AN INTRODUCTION TO THE FUNDAMENTAL GROUP

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Abstract. This paper seeks to introduce the reader to the fundamental group and then show some of its immediate applications by calculating the fundamental group of a circle and by offering a topological proof of the Fundamental Theorem of Algebra. We assume that the reader is familiar with the ideas of connectedness, compactness, and continuity.

Contents

1. Introduction

We use the fundamental group as a tool to tell different spaces apart. For example, what makes a torus different from a double torus, or an n-fold torus? Or how is a coffee cup like a donut? To explore this, we define an algebraic group called the fundamental group of a space. This group consists of different equivalence classes of loops in that space. A function $f : I \to X$ is a loop if it is continuous and has a base point $f(0) = f(1)$. Two loops are equivalent under homotopy if one loop can be bent or stretched without breaking until it becomes the other loop. The fundamental group has a group operation $*$ that glues two classes of loops together. We introduce some theory about covering spaces to help us calculate fundamental groups. In our proof of the Fundamental Theorem of Algebra, we assume that a complex polynomial f has no roots and that its target space is the punctured plane $\mathbb{C} \setminus \{0\}$. From there we derive a contradiction by comparing how the polynomial f and the polynomial $g = x^n$ act upon the fundamental group of S^1.

2. Notation

Throughout the paper, we shall let I denote the closed interval $[0, 1]$. We shall also let X and Y be topological spaces. Given a function $f : I \to X$, we let x_0

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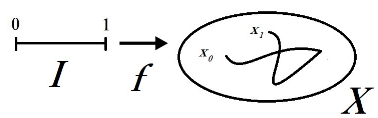
denote $f(0)$ and x_1 denote $f(1)$. We also let $e_x : I \to X$ denote the constant function $e_x(s) = x$. Let S^1 denote the unit circle based at the origin, and let B^2 denote the closed two-dimensional unit ball based at the origin. Let b_0 denote the point $(1, 0)$.

3. Paths and Homotopies

By finding the different homotopy classes of loops in a space, we can better understand that space. In other words, the extent to which a path in a space can be deformed continuously helps us understand what the space is like.

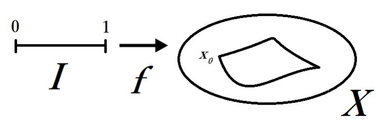
Definition 3.1. Let I denote the unit interval $[0, 1] \subset \mathbb{R}$ and let X be a topological space. A continuous function $f : I \to X$ is called a **path** in X. We call $f(0)$ the **initial point** of f, and we call $f(1)$ the **end point**.

Figure 1. path



Definition 3.2. Let $f : I \to X$ be a path such that $f(0) = f(1) = x_0$. Then, f is a **loop** based at x_0.

Figure 2. loop



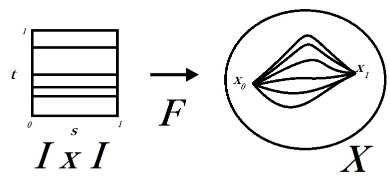
Definition 3.3. Let $f : X \to Y$ and $g : X \to Y$ be continuous. We say that f and g are **homotopic** if there exists a continuous function $F : X \times I \to Y$ such that $\forall x \in X$, $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. The function F is called a **homotopy** between f and g. We denote that f and g are homotopic by writing $f \simeq g$.

We can think of a homotopy as a continuous deformation of one function into another. The continuous function f is transformed continuously until it is the continuous function g. This happens over what we can think of as an interval of time I. If we fix an arbitrary time t, we find that the restriction $F\mid_{X \times \{t\}}$ is a continuous single-variable function with the variable $x \in X$. Also, if we fix a point $x \in X$, we find that $F\mid_{\{x\} \times I}$ is a path.

Definition 3.4. Let f and g be paths sharing the initial point x_0 and the end point x_1. If there exists a homotopy F from f to g such that $\forall t \in I$, $F(0, t) = x_0$ and $F(1, t) = x_1$, then f and g are **path homotopic**. We then write $f \simeq_p g$.

The path homotopy F is thus a continuous deformation of path f into path g that keeps the initial point and end point of the path constant. In more precise language, the functions f_t generated by the path homotopy F are paths from x_0 to x_1. In other words, fix $t \in I$. Let $f_t(x) : I \to X$ be a function such that $f_t(x) = F(x,t)$. Then, f_t is a path from x_0 to x_1.

