SOME THEOREMS AND APPLICATIONS OF RAMSEY THEORY

MATTHEW STEED

Abstract. We present here certain theorems in Ramsey theory and some of their applications. First is Ramsey's Theorem, which concerns the existence of monochromatic complete subgraphs of colored graphs that are large enough. One application is Schur's Theorem, which is used for a result relating to Fermat's Last Theorem. We then present the Hales-Jewett Theorem, which can be used to prove van der Waerden's Theorem and the Gallai-Witt Theorem.

Contents

Ramsey Theory concerns the emergence of order that occurs when structures grow large enough. The first theorem that we present concerns properties of graphs that emerge when the graphs are large enough. We need the following definitions concerning graphs.

Definition 0.1. A graph is a collection of vertices V and edges E, which are pairs of vertices.

Definition 0.2. A simple graph is a graph such that the vertices in the edge pairs are unordered, there is only one edge between any two vertices, and no edge connects a vertex to itself.

Definition 0.3. A complete graph is a simple graph such that E contains every pair of vertices. We denote a complete graph on n vertices by K_n.

Definition 0.4. An r-coloring of the edges of a graph is a function χ that assigns to each edge one of r colors. For convenience, we will usually denote the colors by numbers in the set $\{1, \dots, r\}$.

We will use χ to denote a coloring of any collection of objects, but Ramsey's Theorem concerns edge colorings of graphs. It essentially says that, given r colors, if a complete graph is large enough, then any r-coloring of the edges admits a complete subgraph of a certain size whose edges are all the same color [1].

1. Ramsey Numbers and Ramsey's Theorem

Definition 1.1. The Ramsey number $R(n, k)$ is the minimum positive integer such that for $m \geq R(n, k)$, any red-blue coloring of the edges of the K_m contains either a complete subgraph with blue edges on n vertices or a complete subgraph with red edges on k vertices.

For ease of notation and pictures, we call these colors red and blue rather than 1 and 2. As a first example, $R(2, 2) = 2$. In K_2, there are only two colorings because there is one edge: the edge is red or blue. In either case, there is a complete monochromatic subgraph of size 2. The number $R(3, 3)$ is less trivial. Figure 1 shows that $R(3, 3) > 5$.

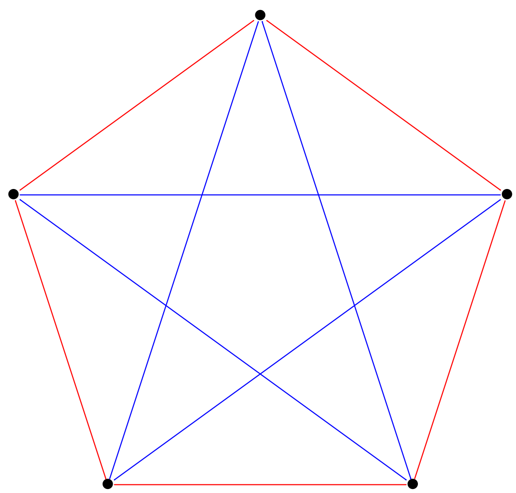

Figure 1. A coloring of K_5 showing that $R(3, 3) > 5$. For any 3 vertices, the edges connecting them are not all the same color.

However, $R(3, 3) \leq 6$ [1]. Indeed, consider a complete graph on 6 vertices with an arbitrary red-blue edge coloring. Consider the first vertex. There are five edges connecting it to the other vertices, so three of them must be the same color. Suppose that they are red. If any of the edges connecting the three corresponding vertices is red, then the two sharing the red edge along with the first vertex form a monochromatic subgraph of size 3. Otherwise, all of the edges connecting the three are blue, so those three form a monochromatic subgraph of size 3. Thus, $5 < R(3, 3) \leq 6$, so $R(3, 3) = 6$.

This definition can be extended to an arbitrary number of colors.

