REU research paper by Gideon Tan

Problem:

In a town with N people, there are different clubs such that the number of people who are members of both of any two given clubs is a multiple of X, and the number of members in each club is also a multiple of X. What is the maximum possible number of clubs, P?

Alternative solution to Babai's solution for X = 2:
Call a town which contains the maximum number of possible clubs which satisfies the above condition a good town.
Given N people, we can group everybody into groups of X, such that for any given group, each club will either contain all the members of that group or none of the members in that group. In this way,

Theorem 1 $P \geq 2^{\lfloor N/X \rfloor}, or \lfloor N/X \rfloor \leq log_2 P$

Consider the matrix formed such that the ith row and jth column is 1 if the jth person is in the ith club and 0 otherwise.

Lemma 1 *Adding any row in the matrix of a good town to every single row of the matrix modulus 2 will result in a matrix of another good town*

Proof: Adding two rows together modulus 2 will give the symmetric difference of the two clubs; if a person was in either both or neither of the 2 clubs, the sum modulo 2 will be 0, which is the same as that person not being in the symmetric difference. If the person is in exactly one of the two clubs, he will be in the symmetric difference and the sum will be 1.
Also, the number of intersections between two clubs = total number of 1's in both rows corresponding to the club - number of 1's in the symmetric difference of the rows.
Since the total number of 1's in each row is a multiple of X, the intersection of two clubs is a multiple of X if and only if the number of 1's in the symmetric difference is a multiple of X. Given 3 rows for 3 clubs C_1, C_2, C_3,
$(\mathrm{C}_{1j} + C_{2j}) + (C_{1j} + C_{3j}) = C_{3j} + C_{2j} + 2 * C_{1j} (mod.2) = C_{3j} + C_{2j} (mod.2)$

Thus, by adding C_{1j} to all the rows of the matrix, we get another matrix of a good town.

For case X = 2, we prove by induction that

Proposition 1 $P = 2^{\lfloor N/X \rfloor}$ *or* $\lfloor N/X \rfloor = log_2 P$

Consider a good town with P clubs, N_x^p people. If the smallest club has 2 members, we remove the 2 members from the town. Since every other club either has both members or neither of the two members, after removing, we will get a good town of N_x^p-2 members with at least $\lceil (P)/2 \rceil$ clubs(since each original club is at most repeated twice). Thus, $N_2^p \geq N_2^{\lceil (P)/2 \rceil} + 2$ or $\lfloor N_2^p/2 \rfloor \geq \lfloor N_2^{\lceil (P)/2 \rceil}/2 \rfloor + 1$. Thus, $\lfloor N_2^p/2 \rfloor \geq log_2 \lceil (P)/2 \rceil + 1 \geq log_2 P$. Combining this with (1), $\lfloor N_2^p/2 \rfloor = log_2 P$

If the smallest club has greater than 2 members, we look at the number of clubs with either both or neither of the first two people in the town. If the number of such clubs < P/2 we pick a row starting with either 1,0 or 0,1 and add it to the matrix of the good town. In this new good town, if the smallest club has 2 members we are done. If the smallest club still has greater than 2 members, the number of clubs which has either both or neither of the first two members will be $\geq$ P/2. Now, we look at all the clubs with both or neither of the first two members. By removing the first two members from these clubs, we will form unique clubs, as if two of the clubs become the same after removing the first two members, the first two members themselves can form a club, which contradicts the fact that the smallest club > 2 members. Thus, we are able to form at least $\lceil (P)/2 \rceil$ clubs with N_2^p -2 people, and similar to above, the inductive step follows.