# HAMILTONIAN SYSTEMS AND NOETHER'S THEOREM

DANIEL SPIEGEL

ABSTRACT. This paper uses the machinery of symplectic geometry to make rigorous the mathematical framework of Hamiltonian mechanics. This framework is then shown to imply Newton's laws and conservation of energy, thus validating it as a physical theory. We look at symmetries of physical systems in the form of Lie groups, and show that the Hamiltonian framework grants us the insight that the existence of a symmetry corresponds to the conservation of a physical quantity, i.e. Noether's Theorem. Throughout the paper we pay heed to the correspondence between mathematical definitions and physical concepts, and supplement these definitions with examples.

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## 1. INTRODUCTION

In 1834, the Irish mathematician William Hamilton reformulated classical Newtonian mechanics into what is now known as Hamiltonian mechanics. While Newton's laws are valuable for their clarity of meaning and utility in Cartesian coordinates, Hamiltonian mechanics provides ease of computation for many other choices of coordinates (as did the Lagrangian reformulation in 1788), tools for handling statistical systems with large numbers of particles, and a more natural transition into quantum mechanics. We refer the reader to chapter 13 of John Taylor's *Classical Mechanics* [3] for an introductory discussion of these uses. This paper, however, will focus on the insights that Hamilton's formalism provides with regards to the conservation of physical quantities. For all these advantages, it is clear why Hamiltonian mechanics is an important area of study for any physics student.

Unfortunately, by the undergraduate level many physics students will not have been exposed to the mathematical tools necessary for fully developing Hamiltonian mechanics. Such tools include manifolds, differential forms, and groups. By assuming basic knowledge of these topics we afford ourselves a concise yet rigorous study of the foundations of Hamiltonian mechanics and its utility in identifying conserved quantities via Noether's theorem, which we prove in the final section.

In general, this paper will follow the organization of Stephanie Frank Singer's *Symmetry in Mechanics* [1]. In the first section we shall introduce the defining components of a Hamiltonian system–a smooth manifold $M$, a symplectic form $\omega$ on $M$, and a smooth real-valued function $H$ on $M$–and show how these define a unique vector field on $M$ that conserves $H$ and follows the trajectory of physical particles. In the second section we will define physical symmetries in terms of Lie group actions on $M$ and illustrate these definitions through the matrix Lie group $SO(3)$. In the third and final section, we will state and prove Noether's theorem in terms of momentum maps and the symmetries defined in section two. We will follow the example of $SO(3)$ to show that if the Hamiltonian system is invariant under the action of $SO(3)$, then Noether's theorem tells us that angular momentum is conserved.

## 2. Hamiltonian Systems

We begin with a foundational assumption of Hamiltonian mechanics: for any particle, the set of all its possible positions is an $n$-dimensional ($n \leq 3$) smooth manifold $C$ in $\mathbb{R}^3$, called *configuration space*. The manifold that we are interested in, however, is the *cotangent bundle* of $C$:

$$T^*C := \{(\mathbf{r}, \mathbf{p}^T) \in \mathbb{R}^3 \times (\mathbb{R}^3)^* : \mathbf{r} \in C \text{ and } \mathbf{p}^T \in (T_\mathbf{r}C)^*\}$$

where $\mathbf{r}$ is a column vector, $\mathbf{p}^T$ is a row vector, and $(T_\mathbf{r}C)^*$ is the dual space of the tangent space of $C$ at $\mathbf{r}$. We note that $T^*C$ is itself a smooth $2n$-dimensional manifold [1], and from now on we shall label such manifolds as $M$. By definition of the tangent space, $T_\mathbf{r}C$ is the set of all possible momentums $\mathbf{p}$ a particle traveling through $\mathbf{r}$ can take, and since we're working in finite dimensions, $(T_\mathbf{r}C)^* \cong T_\mathbf{r}C$, so we can identify each element of $(T_\mathbf{r}C)^*$ as the transpose $\mathbf{p}^T$ of a momentum column vector $\mathbf{p}$. From now on we shall write $\mathbf{p}$ for the row vectors of $(T_\mathbf{r}C)^*$.

In case we want to incorporate $N$ particles into our system, we may just take the cotangent bundle of a Cartesian product of configuration spaces, such that our manifold $M$ has coordinates $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ where $\mathbf{r}_i$ and $\mathbf{p}_i$ represents the position and momentum of the $i$th particle for $i = 1, \ldots, N$. For many systems, such as a free particle, a planetary system, or an ideal harmonic oscillator, we will have $M = \mathbb{R}^{3N} \times (\mathbb{R}^{3N})^*$ or $M = \mathbb{R} \times (\mathbb{R})^*$ for a particle constrained to move in one dimension for example.