Definition 1.2. The Ramsey number $R(n_1, \ldots, n_k)$ is the smallest positive integer such that the following holds: for $m \geq R(n_1, \ldots, n_k)$ and any k-coloring of K_m, there is some $1 \leq i \leq k$ such that there is a complete subgraph of size n_i whose edges are all the ith color.

Ramsey's Theorem says that the Ramsey numbers $R(n_1, \ldots, n_k)$ are finite [1]. We prove this first for the case of two colors $R(n, k)$.

Theorem 1.3. *$R(n, k)$ is finite for all n and k.*

Proof. We prove this using induction on $n + k$. The base case is $n + k = 2$. This is only possible for $n = k = 1$. Any 1-subgraph is trivially monochromatic in every color because there are no edges, so the base case is true.

Now suppose that $R(n-1, k)$ and $R(n, k-1)$ are finite. We claim that $R(n, k) \leq R(n-1, k) + R(n, k-1)$. Let $m = R(n-1, k) + R(n, k-1)$. Consider a randomly colored K_m. Pick a vertex v randomly. Let N be the set of all vertices other than v with a blue edge connecting it to v. Let M be the set of vertices that are connected to v with a red edge. Every vertex is in exactly one of N, M, and $\{v\}$. Then, because there are m vertices, $N + M + 1 = m = R(n-1, k) + R(n, k-1)$. We must then have either $N \geq R(n-1, k)$ or $M \geq R(n, k-1)$.

Suppose that $N \geq R(n-1, k)$. Then, by definition, it has either a blue K_{n-1} or a red K_k. If it has a blue K_{n-1}, add v to the K_{n-1}, and we have blue K_n. If it has a red K_k, we are done. Similarly, the case $M \geq R(n, k-1)$ results in either a blue K_n or a red K_k. Thus, if a graph has at least $R(n-1, k) + R(n, k-1)$ vertices, there is a blue K_n or a red K_k. Hence, $R(n, k) \leq R(n-1, k) + R(n, k-1)$, so it is finite. This completes the induction, so all Ramsey numbers with two colors are finite. □

For an arbitrary number of colors, this is also true.

Theorem 1.4. *$R(n_1, \ldots, n_k)$ is finite for all k and choices of n_i.*

Proof. We induct on the number of colors. The base case $k = 2$ is shown above. Suppose that $R(n_1, \ldots, n_{k-1})$ is finite. We claim that

$$R(n_1, \ldots, n_k) \leq R(n_1, \ldots, n_{k-2}, R(n_{k-1}, n_k)).$$

Suppose we have a graph of size $R(n_1, \ldots, R(n_{k-1}, n_k))$, and we color it with the colors 1 through k. We now identify the $k-1$ and k colors. Then, we either have a complete subgraph of size n_i of color i, for $1 \leq i \leq k-2$, or we have a subgraph of the combined $k-1, k$ color of size $R(n_{k-1}, n_k)$. In the first case, we are done. In the second case, separating the colors again, there is either a subgraph of size n_{k-1} of the color $k-1$ or a subgraph of size n_k of the color k. Thus, $R(n_1, \ldots, n_k) \leq R(n_1, \ldots, R(n_{k-1}, n_k))$. Because $R(n_1, \ldots, R(n_{k-1}, n_k))$ is finite, $R(n_1, \ldots, n_k)$ is finite. This completes the induction. □

2. A Lower Bound on the two-color Ramsey Numbers

We give an exponential lower bound on the Ramsey numbers using the probabilistic method, from [2]. The basic idea is that if n is too small, then graphs without proper monochromatic subgraphs can be chosen with nonzero probability.