FIGURE 3. Path Homotopy



Lemma 3.5 (Pasting Lemma)**.** *Let $f : X \to Y$ be a function mapping space X to space Y. Let $A \cup B = X$, where A and B are both open or both closed subsets of X. If $f\mid_A$ and $f\mid_B$ are both continuous, then f is continuous. The statement is true also if A and B are both open.*

Proof. Let U be an open subset of Y. We have that $f\mid_A^{-1}(U)$ is open in A and $f\mid_B^{-1}(U)$ is open in B. There exist open sets V_A and V_B such that $f\mid_A^{-1}(U) = A \cap V_A$ and $f\mid_B^{-1}(U) = B \cap V_B$.
$f^{-1}(U) = f\mid_A^{-1}(U) \cup f\mid_B^{-1}(U) = (A \cap V_A) \cup (B \cap V_B) = (A \cup B) \cap (V_A \cup V_B) = X \cap (V_A \cup V_B)$
We have that $f^{-1}(U)$ is open in X. Therefore f is continuous. It is also easy to show that f is continuous if sets A and B are open. □

This lemma will be useful for showing that the paths and homotopies we construct are continuous.

Proposition 3.6. *$\simeq$ and $\simeq_p$ are equivalence relations.*

Proof. We need to show $\simeq$ and $\simeq_p$ are reflexive, symmetric, and transitive. Let f, g, h be continuous functions from X to Y.

Reflexivity: Let $F : X \times I \to Y$ be a function such that $\forall t \in I : F(x,t) = f(x)$. The function F is continuous because f is continuous. More specifically, we can show that F is continuous by thinking of F as the composition $f \circ g$ of two continuous functions, the projection $g : X \times I \to X$ and $f : X \to Y$. Thus, F is a homotopy from f to f. Consequently, $f \simeq f$.

If f is a path, then F is a path homotopy from f to f, and $f \simeq_p f$

Symmetry: Suppose $f \simeq g$. Then there exists a homotopy F from f to g. Let $G : X \times I \to Y$ be a function such that $G(x,t) = F(x, 1-t)$. As before, we can show G is continuous by thinking of it as the composition of two continuous functions. Then $G = F \circ j$, where $j(x,t) = (x, t-1)$. For all x, we have $G(x,0) = g(x)$ and $G(x,1) = f(x)$, so G is a homotopy from g to f, and $g \simeq f$.

If $f \simeq_p g$ and F is a path homotopy from f to g, then G as defined above is a

path homotopy from g to f. Then, $g \simeq_p f$.
Although the homotopy relation is symmetric, we will still think of a homotopy F as a function starting from one function f and ending at another function g.

Transitivity: Suppose $f \simeq g$ and $g \simeq h$. There then exists a homotopy F from f to g and a homotopy G from g to h. Define a function $H : X \times I \to Y$:

$$H(x,t) = \begin{cases} F(x,2t) & 0 \le t \le \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} < t \le 1 \end{cases}$$

H is continuous by the Pasting Lemma. We can check this. Let $j : I \times [0, \frac{1}{2}] \to I \times I$ be the function $j(s,t) = (s, 2t)$ and let $k : I \times [\frac{1}{2}, 1] \to I \times I$ be the function $j(s,t) = (s, 2t-1)$. These functions are continuous. Then, $F(x,2t) = (F \circ j)(x,t)$ is continuous and $G(x, 2t-1) = (G \circ k)(x,t)$ is continuous. We have $H = F \circ j$ over $X \times [0, \frac{1}{2}]$ and $H = G \circ h$ over $X \times [\frac{1}{2}, 1]$. On the overlap of the two domains, $X \times \{\frac{1}{2}\}$, $(F \circ j)(x, \frac{1}{2}) = F(x,1) = g(x) = G(x,0) = (G \circ j)(x, \frac{1}{2})$. We have thus satisfied the conditions to show H is continuous by the Pasting Lemma. Also, $H(x,0) = f(x)$ and $H(x,1) = h(x)$. Thus H is a homotopy from f to h, and $f \simeq h$.

If $f \simeq_p g$ and $g \simeq_p h$, and if F and G are path homotopies, then H is a path homotopy, and $f \simeq_p h$. □

Definition 3.7. Let f be a path. Let $[f]$ denote the equivalence class of all paths that are path-homotopic to f.