The second foundational assumption we shall make is that the net force $\mathbf{F}$ on the particle depends only on the particle's position and momentum. This covers many physical forces, such as gravitational, electromagnetic, frictional, and spring forces, so it is often a fair assumption [3]. Newton's second law states

$$\mathbf{F}(\mathbf{r}(t), \mathbf{p}(t)) = \frac{d\mathbf{p}}{dt}$$

Since momentum is defined as

$$\mathbf{p}(t) := m\frac{d\mathbf{r}}{dt}$$

the existence and uniqueness theorem for first order ordinary differential equations tells us that if we specify $\mathbf{r}$ and $\mathbf{p}$ at a given time, then we know the particle's position and momentum at an interval around that time (with some caveats about the continuity of $\mathbf{F}$). Thus every point $(\mathbf{r}, \mathbf{p})$ on our manifold $M$ should define a path through $M$ that corresponds to the trajectory of the particle.

We need a way to find this trajectory, and we can do so given a symplectic form $\omega$ and an infinitely differentiable function $H$ on $M$.

**Definition 2.1.** A *symplectic form* $\omega$ is a closed, nondegenerate 2-form on a manifold $M$. In other words, the following two conditions are satisfied.

(1) $d\omega = 0$. (Closedness)
(2) For all $\mathbf{u} \in M$ and nonzero $\mathbf{v} \in T_{\mathbf{u}}M$, there exists $\mathbf{w} \in T_{\mathbf{u}}M$ such that $\omega_{\mathbf{u}}(\mathbf{v}, \mathbf{w}) \neq 0$. (Nondegeneracy)

As it turns out, every cotangent bundle of a smooth manifold has a canonical symplectic form

$$\omega = \sum dp_i \wedge dr_i$$

or equivalently $\omega = d\mathbf{p} \wedge d\mathbf{r}$ in more condensed notation. A manifold $M$ with a symplectic form $\omega$ is called a *symplectic manifold* $(M, \omega)$. When that manifold is the cotangent bundle of configuration space, $(M, \omega)$ is called *phase space*. For a more complete discussion of the cotangent bundle and it's symplectic structure, see [2].

As for an infinitely differentiable function, we can assign to each point of $M$ a total energy, and assume this function is infinitely differentiable (it is in all realistic cases). We call this function the Hamiltonian and label it $H : M \to \mathbb{R}$. The important fact is that the $\omega$ and $H$ determine a vector field on $M$ along which a particle will typically flow.

**Lemma 2.2.** *Given a $2-$form $\omega$ on an $n-$dimensional smooth manifold $M$, there exists a family of antisymmetric matrices $\{A_{\mathbf{q}}\}_{\mathbf{q} \in M}$ such that for all $\mathbf{q} \in M$ and $\mathbf{u}, \mathbf{v} \in T_{\mathbf{q}}M$, we have*

$$\omega_{\mathbf{q}}(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T A_{\mathbf{q}} \mathbf{v}$$

*Proof.* Since $\omega$ is a 2-form, for every $\mathbf{q} \in M$, $\omega_{\mathbf{q}}$ is bilinear and antisymmetric and can be represented as

$$\omega_p(\mathbf{u}, \mathbf{v}) = \sum_{i<j} \alpha_{ij}(p) \, dx_i \wedge dx_j$$

where $\alpha_{ij}$ are real-valued functions on $M$. Then define the matrices $A_{\mathbf{q}}$ such that

$$A_{\mathbf{q}} = \{\alpha_{ij}(\mathbf{q})\} \text{ for } i < j$$

and the remaining entries are determined by requiring $A_{\mathbf{q}}$ to be antisymmetric. Then the expression $\mathbf{u}^T A_{\mathbf{q}} \mathbf{v}$ also defines a 2-form that agrees with $\omega$ for all the coefficient functions, and so $\omega_{\mathbf{q}}(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T A_{\mathbf{q}} \mathbf{v}$. $\square$

