COBORDISM AND CONFUSION

PETER MAY

We had a great deal of confusion today, some inflicted by me, some by others. I decided to bang out something pedantically rigorous, as least as far as the absolutely precise definition of the objects and the morphisms of the category 2-COB, starting more generally with n-COB. Being precise here will allow us to be a little more informal when we perform verifications later on.

1. What are the objects of the cobordism category?

Well, we want a nice small category, so we should be thinking perhaps of a skeleton of some larger category. Remember from the first talk that identity is a silly notion of sameness, the right notion being isomorphism. The large class of objects we are thinking of is the class of smooth closed oriented $n-1$-manifolds. The right notion of isomorphism is orientation-preserving diffeomorphism, and we agree to choose one such oriented manifold from each isomorphism class and let these be the objects of n-COB.

That much works in general; now focus on the case $n = 2$, so that $n - 1 = 1$. It is silly to think of the circle as having two isomorphism classes, one for each orientation, because that pretends that the only isomorphism between circles is the identity map, which of course does not preserve orientation.

Lemma 1.1. *There is an orientation-preserving diffeomorphism from a circle with one orientation to a circle with the other orientation.*

Proof. Draw little circles of the same size around $(1, 0)$ and $(-1, 0)$ in $\mathbb{R}^2$ and reflect one to the other through the y axis. □

Lemma 1.2. *Every smooth closed oriented* 1*-manifold is isomorphic to a disjoint union of* n *circles, each with its standard orientation.*

That determines the objects of 2-Cob. There is one object, the disjoint union of n oriented circles, for each $n \geq 0$. We write that object as $\mathbf{n}$. Except that no such convenient choice of objects seems likely, we can make a conceptually analogous choice in the case $n > 2$. We shall not bother to introduce notation that would help us talk about the general case.

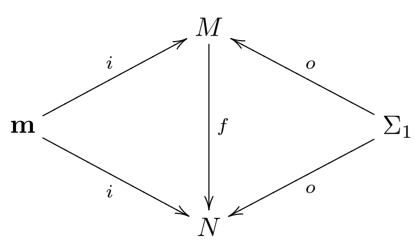
An oriented n-manifold M with boundary induces orientations of its boundary components, but how does it do so? Let Σ be such a boundary component and let $x \in \Sigma$. Choose a basis $[v]$ for the tangent plane $T_x(\Sigma)$ at a point $x \in \Sigma$. The normal bundle of the embedding $\Sigma \subset M$ is trivial, and there are vectors $w \in T_x(M)$ such that $[v, w]$ is a basis for $T_x(M)$. A neighborhood of x in M looks like a half plane $\mathbb{H}^n$, and $T_x(M)$ looks like a copy of $\mathbb{R}^n$, with $\mathbb{H}^n$ inside M and its complementary half-plane outside of M. So the vector w points outside M or points inside M; that is, identifying a small neighborhood of x in M with $0 \in \mathbb{H}^n \subset \mathbb{R}^n$, w is outside $\mathbb{H}^n$ or it is inside $\mathbb{H}^n$.

One way to specify the induced orientation on Σ is to say that $[v]$ is positive if $[v, w]$ is positive in the given orientation of M whenever w is outside M. This is independent of the choice of w. We don't want to focus attention on either this induced orientation or the alternative one that is obtained by saying that $[v]$ is positive if $[v, w]$ is positive in the given orientation of M whenever w is inside M. Note however that these are the only two choices of an orientation of the boundary component Σ.

Definition 1.3. An orientation of the boundary of an oriented n-manifold M with boundary is a choice of orientation of each boundary component. The boundary component Σ is an out-boundary component if the chosen orientation agrees with the induced orientation; it is an in-boundary component if it does not. The disjoint union of the in-boundary components is called the in-boundary. The disjoint union of the out-boundary components is called the out-boundary. When $n = 2$, define the *source* of M to be $\mathbf{m}$, where m is the number of in-boundary components, and define the *target* of M to be $\mathbf{n}$, where n is the number of out-boundary components.

Intuitively, we think of M as a morphism $\mathbf{m} \longrightarrow \mathbf{n}$, but that doesn't really make much sense until we fix a geometric connection between the fixed chosen disjoint unions of circles $\mathbf{m}$ and $\mathbf{n}$ and the in-boundary and out-boundary of M. These are obviously not "the same" in any silly sense. For each n-manifold M with source $\mathbf{m}$ and target $\mathbf{n}$, we choose an isomorphism (in the orientation-preserving diffeomorphism sense above) from $\mathbf{m}$ to the inboundary of M and from $\mathbf{n}$ to the outboundary of M; call these i and o. We use the notations i and o generically, using the same letter for different manifolds. We think of triples (M, i, o) as morphisms from $\mathbf{m}$ to $\mathbf{n}$. There are zillions of them and, again, we don't want to have too many morphisms in 2-COB. We now reduce the number not by choosing one among many isomorphic ones but rather by identifying two if they are suitably equivalent.

Definition 1.4. Triples (M, i, o) and (N, i, o) are equivalent if there is an orientation-preserving diffeomorphism $f\colon M \longrightarrow N$ that makes the following diagram commute.



This specifies an equivalence relation, and we write $[M, i, o]$ (or just $[M]$ or sometimes even M when no confusion seems likely) for the equivalence class of $[M, i, o]$. Intuitively, we think of f as a diffeomorphism that fixes the boundary. The set (it is indeed a set) of equivalence classes $[M, i, o]$ is the set of morphisms $\mathbf{m} \longrightarrow \mathbf{n}$ in the category 2-COB.

For any closed oriented $(n-1)$-manifold Σ, $\Sigma \times I$ has a canonical product orientation. Taking the in-boundary to be $\Sigma \times \{0\}$ and the out-boundary to be $\Sigma \times \{1\}$ and using the obvious maps i_0 and i_1 as i and o, we may define the identity morphism on Σ to be $[M \times I, i, o]$. We must still define composition. To be continued.