Proposition 3.8. *Let $f : X \to Y$ and $g : X \to Y$ be continuous. Let Y be a* ***convex*** *subset of $\mathbb{R}^n$. This means for any $a, b \in Y$, the line segment between a and b is contained in Y. Then there exists a homotopy between f and g called the* ***straight line homotopy*** *$F : X \times I \to Y$, where $F(x,t) = (1-t)f(x) + (t)g(x)$. If f and g are paths with the same initial point and same end point, then F is a path homotopy.*

Proof. Because Y is convex, $F(x,t) \in Y$ for all $t \in I$. It is also evident for all $x \in X$, $F(x,0) = f(x)$ and $F(x,1) = g(x)$. We also know F is continuous, since it is the sum of products of continuous functions. Therefore, F is a homotopy.

If f and g are paths from x_0 to x_1, then for all $t \in I$, $F(0,t) = (1-t)x_0 + tx_0 = x_0$. Similarly, for all $t \in I$, $F(1,t) = x_1$. Then F is a path homotopy. □

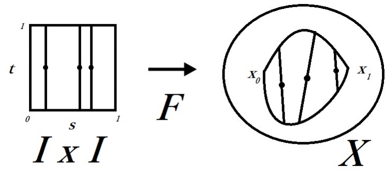
The straight-line homotopy is a convenient homotopy to have. In the straight-line homotopy, the function f is deformed to g at a constant rate. In other words, for all $s \in X$, $f(s)$ transforms continuously to $g(s)$ at a constant rate, or in other words, in a straight line. At time t, the $F(s,t)$ is $t/1$ of the way through the process of changing from $f(s)$ to $g(s)$.

4. The Fundamental Group

Here, we apply algebra to our understanding of loops and homotopies.

Definition 4.1. Let $f : I \to X$ be a path from x_0 to x_1 and let $g : I \to X$ be a path from x_1 to x_2. We define the operation $*$ as:

Figure 4. Straight Line Homotopy

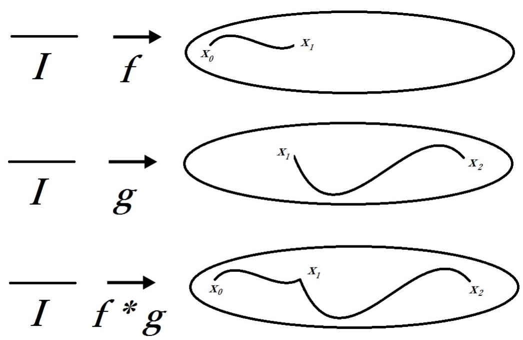


$$f * g = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ g(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$$

$f * g$ is continuous by the Pasting Lemma, so it is a path from x_0 to x_2.

The operation $*$ speeds up two paths and glues the endpoint of one path to the initial point of the other path. This produces a new path that traverses all the points of the original two paths.

Figure 5. $f * g$



Definition 4.2. The operation $*$ can be applied to homotopy classes as well. Once again, let $f : I \to X$ be a path from x_0 to x_1 and let $g : I \to X$ be a path from x_1 to x_2. Define $[f] * [g] = [f * g]$

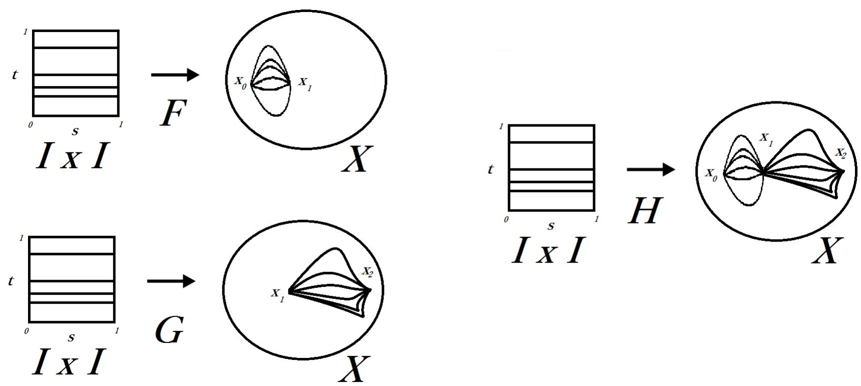
Proposition 4.3. *The operation $*$ between homotopy classes is well-defined.*

Proof. Let $f' \in [f]$ and let $g' \in [g]$. Because $[f'] = [f]$ and $[g'] = [g]$, we want to check $[f'] * [g'] = [f] * [g]$. Because f and f' are path homotopic, there exists a path homotopy F from f to f'. Likewise, there exists path homotopy G from g to g'. Define a function $H : I \times I \to X$ such that:

$$H(s,t) = \begin{cases} F(2s,t) & s \in [0, \frac{1}{2}] \\ G(2s-1,t) & s \in [\frac{1}{2}, 1] \end{cases}$$