**Theorem 2.3.** *Given an infinitely differentiable function $H$ on a smooth manifold $M$ with symplectic form $\omega$, there exists a unique vector field $X_H$ satisfying*

$$dH(\cdot) = -\omega(X_H, \cdot)$$

*Proof.* Let $A_{\mathbf{q}}$ denote the matrices from Lemma 2.2. Since $\omega$ is symplectic, and therefore nondegenerate, $A_{\mathbf{q}}$ is invertible for all $\mathbf{q}$. We define the vector field $X_H$ as

$$X_H(\mathbf{q}) = A_{\mathbf{q}}^{-1}(\nabla H(\mathbf{q}))^T$$

where $\nabla H(\mathbf{q})$ is the row vector gradient of $H$ at $\mathbf{q}$. We show that $X_H$ fulfills the desired condition, using the fact that the inverse of an antisymmetric matrix is antisymmetric:

$$\nabla H \cdot (A_q^{-1})^T = X_H^T$$
$$-\nabla H \cdot A_{\mathbf{q}}^{-1} = X_H^T$$
$$-\nabla H = X_H^T A_{\mathbf{q}}$$

Since $dH$ is the one form that multiplies $\nabla H$ to a vector in the tangent space, this proves that $X_H$ satisfies

$$dH(\cdot) = -\omega(X_H, \cdot)$$

We need to prove $X_H$ is unique. Suppose there exists another vector field $Y_H$ that satisfies the above property. Then by linearity of $\omega$

$$-\omega(X_H - Y_H, \cdot) = -\omega(X_H, \cdot) + \omega(Y_H, \cdot)$$
$$= dH(\cdot) - dH(\cdot)$$
$$= 0$$

But by nondegeneracy of $\omega$, this implies that $X_H - Y_H \equiv 0$, so $X_H$ is unique. $\quad\square$

The significance of the vector field $X_H$ and the symplectic form $\omega$ can be seen through the example of a particle under the influence of conservative forces. This means that the net force can be written as

$$\mathbf{F} = -\nabla U$$

where $U(\mathbf{r})$ is a real-valued potential energy function depending only on the position of the particle. Physics provides many important examples of such systems, including idealized harmonic oscillators and celestial motions.

**Example 2.4.** If the forces on a particle are conservative, the Hamiltonian is

$$H = \frac{\mathbf{p}^2}{2m} + U(\mathbf{r})$$

where the first term is the kinetic energy and the second the potential energy. We use the canonical symplectic form $\omega = d\mathbf{p} \wedge d\mathbf{r}$. Let us work out explicitly what $X_H$ is. We denote

$$X_H = \sum_{i=1}^{3} \left( a_i(\mathbf{r}, \mathbf{p}) \frac{\partial}{\partial r_i} + b_i(\mathbf{r}, \mathbf{p}) \frac{\partial}{\partial p_i} \right) := \mathbf{a}(\mathbf{r}, \mathbf{p}) \frac{\partial}{\partial \mathbf{r}} + \mathbf{b}(\mathbf{r}, \mathbf{p}) \frac{\partial}{\partial \mathbf{p}}$$

Then

$$-\omega(X_H, \cdot) = d\mathbf{r} \wedge d\mathbf{p} \left( \mathbf{a}(\mathbf{r}, \mathbf{p}) \frac{\partial}{\partial \mathbf{r}} + \mathbf{b}(\mathbf{r}, \mathbf{p}) \frac{\partial}{\partial \mathbf{p}}, \cdot \right)$$
$$= \mathbf{a}(\mathbf{r}, \mathbf{p}) \, d\mathbf{p} - \mathbf{b}(\mathbf{r}, \mathbf{p}) d\mathbf{r}$$

We then calculate $dH$ as

$$dH = \frac{\partial H}{\partial \mathbf{r}} d\mathbf{r} + \frac{\partial H}{\partial \mathbf{p}} d\mathbf{p}$$
$$= \nabla U d\mathbf{r} + \frac{\mathbf{p}}{m} d\mathbf{p}$$

Setting the coefficient functions of $dH$ and $-\omega(X_H, \cdot)$ equal to each other yields

$$X_H = \frac{\mathbf{p}}{m}\frac{\partial}{\partial \mathbf{r}} - \nabla U \frac{\partial}{\partial \mathbf{p}}$$

Now let us suppose that $\gamma(t) = (\mathbf{r}(t), \mathbf{p}(t))$ is a curve on $M$ that satisfies the equations of motion, that is:

$$\frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{m} \quad \text{and} \quad \frac{d\mathbf{p}}{dt} = -\nabla U$$

Then clearly $\gamma'(t) = X_H(\gamma(t))$. Likewise if $\gamma(t)$ satisfies $\gamma'(t) = X_H(\gamma(t))$, then $\gamma(t)$ solves the equations of motion. This demonstrates that the symplectic form encodes the equations of motion into our Hamiltonian system, and that the vector field $X_H$ points along physical trajectories. When dealing with nonconservative forces, such as magnetic forces, the symplectic form is not always the canonical one and the Hamiltonian function does not always correspond to total energy. See section 4.3 of [1] for an example using a different symplectic form and chapter 13 of [3] for a more general form of the Hamiltonian function. Ultimately the goal is to choose $\omega$ and $H$ so as to best encode into our system the equations of motion given by Newton's second law.

