Balancing Numbers Problem

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Suppose we have thirteen complex numbers with the following property: if we remove any one of the numbers, the remaining twelve can be split into two sets of 6 numbers each with equal sum.

Claim: all thirteen numbers are equal.

Proof:

1 Integers

I will first prove the claim for integers.

Assume x_1,...,x_{13} $\epsilon\mathbb{Z}$ such that not all x_i are equal.

Let $S = 2r_i + x_i$, where r_i is the sum associated with removing an x_i

Clearly, S is either even or odd.

S is odd $\Rightarrow x_i$ is odd, since $2r_i$ is even.

S is even $\Rightarrow x_i$ is even, for the same reason.

So, either all the $x_i's$ are even, or they are all odd.

Now define $y_i = x_i - x_1$. Observe that the above property still holds for y_i. Furthermore, since $y_1 = x_1 - x_1 = 0$ and 0 is even, y_i is even for all i.

Now define $y_i' = y_i/2$. The same property still holds and since $0/2 = 0$, the numbers are still even. It is now clear that this process can be repeated indefinitely. However, if any nonzero integer is divided through by two enough times, we will come up with an odd number.

$\Rightarrow\Leftarrow$. Since one of the numbers is always zero, all of the numbers should always be even. So all of the numbers must be zero.

$y_i = x_i - x_1 = 0 \Rightarrow x_i = x_1$ for all i. ie. all of the numbers are equal.

2 Rationals

We have now proved the claim for integers. The case for rationals will follow directly from this result. Assume that $p_1/q_1, ..., p_{13}/q_{13}\ \epsilon\mathbb{Q}$ such that they have the property in question. Then clearly $z_i = (p_i/q_i)(q_1...q_{13})$ is an integer. We can apply the above argument to the $z_i's$ to find that $z_1 = z_2 = ... = z_{13}$ But $z_i/(q_1...q_{13}) = p_i/q_i$. So all the p_i/q_i's are equal.

3 Reals

Now we must prove the claim for the reals. Assume $r_1, ..., r_{13}\epsilon\mathbb{R}$ such that they have the property in question. $\exists q_i \epsilon \mathbb{Q}$ such that $q_i \epsilon (r_i - \epsilon, r_i + \epsilon), \forall \epsilon, r_i$

Furthermore, by Dirichlet's Theorem on simultaneous diophantine approximation, $\forall\ 1 \leq i \leq 13$ and $\epsilon > 0, \exists$ integers p_i and $q < 1/(\epsilon^{13})$ such that $|r_i - p_i/q| < \epsilon/q$

ie. all of the reals can be approximated by rationals of a common denominator.

If $-6\epsilon/q < (r_1 + ... + r_6) - \frac{p_1+...+p_6}{q} < 6\epsilon/q$
and $-6\epsilon/q < (r_7 + ... + r_{12}) - \frac{p_1+...+p_{12}}{q} < 6\epsilon/q$
and $r_1 + ... + r_6 = r_7 + ... + r_{12}$, then:

$-12\epsilon/q < \frac{(p_7+...p_{12})-(p_1+...+p_6)}{q} < 12\epsilon/q$
$\Rightarrow -12\epsilon < (p_7 + ...p_{12}) - (p_1 + ... + p_6) < 12\epsilon$

But the p_i's are integers. So set $\epsilon = 1/12$ Then it is clear that $(p_7 + ... + p_{12}) - (p_1 + ... + p_6) = 0$

ie. $(p_7 + ...p_{12}) = (p_1 + ... + p_6)$ and we would come up with something equivalent if we had "picked out" any r_i.

$\Rightarrow p_1 = p_2 = ... = p_{13}$

So $\forall \epsilon > 0$, $|r_i - p/q| < \epsilon/q$ and p/q is the same for each r_i

$\Rightarrow r_1 = r_2 = ... = r_{13}$

4 Complex Numbers

The case for the complex numbers follows directly from the reals. Assume that $c_k = x_k + iy_k \epsilon \mathbb{C}$ for $1 \leq k \leq 13$ such that c_k's have the property in question.

If $c_1 + ... + c_6 = c_7 + ... + c_{12}$, then:

$x_1 + ... + x_6 = x_7 + ... + x_{12}$, and
$y_1 + ... + y_6 = y_7 + ... + y_{12}$

$\Rightarrow$ all of the y_i's are equal and all of the x_i's are equal.
$\Rightarrow$ all of the c_i's are equal. QED

Note furthermore that this proof did not rely on the number thirteen other than its being an odd number. The claim should therefore be true for any group of n numbers where n is odd that have the property in question.