Now we show that H is a path homotopy between $f * g$ and $f' * g'$, which are paths from x_0 to x_2. We know H is continuous by the Pasting Lemma. For all $t \in I$ we have $H(0,t) = F(0,t) = x_0$ and $H(1,t) = G(1,t) = x_2$. For all $s \in I$ we have $H(s,0) = (f * g)(s)$ and $H(s,1) = (f' * g')(s)$. The function H is therefore a path homotopy between $f * g$ and $f' * g'$. Thus, $[f' * g'] = [f * g]$. This means $[f'] * [g'] = [f] * [g]$. Therefore $*$ is well-defined. $\square$

FIGURE 6. $[f] * [g]$ is well-defined



Proposition 4.4. *Let $f : I \to X$ be a path from x_0 to x_1. Let $g : I \to X$ be a path from x_1 to x_2. Let $h : I \to X$ be a path from x_2 to x_3. Let $\bar{f} : I \to X$ be the continuous function such that $\bar{f}(s) = f(1-s)$.*

The operation $*$ between homotopy classes has the following properties:

- ***Existence of Identity Elements**: $[e_{x_0}] * [f] = [f]$ and $[f] * [e_{x_1}] = [f]$*
- ***Existence of Inverses**: $[f] * [\bar{f}] = [e_{x_0}]$ and $[\bar{f}] * [f] = [e_{x_1}]$*
- ***Associativity**: $([f] * [g]) * h = [f] * ([g] * [h])$*

Proof. **Identity Elements**: We want to show $(e_{x_0} * f) \simeq_p f$. Let $e_0 : I \to I$ be the constant path at 0 and let $i : I \to I$ be the identity path. Then $f = f \circ i$ and $e_{x_0} * f = f \circ (e_0 * i)$. i and $e_0 * i$ are paths in I from 0 to 1. Because I is convex, the straight line path homotopy F exists between i and $e_0 * i$, and $f \circ F$ is a path homotopy between $f \circ i$ and $f \circ (e_0 * i)$. Thus $f \circ i \simeq_p f \circ (e_0 * i)$. Then $f \simeq_p e_{x_0} * f$, and $[f] = [e_{x_0}] * [f]$.

It follows similarly $[f] * [e_{x_1}] = [f]$.

Inverses: We want to show $(f * \bar{f}) \simeq_p e_{x_0}$.
We know $f * \bar{f} = (f \circ i) * (f \circ \bar{i}) = f \circ (i * \bar{i})$. Similarly, we know $e_{x_0} = f \circ e_0$
We know $i * \bar{i}$ and e_0 are loops in I based at 0. Because I is convex, there exists the straight line path homotopy F between $i * \bar{i}$ and e_0. Then $f \circ F$ is a path homotopy between $f \circ (i * \bar{i})$ and $f \circ e_0$. Therefore, as before, we find $[f] * [\bar{f}] = [e_{x_0}]$. By similar reasoning, $[\bar{f}] * [f] = [e_{x_1}]$.

Associativity: We want to show $(f * g) * h = f * (g * h)$.

$$(f * g) * h = \begin{cases} f(4s) & s \in [0, \frac{1}{4}] \\ g(4s) & s \in [\frac{1}{4}, \frac{1}{2}] \\ h(2s) & s \in [\frac{1}{2}, 1] \end{cases} \qquad f * (g * h) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ g(4s) & s \in [\frac{1}{2}, \frac{3}{4}] \\ h(4s) & s \in [\frac{3}{4}, 1] \end{cases}$$

Let $k : I \to I$ be a map:

$$k(s) = \begin{cases} \frac{s}{2} & s \in [0, \frac{1}{2}] \\ s - \frac{1}{4} & s \in [\frac{1}{2}, \frac{3}{4}] \\ 2s - 1 & s \in [\frac{3}{4}, 1] \end{cases}$$

Then $\big((f * g) * h\big) \circ k = f * (g * h) = \big(f * (g * h)\big) \circ i$. Because k and i are both paths from 0 to 1 in the convex space I, k and i are path homotopic. Let F be a path homotopy from k to i. Then, $\big((f * g) * h\big) \circ F$ is a path homotopy from $\big((f*g)*h\big) \circ k = f*(g*h)$ to $\big((f*g)*h\big) \circ i = (f*g)*h$. Thus, $f*(g*h) \simeq_p (f*g)*h$. Then $([f] * [g]) * h = [f] * ([g] * [h])$

$\square$

Now that we know $*$ is associative, we know $[f_1] * [f_2] * \cdots * [f_n]$ is well-defined.