Despite the ambiguity in choice of $\omega$ and $H$, the vector field $X_H$ always points along the direction of constant $H$, i.e. $X_H$ is tangent to the level sets of $H$. We take advantage of the partial differentiation notation for $X_H$ to define

$$X_H H := \mathbf{a}(\mathbf{r}, \mathbf{p})\frac{\partial H}{\partial \mathbf{r}} + \mathbf{b}(\mathbf{r}, \mathbf{p})\frac{\partial H}{\partial \mathbf{p}} = dH(X_H)$$

**Corollary 2.5.** *Given a Hamiltonian system $(M, \omega, H)$ and Hamiltonian vector field $X_H$, we have*

$$X_H H = 0$$

*Proof.* The proof is immediate from the definitions.

$$X_H H = dH(X_H) = -\omega(X_H, X_H) = 0$$

where $\omega(X_H, X_H) = 0$ since $\omega$ is antisymmetric.                    $\square$

Therefore conservation of energy is equivalent to the statement that $X_H$ points along physical trajectories. We would like to know when energy is conserved, and whether there are other physical quantities that are conserved along trajectory of our particle. The answers to these questions are given by Noether's theorem. We understand Noether's theorem physically through the lens of symmetry, and we understand symmetry mathematically through the language of Lie group actions, which we develop in the next section.

## 3. Symmetries and Lie Groups

In physics, a *symmetry* is a change of coordinates that leaves the equations of motion unchanged. Recall that the equations of motion are encoded in our Hamiltonian system $(M, \omega, H)$, and so we can equivalently say that a symmetry is a change of coordinates that leaves our Hamiltonian system unchanged.

**Definition 3.1.** A *symmetry* of a Hamiltonian system $(M, \omega, H)$ is a function $S : M \to M$ such that

$$S^*\omega = \omega \quad \text{and} \quad S^*H = H$$

where $S^*$ is the pullback of $\omega$ or $H$ by $S$.

We can make powerful statements about conserved quantities when we have entire collections of symmetries. Consider a set of symmetries, which we construe as a changes of coordinates. We are changing coordinates from some initial reference frame which can be represented as the identity function, so we can include the identity in our set of symmetries. For each change of coordinates we should be able to change back, and by definition of the pullback we should be able to compose symmetries and get another symmetry. Note also that since symmetries are functions, composition is an associative operation. Thus, collections of symmetries can naturally be modeled by groups. In particular, we want these collections to have some smoothness to them, so we typically model them as *Lie groups*.

**Definition 3.2.** A *Lie group* is a smooth manifold $G$ with an operation $\cdot : G \times G \to G$ such that $(G, \cdot)$ is a group, and such that multiplication and inversion are smooth functions.

**Example 3.3.** Consider the set of invertible, $n \times n$ real matrices, called $GL(n, \mathbb{R})$. Consider a coordinate map $\phi : U \subset \mathbb{R}^{n^2} \to GL(n, \mathbb{R})$ that assigns a unique component of an $n^2$-vector to each entry of an $n \times n$ matrix, where $U$ is the set $\mathbf{v} \in \mathbb{R}^{n^2}$ such that $\det(\phi(\mathbf{v})) \neq 0$. Note that $\phi$ is bijective. Since $\det(\phi(\mathbf{v}))$ is a polynomial in the entries of $\mathbf{v}$, $U$ is an open set. Therefore $GL(n, \mathbb{R})$ is a smooth manifold. Since the determinant distributes over matrix multiplication and matrix multiplication is associative, matrix multiplication makes $GL(n, \mathbb{R})$ into a group. Since multiplication and inversion are polynomial and rational operations, they are smooth operations. Therefore $GL(n, \mathbb{R})$ satisfies the definition of a Lie group.

