of $X^c$ and let $x \in p(U)$. Then there exists $O$ open around $x$ such that $O$ is evenly covered, and $p^{-1}(O) = \bigcup_{j \in J} O_j^c$ for $O_j^c \in X^c$ pairwise disjoint.

Then $p(O_j^c \cap U) \subseteq p(U)$ is open in $O$ for some $O_j^c$ with $p^{-1}(x) \subseteq O_j^c \subseteq U$, since $p$ is a homeomorphism. Thus, For all $x \in p \mid U$ there is an open neighborhood around $x$, namely $p(O_j^c \cap U)$. □

**Exercise 1.2**
(1) Prove: $X$ has the quotient topology with respect to the covering map $p$.

(2) Let $G$ be a group and $X$ be Hausdorff. Prove: If the action of $G$ on $X$ is free, then the action is properly discontinuous (Hint: Consider the intersection of the open neighborhoods of $\{g \cdot x \mid g \in G\}$ where $x \in X$).

**Theorem (Path Lifting)**
Let $f : [0,1] \mapsto X$ be a continuous map and $p : X^c \mapsto X$ be a covering. Suppose $x_0^c \in X^c$ such that $p(x_0^c) = f(0)$. Then there exists a unique lift $f_l$ of $f$ such that $f_l(0) = x_0^c$.

*Proof* : For each $x \in X$, let $U_x$ be an evenly covered open neighborhood of $x$. Then $\bigcup_{x \in X} f^{-1}(U_x)$ covers [0,1].

Because [0,1] is compact, there exists a finite subset of $\bigcup_{x \in X} f^{-1}(U_x)$ of the form $\bigcup_{i=1}^{n} I_i$ such that $[0, b_1) = I_1, (a_n, 1] = I_n, (a_i, b_i) = I_i$ with $b_{i+1} < a_i$.

Let $t_i \in [0,1]$ such that $a_{i+1} < t_i < b_i$ for $i < n$. Note that, for each $i$, $f([t_i, t_{i+1}]) \subset f(I_i) \in \{U_x \mid x \in X\}$ with $p \mid p^{-1}[t_i, t_{i+1}] : X^c \mapsto X$ a homeomorphism.

We proceed inductively to prove the existence/uniqueness $f_l$, the lift of $f$. Let $f_l(0) = x_0^c$. Then $f_l(s)$ is defined and unique on $s = 0$. Suppose, now, that $f_l$ is defined and unique on $[0, t_i]$ such that $p \circ f_l(t_i) = f(t_i)$. Because $p$ is a homeomorphism on $[t_i, t_{i+1}]$, there is a unique $\gamma : [t_i, t_{i+1}] \mapsto X^C$ such that $p \circ \gamma = f$. Let $f_l = \gamma$ on $[t_i, t_{i+1}]$, and define $f_l$ as before on $[0, t_i]$. Then $f_l$ is defined and unique on $[0, t_{i+1}]$. □

**Corollary to the Path Lifting Theorem**
Let $f : [0,1] \times [0,1] \mapsto X$ be a continuous map and $p : X^c \mapsto X$ be a covering. Suppose $x_0^c \in X^c$ such that $p(x_0^c) = f(0,0)$. Then there exists a unique lift $f_l$ of $f$ such that $f_l(0,0) = x_0^c$.

*Proof* : Left as an exercise (Hint: Consider the cross products of the intervals $[t_i, t_{i+1}]$ and apply the method described in the proof of the Path Lifting Theorem).

**Theorem 1.3**
Let $p : X^c \mapsto X$ be a covering, and suppose that $f_1$ and $f_2$ are two lifts of $f : Y \mapsto X$ with $Y$ connected. If $f_1(y_0) = f_2(y_0)$ for some $y_0 \in Y$, then $f_1 = f_2$.

*Proof* : Let $\Omega$ be the set of all $y \in Y$ such that $f_1(y) = f_2(y)$. We prove $\Omega$ is both open and closed.

Let $y \in \Omega$. Then there is an open neighborhood $U$ of $f(y)$ such that $p^{-1}(f(y)) = \bigcup_{j \in J} O_j$ where the $O_j$ are pairwise disjoint, open sets in $X^c$ mapped homeomorphically into $U$ by $p$.

Then $f_1(y) = f_2(y)$ and $p \circ f_1(y) = p \circ f_2(y) = f(y) \Rightarrow$ there exists $O_i \in \bigcup_{j \in J} O_j$, $f_1(y) = f_2(y) \in O_i$ $\Rightarrow {f_1}^{-1}(O_i) \cap {f_2}^{-1}(O_i)$ is an open cover of $y$.