Theorem 2.1. $2^{\frac{k}{2}} \leq R(k, k)$

Proof. Consider a complete simple graph of n vertices with a random red-blue two coloring of the edges. A complete subgraph of size k has $\binom{k}{2}$ edges. Then, for a color assigned to edges with probability p, the probability that a K_k is entirely that color is $p^{-\binom{k}{2}}$. In our case, because there are two colors assigned with probability $\frac{1}{2}$, the probability that the subgraph is a specific color is $2^{-\binom{k}{2}}$. The probability that it is monochromatic in either color is $2^{1-\binom{k}{2}}$. There are $\binom{n}{k}$ ways of choosing a subgraph of size k. Thus, the probability that there is a monochromatic subgraph is less than or equal to $\binom{n}{k}2^{1-\binom{k}{2}}$. Also,

$$\begin{aligned}\binom{n}{k}2^{1-\binom{k}{2}} &= \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} \\ &\leq \frac{n^k}{k!}\frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}}.\end{aligned}$$

Suppose that $n \leq 2^{\frac{k}{2}}$. Then,

$$\begin{aligned}\binom{n}{k}2^{1-\binom{k}{2}} &\leq \frac{2^{\frac{k^2}{2}}}{k!}\frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} \\ &= \frac{2^{1+\frac{k}{2}}}{k!} \\ &< 1.\end{aligned}$$

If $n < 2^{\frac{k}{2}}$, then, with positive probability, there is no monochromatic subgraph of size k. Thus, $R(k,k) \geq 2^{\frac{k}{2}}$. □

Let $l < m$. Because $R(l,l) \leq R(l,m) \leq R(m,m)$, $2^{\frac{l}{2}} \leq R(l,m)$, so all of the Ramsey numbers have exponential lower bound

This result is not very sharp. For example, $R(4,4) = 18$, but the bound only gives $4 \leq R(4,4)$. The exact value of $R(5,5)$ is not known, but it is known that $43 \leq R(5,5) \leq 49$. The bound gives $2^{\frac{5}{2}} \approx 5.66 \leq R(5,5)$. However, this method gives an exponential bound, showing that the Ramsey numbers grow very quickly.

3. Schur's Theorem

Schur's Theorem [3] is a corollary of Ramsey's Theorem that can be used to disprove Fermat's Last Theorem in $\mathbb{Z}_p$ for prime p large enough. In the following, we use the notation $[n] = \{1, \ldots, n\}$.

Theorem 3.1 (Schur)**.** *Let r be a positive integer. Then, there exists a positive integer $S(r)$ such that if $[S(r)]$ is colored with r colors, then there exist x, y, z of the same color satisfying $x + y = z$.*

Proof. Let

$$S(r) = R(\underbrace{3, \ldots, 3}_{r \text{ 3's}})$$

be the Ramsey number with r colors, which exists by Ramsey's Theorem. Color $[S(r)]$ with r colors. Consider the complete graph $K_{S(r)}$. Label the vertices with the natural numbers up to $S(r)$. Color the edge ab with the color that $|a-b|$ has in the initial coloring of $[S(r)]$. This gives an r-coloring of $K_{S(r)}$. By definition of the Ramsey numbers, there is a monochromatic subgraph of size 3. Let the vertices of the monochromatic triangle be a, b, and c, with $a > b > c$. Then, $a-b$, $b-c$, and $a-c$ are all the same color, so let $x = a-b$, $y = b-c$, and $z = a-c$. Then, $x + y = z$. □

Because $R(3,3) = 6$, as shown above, any two coloring of $[6]$ admits x, y, and z such that $x + y = z$. However, these need not be distinct. The proof above allows them to be not necessarily distinct because they are defined as differences of the elements in the set. The two coloring of $[6]$ in Figure 2 provides an example where x and y are not distinct.

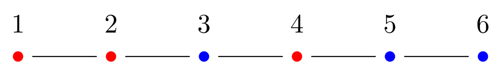

FIGURE 2. Coloring of $[6]$ showing that distinct x and y are not guaranteed with the bounds from the given proof of Schur's Theorem.

Its application to Fermat's Last Theorem is the following.