We state without proof that any closed subgroup of a Lie group is also a Lie group. It follows that any closed matrix group (with matrix multiplication as the group operation) is a Lie group. Matrix groups in particular are of interest in Hamiltonian mechanics because they correspond to linear changes of coordinates, and transformations between inertial reference frames can be described by linear changes of coordinates on phase space.

We give two important examples of Lie groups–the group $(\mathbb{R}^3, +)$ and the matrix group $SO(3)$.

**Example 3.4.** The group $(\mathbb{R}^3, +)$ is 3-space where group "multiplication" is is just vector addition, and obviously satisfies the definition of a Lie group.

**Example 3.5.** The group $SO(3)$ is defined as

$$SO(3) = \{A \in M_{3 \times 3}(\mathbb{R}) : \det(A) = 1 \text{ and } A^T A = AA^T = I\}$$

Consider a sequence of matrices $\{A_n\} \subset SO(3)$ converging (componentwise) to $A$. Since the determinant is continuous $\det(A_n) \to \det(A)$ so $\det(A) = 1$. Furthermore we can take limits in the orthogonality condition $A_n^T A_n \to A^T A$, so $A^T A = I$ and so also $AA^T = I$. Thus $SO(3)$ is a closed subgroup of $GL(n, \mathbb{R})$, and is therefore a Lie group.

But we are concerned with symmetries, which are functions on the manifold, not just the groups themselves. We can transform groups into possible symmetries by defining *group actions*.

**Definition 3.6.** Given a group $G$ and a manifold $M$, an *action* of $G$ on $M$ is a function $S_g : M \to M$ satisfying the following properties:

(1) If $I$ is the identity element of $G$, then $S_I$ is the identity function on $M$.
(2) Composition of actions respects group multiplication. That is, for any $g, h \in G$, we have $S_g \circ S_h = S_{gh}$.

Given a cotangent bundle $M$, we can define actions for the above two examples. For $G = (\mathbb{R}^3, +)$, we define:

$$S_{\mathbf{g}}(\mathbf{r}, \mathbf{p}) = (\mathbf{r} + \mathbf{g}, \mathbf{p})$$

For $G = SO(3)$, we define:

$$S_g(\mathbf{r}, \mathbf{p}) = (g\mathbf{r}, \mathbf{p}g^T)$$

The significance of these group actions is that there are many applications for which these groups are symmetries of Hamiltonian systems, as defined in Definition 3.1.

**Example 3.7.** For both of the above groups, we claim $S_g$ is a symmetry for all group elements $g$ when the Hamiltonian system is that of a free particle, i.e.

$$M = \mathbb{R}^3 \times (\mathbb{R}^3)^*$$
$$\omega = d\mathbf{p} \wedge d\mathbf{r}$$
$$H = \frac{\mathbf{p}^2}{2m}$$

For $(\mathbb{R}^3, +)$, we see that

$$S_{\mathbf{g}}^* \omega = (S_{\mathbf{g}}^* d\mathbf{p}) \wedge (S_{\mathbf{g}}^* d\mathbf{r}) = \frac{\partial}{\partial \mathbf{p}}[\mathbf{p}]d\mathbf{p} \wedge \frac{\partial}{\partial \mathbf{r}}[\mathbf{r} + \mathbf{g}]d\mathbf{r} = d\mathbf{p} \wedge d\mathbf{r}$$

and

$$S_{\mathbf{g}}^* H = H \circ S_{\mathbf{g}} = H(\mathbf{r} + \mathbf{g}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} = H$$

For $SO(3)$, we have

$$S_g^* \omega = \frac{\partial}{\partial \mathbf{p}}[\mathbf{p}g^T]d\mathbf{p} \wedge \frac{\partial}{\partial \mathbf{r}}[g\mathbf{r}]d\mathbf{r} = g^T g \, d\mathbf{p} \wedge d\mathbf{r} = d\mathbf{p} \wedge d\mathbf{r}$$

and

$$S_g^* H = H(g\mathbf{r}, \mathbf{p}g^T) = \frac{1}{2m}[\mathbf{p}g^T g\mathbf{p}] = \frac{\mathbf{p}^2}{2m}$$

which proves the claim.

Since Lie groups are smooth manifolds, each member of a Lie group comes equipped with a tangent space. The tangent space at the identity is of particular interest.

**Definition 3.8.** The *Lie algebra* $\mathfrak{g}$ of a Lie group $G$ is the tangent space $T_e G$ of $G$ at the identity element $e$, together with a canonical bracket operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket.