Let $b \in {f_1}^{-1}(O_i) \cap {f_2}^{-1}(O_i)$. Then $f_1(b)$ and $f_2(b)$ are both in $O_i$ and $p$ is a homeomorphism on $O_i$. Thus $f(b) = p \circ f_1(b) = p \circ f_2(b)$, $f_1(b) = f_2(b) \Rightarrow b \in \Omega \Rightarrow {f_1}^{-1}(O_i) \cap {f_2}^{-1}(O_i)$ is an open neighborhood around $y$ that is contained in $\Omega$. $\Omega$ is open.

Suppose $y \notin \Omega$. Then, there exist $O_m, O_n \in X^c$ such that $f_1(y) \in O_m$ and $f_2(y) \in O_n$. Then ${f_1}^{-1}(O_i) \cap {f_2}^{-1}(O_i)$ is an open neighborhood around $y$ that is contained in the complement of $\Omega$. Thus, $\Omega$ is closed. $\square$

## 2 The Generalized Lens Space

**Construction**
Consider the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ where $n \in \mathbb{N}$. In this case, $S^{2n+1} = \left\{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^{n} |z_n|^2 = 1\right\}$

The Generalized $n$-Lens Space, denoted $L(p, q_1, q_2, \ldots, q_n)$ where $p \in \mathbb{N}$ and $p$ is prime to $q_i$ for $i \leq n$, is the quotient space $S^{2n+1}/\mathbb{Z}_p$ where $\mathbb{Z}_p$ acts on $S^{2n+1}$ via the following:

Let $g \in \mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ and let $(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$.

Then $g \cdot (z_0, z_1, \ldots, z_n) = (e^{(2\pi gi/p)} z_0, e^{(2\pi giq_1/p)} z_1, e^{(2\pi giq_2/p)} z_2, \ldots, e^{(2\pi giq_n/p)} z_n)$.

**Exercise 2.0**
(1) Prove: The above action is free and well-defined.

(2) Prove: $L(2, \overbrace{1, 1, \ldots, 1}^{n}) = \mathbb{R}P^{2n+1}$.

**Theorem 2.1**
Let $X$ be a path connected topological space, and let $G$ be a group whose action on $X$ is properly discontinuous. Also, let $x_0 \in X$ and let $p : X \mapsto X/G$ be the canonical projection from $X$ to $X/G$, with $p(x_0) \in X/G$. Define $\phi : \pi(X/G, p(x_0)) \mapsto G$ by: $\phi(f) = g \in G$ such that $g \cdot x_0 = l_{x_0} f(1)$, where $f \in \pi(X/G, p(x_0))$ and $l_{x_0} f$ is the homotopy class of lifts from $[0, 1]$ to $X$ of representatives of $f$ based at $x_0$. Then $\phi$ is a homomorphism.

*Proof* : Suppose $[f_1]$, $[f_2] \in X/G$ are based at $p(x_0)$, and let $f_1$, $f_2$ be representatives of their respective homotopy classes. Suppose $\phi(f_1) = x_1 = g_1 \cdot x_0$ and $\phi(f_2) = x_2 = g_2 \cdot x_0$, where $g_1$, $g_2 \in G$.

Then,

$$p(g_1 \cdot l_{x_0}(f_2)) = f_2$$
$$\Rightarrow l_{x_1}(f_2) = g_1 \cdot l_{x_0}(f_2)$$
$$\Rightarrow l_{x_0}(f_1 \circ f_2) = l_{x_0}(f) \circ l_{x_1}(f_2)$$

which further implies

$$\phi([f_1] \cdot [f_2]) = \phi([f_1 \circ f_2])$$
$$= k \in G \text{ such that: } k \cdot x_0 = g_1 \cdot x_2 = g_1 \cdot (g_2 \cdot x_0) = (g_1 \cdot g_2) \cdot x_0.$$
$$= (g_1 \cdot g_2)$$
$$= \phi([f_1]) \cdot \phi([f_2]).$$

□

**Corollary 2.1**
Let $p_*$ be the induced homomorphism of the fundamental groups $\pi(X, x_0)$ and $\pi(X/G, p(x_0))$. Then $\pi(X/G, p(x_0))/p_*\pi(X, x_0) \cong G$.

*Proof* : Left as an exercise (Hint: Consider the kernel of $\phi$, and show that $\phi$ is a surjective map).

**Theorem 2.2**
$\pi(L(p, q_1, q_2, \ldots, q_n)) \cong \mathbb{Z}_p$.

*Proof* : By Exercises 1.2.2 and 2.0.1, the canonical map $m : S^{2n+1} \mapsto S^{2n+1}/\mathbb{Z}_p$ is a covering map. Thus, by Corollary 2.1, $\pi(S^{2n+1}/\mathbb{Z}_p, m(x_0))/m_*\pi(X, x_0) \cong \pi(S^{2n+1}/\mathbb{Z}_p, m(x_0)) \cong G$. □

# 3 Bibliography


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