Definition 4.5. A **group** is a set G paired with an operation $\cdot$ that combines any two elements $a, b \in G$ to make a third element $a \cdot b$. To be a group, $(G, \cdot)$ must satisfy four properties:

(1) **Closure**: If $a, b \in G$ then $a \cdot b \in G$

(2) **Associativity**: If $a, b, c \in G$ then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(3) **Existence of Identity Element**:
There exists $e \in G$ such that if $a \in G$ then $a \cdot e = e \cdot a = a$

(4) **Existence of Inverses**:
If $a \in G$ then there exists $a^{-1} \in G$ such that $a^{-1} \cdot a = a \cdot a^{-1} = e$

Definition 4.6. Let X be a topological space. Let $\pi_1(X, x_0)$ denote the set of all equivalence classes $[f]$ of loops in X based at x_0.

Proposition 4.7. *$\pi_1(X, x_0)$, when paired with the operation $*$, is a group. We call $\pi_1(X, x_0)$ the fundamental group.*

Proof. To show $\pi_1(X, x_0)$ is a group, we will want to show closure, associativity, the existence of an identity element, and the existence of inverses. Most of this is evident and follows from Proposition 4.4. $\square$

Definition 4.8. Let X be a topological space and let p be a path in X from x_0 to x_1. The path p induces a function $\hat{p} : \pi_1(X, x_0) \to \pi_1(X, x_1)$, defined by

$$\hat{p}([f]) = [\bar{p}] * [f] * [p]$$

$\hat{p}$ is well-defined because $*$ is well-defined and associative. Note that $\hat{p}([f])$ is a class of loops based at x_1.

Definition 4.9. Let G and G' be groups. Let $f : G \to G'$ be a function such that if $a, b \in G$ then $f(a \cdot b) = f(a) \cdot f(b)$. We say f is a **homomorphism**. If f is bijective, then f is also called an **isomorphism**. If f is an isomorphism, then its inverse f^{-1} is also an isomorphism.

Proposition 4.10. *$\hat{p}$ is an isomorphism.*

Proof. First we will show $\hat{p}$ is a homomorphism. Let $[f], [g] \in \pi_1(X, x_0)$. Recall that $*$ is associative, so:

$$\begin{aligned}\hat{p}([f] * [g]) &= [\bar{p}] * [f] * [g] * [p] \\ &= [\bar{p}] * [f] * ([e_{x_0}] * [g]) * [p] \\ &= [\bar{p}] * [f] * (([p] * [\bar{p}]) * [g]) * [p] \\ &= ([\bar{p}] * [f] * [p]) * ([\bar{p}] * [g] * [p]) \\ &= \hat{p}([f]) * \hat{p}([g])\end{aligned}$$

Now, we show $\hat{p}$ is bijective. Let $[f'] \in \pi_1(X, x_1)$. Then $[p]*[f']*[\bar{p}]$ is the homotopy class of a loop in X based at x_0. Then

$$\hat{p}\big([p] * [f'] * [\bar{p}]\big) = [\bar{p}] * ([p] * [f'] * [\bar{p}]) * [p] = [e_{x_1}] * [f'] * [e_{x_1}] = [f']$$

Therefore, $\hat{p}$ is surjective. Now, let $\hat{p}([f]) = \hat{p}([g])$. Then $[\bar{p}]*[f]*[p] = [\bar{p}]*[g]*[p]$. Then $[p] * ([\bar{p}] * [f] * [p]) * [p] = [p] * ([\bar{p}] * [g] * [p]) * [p]$. Then $[f] = [g]$. Thus, $\hat{p}$ is injective. Therefore $\hat{p}$ is an isomorphism. Note that $\hat{p}$ and $\hat{\bar{p}}$ are inverses. □

Corollary 4.11. *If X is a space and there exists a path between $x_0 \in X$ and $x_1 \in X$, then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.*

Proof. Let p be a path from x_0 to x_1. Then, $\hat{p}$ is an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$. □

Thus the fundamental group does not depend on the base point up to isomorphism.