By "bracket operation" we mean a bilinear function $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:

(1) $[\xi, \xi] = 0$ for all $\xi \in \mathfrak{g}$.
(2) $[\xi, [\eta, \zeta]] + [\zeta, [\xi, \eta]] + [\eta, [\zeta, \xi]] = 0$ for all $\xi, \eta, \zeta \in \mathfrak{g}$.

For our purposes we will not need the bracket with which the Lie algebra is equipped, but we remark that every Lie group $G$ is endowed with a canonical bracket operation satisfying the above criteria. See Chapter 3 of [4] for more on Lie algebras.

The elements of $\mathfrak{g}$ correspond to infinitesimal coordinate changes we can make to our reference frame. We can compute the Lie algebra by remembering that the elements of the tangent space are exactly the derivatives of curves through $G$ at the tangent point, and that the tangent space has the same dimension as the manifold.

**Example 3.9.** For $(\mathbb{R}^3, +)$, it is clear that $\mathfrak{g} = \mathbb{R}^3$ considering the curves $f_i(t) = t\mathbf{e}_i$.

**Example 3.10.** We notate the Lie algebra of $SO(3)$ by $so(3)$. We want to show that

$$so(3) = \{\xi \in M_{3\times3}(\mathbb{R}) : -\xi = \xi^T\}$$

Consider a path $A(t)$ through $SO(3)$ such that $A(0) = I$. Differentiating the equation

$$A(t)^T A(t) = I$$

yields

$$A'(0)^T + A'(0) = 0$$

so $A'(0)^T = -A'(0)$. This shows that $so(3)$ is a subset of the antisymmetric matrices. Now consider an antisymmetric matrix $\xi$ and the path

$$f(t) = e^{t\xi}$$

We first need to show that $f(t)$ is indeed a path through $SO(3)$. Recall that $e^{t\xi}$ is defined for matrices by a Taylor expansion, and that transposition distributes over addition and taking powers of matrices. Therefore

$$(e^{t\xi})^T e^{t\xi} = e^{t\xi^T} e^{t\xi} = e^{-t\xi} e^{t\xi} = I$$

Now fix $\xi$ and $t$ and define for $s \in [0, t]$

$$\eta(s) = e^{s\xi}$$

Since matrix exponentiation and the determinant are continuous, we know that $\det(\eta(s))$ is continuous in $s$. By orthogonality of $e^{t\xi}$ we know that $\det(\eta(s))$ must be 1 or $-1$ for all $s$. But since $\det(\eta(0)) = 1$, we know that $\det(\eta(t)) = 1$ by the intermediate value theorem. Therefore $e^{t\xi} \in SO(3)$. Taking the derivative gives $f'(0) = \xi$, so $\xi \in so(3)$.

Given a Lie algebra element $\xi$, there exists a natural path through the corresponding Lie group in the direction of $\xi$. For a proof of the following proposition, see [4].

**Proposition 3.11.** *For each element $\xi$ of the Lie algebra $\mathfrak{g}$ of a Lie group $G$, there exists a unique function $f_\xi : \mathbb{R} \to G$ such that $f_\xi(s+t) = f_\xi(s) \cdot f_\xi(t)$ for all $s, t \in \mathbb{R}$ and such that*

$$f_\xi'(0) = \xi$$

We call the image of the above $f_\xi$ the one-parameter subgroup of $\xi$. The properties of the map $f_\xi$ are akin to those of the exponential map, and so we define the exponential of an arbitrary Lie algebra element $\xi$ as

$$e^{s\xi} := f_\xi(s)$$

For $(\mathbb{R}^3, +)$, one can verify that $e^{s\xi} = s\xi$ and for $SO(3)$, exponentiation is done by matrix exponentiation via the Taylor expansion.

The one-parameter subgroup associates a Lie algebra element with a natural path through a Lie group. If that Lie group acts on a manifold $M$, then the one-parameter subgroup gives a natural path on $M$. We can differentiate along this path to obtain a vector field on $M$ associated with a Lie algebra element.