Corollary 3.2. *Fix $m \in \mathbb{N}$. There exists q such that for primes $p \geq q$, the equation $x^m + y^m = z^m$ has a solution in $\mathbb{Z}_p$.*

Proof. Let $q = S(m) + 1$. Let g be a generator of $\mathbb{Z}_p^*$, which exists because the group is cyclic. For any element $x \in \mathbb{Z}_p^*$, we can write $x = g^a$. We write $a = mj + i$ for $0 \leq i < m$, so that $x = g^{mj+i}$. We color the elements of $\mathbb{Z}_p^*$ with m colors where x has the ith color if $x = g^{mj+i}$. By Schur's Theorem, there exist a, b and c of the same color such that $a + b = c$. That is, the exponents of a, b, and c are congruent modulo m. Thus,

$$g^{mj_a+i} + g^{mj_b+i} = g^{mj_c+i}.$$

Let $x = g^{j_a}$, $y = g^{j_b}$, and $z = g^{j_c}$. Then, multiplying by g^{-i} above, we have $x^m + y^m = z^m$. □

4. The Hales-Jewett Theorem

Another of the major theorems in Ramsey theory is the Hales-Jewett Theorem. Before we give its statement, we need a few definitions [4].

Definition 4.1. A t-character alphabet is a set of t elements.

Unless stated otherwise, we will assume that our t-character alphabet is $\{0, 1, \ldots, t-1\}$.

Definition 4.2. An n-character word is a string of n characters from the alphabet. For example, 01321 is a 5-character word from the alphabet $\{0, 1, 2, 3\}$.

Definition 4.3. The set of all n-character words based on a t-character alphabet is denoted by $(\{0\} \cup [t-1])^n$.

Definition 4.4. Two words are called neighbors if they differ in only one character and that character is 0 in one word and 1 in the other. For example, 04302 and 04312 are neighbors.

Definition 4.5. A root τ of an alphabet A is a word from the alphabet $A \cup \{*\}$ that contains at least one $*$. For example, $\tau = 01*2*3$ is a 6-character root of $\{0, 1, 2, 3\}$.

We will think of the $*$'s as wildcards that can be replaced with characters in the alphabet, and all $*$'s in a word must be replaced by the same character. We denote by $\tau(a)$ the word formed from the root τ with the wildcards replaced by a. In the above example, $\tau(0) = 010203$, $\tau(1) = 011213$, and so on.

The alphabets are also equipped with a concatenation operation that concatenates words. For example, 012 and 041 can be concatenated to form the 6-character word 012041. Let τ_1, …, τ_n be n roots. We can concatenate them to form a root $\tau = \tau_1 \cdots \tau_n$, and we allow it to take n-character words as inputs, where the ith character replaces the wildcards in the ith root. For example, let $a = a_1 \cdots a_n$ be an n-character word. Then, $\tau(a) = \tau_1(a_1) \cdots \tau_n(a_n)$.

Definition 4.6. Given a root τ, the set of words $\{\tau(0), \tau(1), \ldots, \tau(t-1)\}$ is called the combinatorial line generated by τ.

We are now ready to state the Hales-Jewett theorem.

Theorem 4.7 (Hales-Jewett)**.** *Given an alphabet of length t and r colors, there exists an integer $HJ(r,t)$ such that for $N \geq HJ(r,t)$ and any r-coloring of words of length N, there exists a monochromatic combinatorial line. That is, there exists a root τ such that the combinatorial line generated by τ is monochromatic.*

If the alphabet is given by $\{0, \ldots, t-1\}$, we can think of words of length n as coordinates in $\mathbb{R}^n$, with all of the words lying on the lattice points of a cube of side length $t-1$. Figure 3 shows some words of length 3 in the alphabet $\{0, 1, 2\}$. However, not all lines in the cube are combinatorial lines. One such line is also shown in Figure 3.

In this view, the combinatorial lines are lines in the cube whose changing coordinates increase by 1 at each step. The Hales-Jewett theorem states that, given a t-character alphabet and r colors, if the dimension is large enough, there exists a monochromatic combinatorial line.

