Lecture 1

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1 Introduction

We begin with a definition and a theorem. Do not worry if you do not understand; everything will be defined and explained in due course.

Definition 1.1 *An n-Topological Quantum Field Theory (n-TQFT) is a symmetric monoidal functor* $F : \textbf{n-Cob} \to \textbf{VectK}$.

Theorem 1.2 *The category of 2-TQFTs is equivalent to the category of commutative Frobenius K-algebras.*

Proof: Left to the reader. $\square$

Just kidding. Understan4ding this definition and proving this theorem will be the main subject of this course. First, we need to know some category theory. The next sections will provide some useful definitions.

2 Categories, etc...

Definition 2.1 *An algebra A over a field K is a vector space A over K together with an associative multiplication* $A \times A \to A$, *written* $(a, b) \to ab$, *such that* $\forall a, b \in A$ *and* $\forall k \in K$, $(ka)b = k(ab) = a(kb)$.

Definition 2.2 *A category* $\mathfrak{C}$ *is a collection of objects (X,Y,Z,...) denoted* $Ob(\mathfrak{C})$, *together with, for each pair (X,Y) of objects of* $\mathfrak{C}$, *a set of morphisms (maps)* $f : X \to Y$ *denoted* $\mathfrak{C}(X, Y)$ *satisfying the following: For each object X of* $\mathfrak{C}$ *there is a given identity morphism* $1_X : X \to X$ *and for each triple (X,Y,Z) of objects of* $\mathfrak{C}$ *and pair of morphisms* $f : X \to Y$, $g : Y \to Z$ *there is a composition law* $\circ : \mathfrak{C}(Y, Z) \times \mathfrak{C}(X, Y) \to \mathfrak{C}(X, Z)$, *written* $(g, f) \to g \circ f$, *such that* $1_Y \circ f = f = f \circ 1_X$ *and* $h \circ (g \circ f) = (h \circ g) \circ f$ *for any morphism h with domain Z. Remark: We do not require that* $Ob(\mathfrak{C})$ *be a set; it may be a proper class. If it is a set, we say that the category is small.*

Example: The collection of all sets is a category denoted **SET**. Its morphisms are functions.

Example: The collection of all groups is a category denoted **GRP**. Its morphisms are group homomorphisms.

Example: The collection of all topological spaces is a category denoted **TOP**. Its morphisms are continuous functions.

Example: A monoid is a set M with an associative binary operation and an identity element. Note that in a category $\mathfrak{C}$ the composition law $\circ$ on $\mathfrak{C}(X, X)$ is such a binary operation with identity element 1_X. Therefore a monoid is a category with one object.

Definition 2.3 *A morphism $f : X \to Y$ in a category $\mathfrak{C}$ is called an isomorphism if there is a morphism $g : Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.*

Exercise: If a morphism f has a left inverse and a right inverse then it is an isomorphism and the left and right inverses coincide.

Definition 2.4 *A groupoid is a category in which every morphism is an isomorphism.*

3 Functors

A morphism of categories is called a functor.

Definition 3.1 *Let $\mathfrak{C}, \mathfrak{D}$ be categories. A functor $F : \mathfrak{C} \to \mathfrak{D}$ consists of a rule that assigns for each object X of $\mathfrak{C}$ an object FX of $\mathfrak{D}$, together with, for each pair (X,Y) of objects of $\mathfrak{C}$, a function $F : \mathfrak{C}(X, Y) \to \mathfrak{D}(FX, FY)$, written $f \to Ff$, such that $F(1_X) = 1_F X$ and $F(g \circ f) = Fg \circ Ff$.*

Example: The abelianization of a group G is the group G/[G,G] where [G,G] is the commutator subgroup, i.e. the subgroup generated by $\{ghg^{-1}h^{-1} \mid g, h \in G\}$. Abelianization defines a functor $A : \mathbf{GRP} \to \mathbf{AB}$ where **AB** is the category of abelian groups.

Definition 3.2 *A functor $F : \mathfrak{C} \to \mathfrak{D}$ is said to be faithful if the function $F : \mathfrak{C}(X, Y) \to \mathfrak{D}(FX, FY)$ is injective for every pair (X,Y) of objects of $\mathfrak{C}$.*

Definition 3.3 *A functor $F : \mathfrak{C} \to \mathfrak{D}$ is said to be full if the function $F : \mathfrak{C}(X, Y) \to \mathfrak{D}(FX, FY)$ is surjective for every pair (X,Y) of objects of $\mathfrak{C}$.*

Definition 3.4 *A functor $F : \mathfrak{C} \to \mathfrak{D}$ is said to be an isomorphism of categories if there is a functor $G : \mathfrak{D} \to \mathfrak{C}$ such that FG is the identity functor on $\mathfrak{D}$ and GF is the identity functor on $\mathfrak{C}$.*

Definition 3.5 *A functor $F : \mathfrak{C} \to \mathfrak{D}$ is said to be essentially surjective if, for every object Y of $\mathfrak{D}$, there is an object X of $\mathfrak{C}$ and an isomorphism $FX \cong Y$.*

Definition 3.6 *A functor $F : \mathfrak{C} \to \mathfrak{D}$ is said to be an equivalence of categories if it is full, faithful, and essentially surjective.*

Definition 3.7 *The skeleton of a category $\mathfrak{C}$ is a full subcategory (a subcollection of Ob($\mathfrak{C}$) together with all of the morphisms from $\mathfrak{C}$) of $\mathfrak{C}$ which contains exactly one object from each isomorphism class of objects of $\mathfrak{C}$.*

4 Natural Transformations

Naturally, there are also morphisms of functors.

Definition 4.1 *Let $F, F' : \mathfrak{C} \to \mathfrak{D}$ be functors. A natural transformation $\eta : F \to F'$ is a collection of maps $\eta_X : FX \to F'X$, one for each object X of $\mathfrak{C}$, such that $\forall f : X \to Y$ in $\mathfrak{C}$ the following diagram commutes:*

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ F'X & \xrightarrow[F'f]{} & F'Y \end{array}$$

Definition 4.2 *A natural transformation η is said to be a natural isomorphism if each of the maps η_X is an isomorphism.*

Definition 4.3 *A functor $F : \mathfrak{C} \to \mathfrak{D}$ is said to be an equivalence of categories if there is a functor $G : \mathfrak{D} \to \mathfrak{C}$ and natural isomorphisms $\eta : FG \to Id_{\mathfrak{D}}$ and $\nu : GF \to Id_{\mathfrak{C}}$.*

Example: Let V be a vector space over a field K. V is naturally isomorphic to its double dual space, i.e. the space Hom(Hom(V,K),K).