**Definition 3.12.** Given a Lie group $G$ acting on a manifold $M$ with actions $S_g$, for each element $\xi \in \mathfrak{g}$, the vector field $\xi_M$ associated to $\xi$ is given by

$$\xi_M(m) = \frac{d}{dt}\bigg|_{t=0} S_{e^{t\xi}}(m)$$

**Example 3.13.** We compute $\xi_M$ for $(\mathbb{R}^3, +)$ and $\xi \in \mathbb{R}^3$.

$$\xi_M(\mathbf{r}, \mathbf{p}) = \frac{d}{dt}\bigg|_{t=0} (\mathbf{r} + t\xi, \mathbf{p}) = \xi\frac{\partial}{\partial \mathbf{r}}$$

**Example 3.14.** For $SO(3)$, given an antisymmetric matrix $\xi \in so(3)$ we find

$$\xi_M(\mathbf{r}, \mathbf{p}) = \frac{d}{dt}\bigg|_{t=0} (e^{t\xi}\mathbf{r}, \mathbf{p}e^{t\xi^T}) = (\xi\mathbf{r}, \mathbf{p}\xi^T) = \xi\mathbf{r}\frac{\partial}{\partial \mathbf{r}} - \mathbf{p}\xi\frac{\partial}{\partial \mathbf{p}}$$

As we shall see in the next section, these vector fields play a crucial role in developing the momentum map needed for Noether's theorem.

## 4. Noether's Theorem

In the Hamiltonian formulation, Noether's theorem uses the mathematical language of the previous section to formalize the notion that symmetries in physical systems cause the conservation in time of certain quantities. Since our manifold $M$ describes the state of the system, these conserved quantities will naturally be the value of a function on $M$. In many examples in physics the conserved quantity is linear or angular momentum, and so we call said function on $M$ a *momentum map*.

**Definition 4.1.** Let $(M, \omega, H)$ be a Hamiltonian system and $G$ a Lie group acting on $M$ with Lie algebra $\mathfrak{g}$. Let $\Phi : M \to \mathfrak{g}^*$ be a smooth map and for each $\xi \in \mathfrak{g}$ define $\Phi^\xi : M \to \mathbb{R}$ as

$$\Phi^\xi(m) = (\Phi(m))(\xi)$$

Then $\Phi$ is a *momentum map* if for each $\xi \in \mathfrak{g}$

$$-\omega(\xi_M, \cdot) = d\Phi^\xi(\cdot)$$

Recalling Theorem 2.3 and Corollary 2.4, we see that $\Phi$ is a momentum map if and only if for all $\xi \in \mathfrak{g}$, we have $\xi_M = X_{\Phi^\xi}$ where $X_{\Phi^\xi}$ is the vector field from Theorem 2.3.

Continuing our examples, we show that linear momentum and angular momentum are the momentum maps for the translation action of $(\mathbb{R}^3, +)$ and rotation action of $SO(3)$, assuming $\omega = d\mathbf{p} \wedge d\mathbf{r}$.

**Example 4.2.** The Lie algebra of $(\mathbb{R}^3, +)$ is $\mathbb{R}^3$, so we represent dual space elements as row vectors. We claim $\Phi : M \to (\mathbb{R}^3)^*$ defined as

$$\Phi(\mathbf{r}, \mathbf{p}) = \mathbf{p}$$

is the momentum map for the translation action (for a multiparticle system we sum the momentums). Given arbitrary $\xi \in \mathbb{R}^3$, we have

$$-\omega(\xi_M, \cdot) = -\omega\left(\xi \frac{\partial}{\partial \mathbf{r}}, \cdot\right) = \xi \, d\mathbf{p}$$

Furthermore we have $\Phi^\xi(\mathbf{r}, \mathbf{p}) = \mathbf{p}\xi$, so

$$d\Phi^\xi = \xi \, d\mathbf{p}$$

which proves $\Phi(\mathbf{r}, \mathbf{p}) = \mathbf{p}$ is the momentum map of $(\mathbb{R}^3, +)$.

**Example 4.3.** For $SO(3)$, we first note that any linear functional $\eta$ on the space of antisymmetric matrices can itself be represented as an antisymmetric matrix if we consider it as the following map.

$$\xi \mapsto -\frac{1}{2}\mathrm{tr}\begin{pmatrix} 0 & -\eta_z & \eta_y \\ \eta_z & 0 & -\eta_x \\ -\eta_y & \eta_x & 0 \end{pmatrix}\begin{pmatrix} 0 & -\xi_z & \xi_y \\ \xi_z & 0 & -\xi_x \\ -\xi_y & \xi_x & 0 \end{pmatrix} = \eta_x\xi_x + \eta_y\xi_y + \eta_z\xi_z$$