One consequence is that tic-tac-toe always has a winner when played in a large enough dimension. The size of the alphabet corresponds to the size of the board, and the number of colors corresponds to the number of players. Words correspond to places where the player can play because they are the coordinates of playable positions. Combinatorial lines are winning positions. The classic game is played on a board of size three with two players. Because $HJ(2,3) = 4$, the standard tic-tac-toe game always has a winner in 4 dimensions. However, lines that win in tic-tac-toe are more general than combinatorial lines, so the minimum possible value of $HJ(r,t)$ does not necessarily give the best bound.

4.1. **Proof of the Hales-Jewett Theorem.**

We prove the theorem using induction on the number of characters in the alphabet [4]. For $t = 1$, $HJ(r,1) = 1$ because there is only one word. The single word is a line, and it is monochromatic. We now assume that $HJ(r,t-1)$ exists and let $n = HJ(r,t-1)$. We need the following lemma

Lemma 4.8. *Define the following sequence of numbers inductively:*

$$N_1 = r^{t^n}$$

$$N_i = r^{t^n + \sum_{k=1}^{i-1} N_k},$$

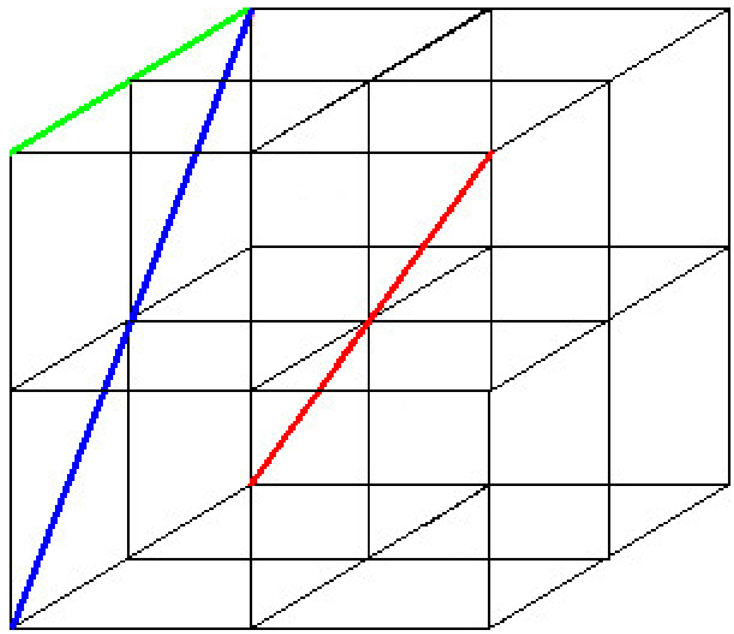

FIGURE 3. The red line is the combinatorial line corresponding to the root $***$. The green line is a combinatorial line corresponding to the root $02*$. The blue line is a geometric line going through the points $(0,2,0)$, $(0,1,1)$, and $(0,0,2)$. It is not a combinatorial line.

for $2 \leq i \leq n$. Also, define

$$N = N_1 + \cdots + N_n = r^{t^n} + \sum_{i=2}^{n} r^{t^n + \sum_{k=1}^{i-1} N_k}.$$

Then, given an r-coloring of $(\{0\} \cup [t-1])^N$, there exists a concatenated root $\tau = \tau_1 \cdots \tau_n$ satisfying the following conditions:

(a) τ_i is an N_i-character root. Thus, τ is an N character root.

(b) For any two neighbors x and y in $(\{0\} \cup [t-1])^n$, $\tau(x)$ and $\tau(y)$ have the same color.

Proof. Let χ be the r coloring of $(\{0\} \cup [t-1])^n$. We prove this using backward induction on i. We assume that we have found $\tau_{i+1}, \ldots, \tau_n$ such that neighbors that differ in the kth character, $i+1 \leq k \leq n$ are the same color, and we find τ_i such that neighbors that differ in the ith character are the same color.