Using this representation we claim that the momentum map for $SO(3)$ is

$$\Phi(\mathbf{r}, \mathbf{p}) = (\mathbf{rp} - (\mathbf{rp})^T) = \begin{pmatrix} 0 & r_xp_y - r_yp_x & r_xp_z - r_zp_x \\ r_yp_x - r_xp_y & 0 & r_yp_z - r_zp_x \\ r_zp_x - r_xp_z & r_zp_y - r_yp_z & 0 \end{pmatrix}$$

Given an arbitrary antisymmetric matrix $\xi$, we have

$$-\omega(\xi_M, \cdot) = -\omega\left(\xi\mathbf{r}\frac{\partial}{\partial \mathbf{r}} - \mathbf{p}\xi\frac{\partial}{\partial \mathbf{p}}\right) = \xi\mathbf{r} \, d\mathbf{p} + \mathbf{p}\xi \, d\mathbf{r}$$

and

$$\begin{aligned}
d\Phi^\xi &= d\left(-\frac{1}{2}\mathrm{tr}[\mathbf{rp}\xi - (\mathbf{rp})^T\xi]\right) \\
&= d\left[(r_yp_z - r_zp_y)\xi_x + (r_zp_x - r_xp_z)\xi_y + (r_xp_y - r_yp_x)\xi_z\right] \\
&= (p_y\xi_z - p_z\xi_y)dr_x + (p_z\xi_x - p_x\xi_z)dr_y + (p_x\xi_y - p_y\xi_z)dr_z \\
&\quad + (r_z\xi_y - r_y\xi_z)dp_x + (r_x\xi_z - r_z\xi_x)dp_y + (r_y\xi_x - r_x\xi_y)dp_z \\
&= \mathbf{p}\xi \, d\mathbf{r} + \xi\mathbf{r} \, d\mathbf{p}
\end{aligned}$$

so $\Phi(\mathbf{r}, \mathbf{p}) = (\mathbf{rp} - (\mathbf{rp})^T)$ is the momentum map associated with the rotation action. The entries of the matrix $(\mathbf{rp} - (\mathbf{rp})^T)$ are the components of the vector $\mathbf{r} \times \mathbf{p}$, which is the definition of angular momentum.

The existence of a momentum map is related to the condition that the Lie group action preserve the symplectic form [1]. If we allow ourselves to assume the existence of the momentum map, then the only further condition we need to obtain a conserved quantity is that the group action preserve the Hamiltonian.

**Theorem 4.4** (Noether's Theorem). *Given a Hamiltonian system $(M, \omega, H)$ and a Lie group $G$ acting on $M$ with associated momentum map $\Phi$, if $H \circ S_g = H$ for all $g \in G$, then $\Phi$ is conserved on integral curves of $X_H$.*

*Proof.* The proof follows almost directly from the definitions. We let $\gamma(t)$ be an integral curve of $X_H$, i.e.

$$\dot\gamma(t) = X_H(\gamma(t))$$

and then take the time derivative of $\Phi^\xi$ along $\gamma$ for an arbitrary $\xi \in \mathfrak{g}$.

| | |
|---|---|
| (Chain rule) | $\dfrac{d}{dt}[\Phi^\xi(\gamma(t))] = d\Phi^\xi_\gamma(\dot{\gamma})$ |
| (Def 4.1) | $= -\omega_\gamma(\xi_M(\gamma), \dot{\gamma})$ |
| (Def of $\gamma$) | $= \omega_\gamma(X_H(\gamma), \xi_M(\gamma))$ |
| (Thm 2.3) | $= dH_\gamma(\xi_M(\gamma))$ |
| (Def 3.6) | $= dH_\gamma\left(\left.\dfrac{d}{ds}\right|_{s=0} S_{e^{s\xi}}(\gamma)\right)$ |
| (Chain Rule) | $= \left.\dfrac{d}{ds}\right|_{s=0} H(S_{e^{s\xi}}(\gamma))$ |
| ($H \circ S_g = H$) | $= \left.\dfrac{d}{ds}\right|_{s=0} H(\gamma)$ |
| | $= 0$ $\qquad\qquad\qquad\square$ |

Looking back at our two examples, Noether's theorem tells us that if we have a Hamiltonian that is translation invariant, then linear momentum is conserved. Likewise if we have a Hamiltonian that is rotation invariant, then angular momentum is conserved. In physics, the conservation of these quantities is frequently a boon in solving the equations of motion. With conservation laws being such powerful tools, it is clear that an understanding of Noether's theorem is enlightening and useful for any physicist.

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