Suppose we have $\tau_{i+1}, \ldots, \tau_n$. For convenience, we define

$$M_{i-1} = \sum_{k=1}^{i-1} N_k,$$

which is the length of the first part of the concatenated root before τ_i. For $0 \leq k \leq N_i$, we define the following words:

$$w_k = \underbrace{0 \ldots 0}_{k} \underbrace{1 \ldots 1}_{N_i - k}.$$

We also define the following r-coloring χ_k on $(\{0\} \cup [t-1])^{M_{i-1}+n-i}$:

$$\chi_k(a_1 \cdots a_{M_{i-1}} b_{i+1} \cdots b_n) = \chi(a_1 \cdots a_{M_{i-1}} w_k \tau_{i+1}(b_{i+1}) \cdots \tau_n(b_n)).$$

This produces $N_i + 1$ colorings: $\chi_0, \ldots, \chi_{N_i}$. However, the total number of colorings of $(\{0\} \cup [t-1])^{M_{i-1}+n-i}$ is the number of colors r raised to the number of words, which is $t^{M_{i-1}+n-i}$. Thus, the number of colorings is

$$\begin{aligned} r^{t^{M_{i-1}+n-i}} &\leq r^{t^{n+M_{i-1}}} \\ &= r^{t^{n+\sum_{k=1}^{i-1} N_k}} \\ &= N_i. \end{aligned}$$

Then, we have generated more colorings than the total number of possible colorings, so two of them must be the same. Let χ_k and χ_l be identical colorings, with $k < l$. We now define τ_i:

$$\tau_i = \underbrace{0 \cdots 0}_{k} \underbrace{* \cdots *}_{l-k} \underbrace{1 \cdots 1}_{N_i - l}.$$

This satisfies the first part of the lemma because it has length N_i. It also satisfies the second part. Let x and y be neighbors in $(\{0\} \cup [t-1])^n$ that differ in the ith coordinate. Then,

$$\tau(x) = \tau_1(x_1) \cdots \tau_{i-1}(x_{i-1}) \tau_i(0) \tau_{i+1}(x_{i+1}) \cdots \tau_n(x_n)$$

and

$$\tau(y) = \tau_1(x_1) \cdots \tau_{i-1}(x_{i-1}) \tau_i(1) \tau_{i+1}(x_{i+1}) \cdots \tau_n(x_n).$$

Also, notice that

$$\begin{aligned} \tau_i(0) &= \underbrace{0 \cdots 0}_{k} \underbrace{0 \cdots 0}_{l-k} \underbrace{1 \cdots 1}_{N_i - l} \\ &= w_l, \end{aligned}$$

and, similarly, $\tau_i(1) = w_k$.

We now show that $\tau(x)$ and $\tau(y)$ are the same color under χ. First, from the definition of χ_l,

$$\begin{aligned} \chi(\tau(x)) &= \chi(\tau_1(x_1) \cdots \tau_{i-1}(x_{i-1}) \tau_i(0) \tau_{i+1}(x_{i+1}) \cdots \tau_n(x_n)) \\ &= \chi(\tau_1(x_1) \cdots \tau_{i-1}(x_{i-1}) w_l \tau_{i+1}(x_{i+1}) \cdots \tau_n(x_n)) \\ &= \chi_l(\tau_1(x_1) \cdots \tau_{i-1}(x_{i-1}) x_{i+1} \cdots x_n) \end{aligned}$$

Because χ_l and χ_k are identical, we can switch them. Thus,

$$\begin{aligned} \chi(\tau(x)) &= \chi_k(\tau_1(x_1) \cdots \tau_{i-1}(x_{i-1}) x_{i+1} \cdots x_n) \\ &= \chi(\tau_1(x_1) \cdots \tau_{i-1}(x_{i-1}) w_k \tau_{i+1}(x_{i+1}) \cdots \tau_n(x_n)) \\ &= \chi(\tau_1(x_1) \cdots \tau_{i-1}(x_{i-1}) \tau_i(1) \tau_{i+1}(x_{i+1}) \cdots \tau_n(x_n)) \\ &= \chi(\tau(y)). \end{aligned}$$

Thus, neighbors with differing indices from i to n are the same color. This completes the proof of the lemma. □

We now claim that the integer N satisfies the conditions of the Hales-Jewett theorem for r colors and a t-character alphabet, i.e., $HJ(r,t) \leq N$. Let χ be an r-coloring of $(\{0\} \cup [t-1])^N$. By the lemma above, there exists a root τ of length N such that $\chi(\tau(x)) = \chi(\tau(y))$ for any two neighbors x and y in $(\{0\} \cup [t-1])^n$. We now consider the alphabet $T = [t-1] = \{1, \dots, t-1\}$, which has $t-1$ characters. We define a coloring χ^* on T^n by

$$\chi^*(x) = \chi(\tau(x)).$$

By the induction hypothesis, since $HJ(r, t-1) = n$, there exists a root σ corresponding to a monochromatic line in T^n under χ^*.

We claim that $\tau(\sigma)$ is a root in $(\{0\} \cup [t-1])^N$ that corresponds to a monochromatic line under χ. Note that the wildcards in τ_i get replaced with σ_i. By the construction of τ in the proof of the lemma, each τ_i contains some wildcards. Because σ is a root, some σ_j is a wildcard. Then, $\tau(\sigma)$ contains the wildcards that were in τ_j, so $\tau(\sigma)$ is a nontrivial root.

Let $1 \leq i, j \leq n$. Then, $\chi^*(\sigma(i)) = \chi^*(\sigma(j))$. By the definition of χ^* above, we have

$$\chi(\tau(\sigma(i))) = \chi(\tau(\sigma(j))).$$

Then, for the line to be monochromatic, we need $\chi(\tau(\sigma(0))) = \chi(\tau(\sigma(1)))$. Suppose that $\tau(\sigma)$ has one wildcard. Then, $\tau(\sigma(0))$ and $\tau(\sigma(1))$ are neighbors, so they are the same color. If $\tau(\sigma)$ has more than one wildcard, we go through a series of steps changing the differing indices one by one. For example, with three, we would do the following:

$$\begin{aligned}\chi(\tau(\sigma(0))) &= \chi(\tau_{11} \cdots 0 \cdots 0 \cdots 0 \cdots \tau_{nN_n}) \\ &= \chi(\tau_{11} \cdots 0 \cdots 0 \cdots 1 \cdots \tau_{nN_n}) \\ &= \chi(\tau_{11} \cdots 0 \cdots 1 \cdots 1 \cdots \tau_{nN_n}) \\ &= \chi(\tau_{11} \cdots 1 \cdots 1 \cdots 1 \cdots \tau_{nN_n}) \\ &= \chi(\tau(\sigma(1))).\end{aligned}$$

We can do this with any number of wildcards. Thus, we have found a monochromatic line for an arbitrary r-coloring. Then, $HJ(r, t)$ is finite. This completes the proof of the Hales-Jewett Theorem.

5. Some Applications of Hales-Jewett

5.1. **Van der Waerden's Theorem.** Van der Waerden's Theorem is a theorem concerning monochromatic arithmetic progressions. Originally, van der Waerden wanted to solve the problem of showing that if $\mathbb{N}$ is two colored, then one color contains arbitrarily long arithmetic progressions. He proved this by proving the following more general theorem [4].

Theorem 5.1 (van der Waerden)**.** *Given natural numbers r and l, there exists a natural number $W(r, l)$ such that if set $\{1, 2, \dots, W(r, l)\}$ is colored with r colors, there exists a monochromatic arithmetic progression of length l.*

Proof. Let $N = HJ(r, l)$. We claim that $W(r, l) = N(l-1)+1$ satisfies the conditions of the theorem. Give $(\{0\} \cup [N(l-1)])$ an r-coloring. Define the map f from the set of words $(\{0\} \cup [l-1])^N$ to $(\{0\} \cup [N(l-1)])$ by

$$f(a_1, \dots, a_N) = a_1 + \cdots + a_l,$$

which takes on maximum value $N(l-1)$. Color the words w with the color given to $f(w)$. This defines an r-coloring on the cube. By the Hales-Jewett theorem, there exists a monochromatic combinatorial line, $\{\tau(0), \dots, \tau(l-1)\}$. Because $f(\tau(m)) - f(\tau(m-1))$ is equal to the number of wildcards in τ, the sequence

$f(\tau(0)), \ldots, f(\tau(l-1))$ is an arithmetic progression of length l. Given any coloring of $[N(t-1)+1]$, we can shift to $(\{0\} \cup [N(l-1)])$ to find the progression then shift back to $[N(l-1)+1]$. □

To see how the original problem follows, color $\mathbb{N}$ with two colors. Let A_k be a monochromatic arithmetic progression of length k in $[W(2,k)]$, which exists by the theorem. By the pigeonhole principle, one color must contain infinitely many A_k's. Thus, it contains arbitrarily long arithmetic progressions.

5.2. **Gallai-Witt Theorem.** Another result that follows from the Hales-Jewett Theorem is the Gallai-Witt Theorem [4]. We first need the following definition.

Definition 5.2. Given a set of vectors V, a homothetic copy of V is a set $u+\lambda V = \{u+\lambda v_i : v_i \in V\}$, where u is a vector and λ is a scalar.

A homothetic copy is a translation and dilation.

Theorem 5.3 (Gallai-Witt)**.** *Color $\mathbb{Z}^m$ with r colors. Then, for any finite subset of vectors V, there exists a monochromatic homothetic copy of V.*

Proof. Color $\mathbb{Z}^m$ with r colors. Let $V = \{v_1, \ldots, v_t\}$ contain t vectors. We will consider V as the underlying alphabet. Let $N = HJ(r,t)$. Define a function $f : V^N \to \mathbb{Z}^m$ by $f(x) = x_1 + \cdots + x_N$, where $x \in V^N$ and $x = (x_1, \ldots, x_N)$, with $x_i \in V$. Note that V^N is the set of words of length N. Color this set such that the color of $w \in V^N$ is the same as that of $f(w) \in \mathbb{Z}^m$. This induces an r-coloring on the cube. By the Hales-Jewett theorem, there is a monochromatic line $\{\tau(v_1), \ldots, \tau(v_t)\}$. Let $\tau = \tau_1, \ldots, \tau_N$. Let λ be the number of wildcards in τ, and let $I = \{i : \tau_i \neq *\}$. Then, $f(\tau(v_i)) = \sum_{i \in I} \tau_i + \lambda v_i$. Thus, with $u = \sum_{i \in I} \tau_i$,

$$\{f(\tau(v_1)), \ldots, f(\tau(v_t))\} = \left\{ \sum_{i \in I} \tau_i + \lambda v_i : v_i \in V \right\} = \{u + \lambda v_i : v_i \in V\}.$$

This is a monochromatic homothetic copy of V. □

Acknowledgments. It is a pleasure to thank my mentor, Tori Akin, for helping me each week and commenting on earlier drafts and ideas to help me put this paper together.

References

[1] Ronald Graham, Bruce Rothschild and Joel Spencer. Ramsey Theory. John Wiley & Sons, Inc. 1980.
[2] Noga Alon and Joel Spencer. The Probabilistic Method. John Wiley & Sons, Inc. 2000.
[3] Jacob Fox. Lecture 5: Ramsey Theory. http://math.mit.edu/ fox/MAT307-lecture05.pdf.
[4] Marcus Näslund. The Hales-Jewett theorem and its application to further generalisations of m, n, k-games. http://uu.diva-portal.org/smash/get/diva2:631339/FULLTEXT01.pdfd