Definition 4.12. A space X is **simply connected** if X is path connected and for some $x_0 \in X$, $\pi_1(X, x_0)$ contains only one element, $[e_{x_0}]$. The fundamental group is then called **trivial**. It follows from Corollary 4.11 that for all $x \in X$, $\pi_1(X, x)$ is the trivial group.

A simply connected space is a path connected space that has no "holes" that pass through the entire space. Such a hole would prevent some loops from being shrunk continuously into a single point. For all points in a simply connected space, the fundamental group based at that point is trivial. Thus, given any loop based at any point in the space, that loop can be continuously deformed into a constant loop based somewhere along the original loop.

Lemma 4.13. *Let X be simply connected. Then any two paths in X that have the same initial point and the same end point are path homotopic.*

Proof. Let f and g be paths in X from x_0 to x_1. Then $[f] * [\bar{g}]$ is a class of loops based at x_0. Therefore $[f] * [\bar{g}] \in \pi_1(X, x_0)$. Because $\pi_1(X, x_0)$ is the trivial group, $[f] * [\bar{g}] = [e_{x_0}]$. Then $[f] * [\bar{g}] * [g] = [e_{x_0}] * [g]$. Consequently, $[f] * [e_{x_1}] = [g]$. Therefore $[f] = [g]$. Thus $f \simeq_p g$. □

FIGURE 7. Example of Simply Connected Space

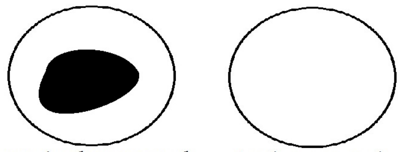
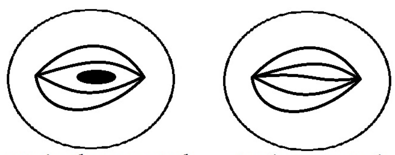


FIGURE 8. Path Homotopy in a Simply Connected Space



Definition 4.14. Let $h : X \to Y$ be a continuous map such that $h(x_0) = y_0$. To denote this, we say $h : (X, x_0) \to (Y, y_0)$ is a continuous map. We define the **homomorphism induced by h** as $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

$$h_*([f]) = [h \circ f]$$

The function h_* sends loops in X based at x_0 and sends them to loops in Y based at y_0. We shall check that h_* is well-defined. Let $g \in [f]$. Then there exists a path homotopy F between g and f. The path homotopy F "follows along" with f and g when composed with h. In other words, $h \circ F$ is a path homotopy between $h \circ g$ and $h \circ f$. Consequently, $[h \circ g] = [h \circ f]$. As we hoped, we have $h_*([g]) = h_*([f])$. We shall now check that h_* is a homomorphism.

$$h_*([f]*[g]) = h_*([f*g]) = [h\circ(f*g)] = [(h\circ f)*(h\circ g)] = [h\circ f]*[h\circ g] = h_*([f])*h_*([g])$$

Theorem 4.15. *Let $g : (X, x_0) \to (Y, y_0)$ and $h : (Y, y_0) \to (Z, z_0)$ be continuous. Then $(h \circ g)_* = h_* \circ g_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.*

Proof. Let f be a loop in X based at x_0. Then

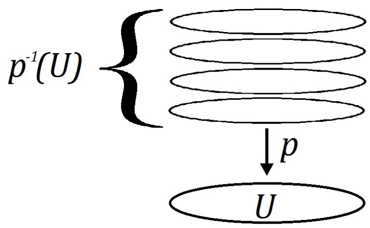
$$(h \circ g)_*([f]) = [(h \circ g) \circ f] = [h \circ (g \circ f)] = h_*([g \circ f]) = (h_* \circ g_*)([f])$$
$$i_*([f]) = [i \circ f] = [f]$$

□

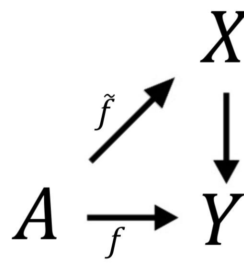
Corollary 4.16. *If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then h_* is an isomorphism.*

Proof. Because h is a homeomorphism, $h^{-1} : (Y, y_0) \to (X, x_0)$ is a continuous function. We have

$$h_* \circ (h^{-1})_* = (h \circ h^{-1})_* = (id_Y)_* = id_{\pi_1(Y, y_0)}$$



Even Covering



Lifting

$$(h^{-1})_* \circ h_* = (h^{-1} \circ h)_* = (id_X)_* = id_{\pi_1(X,x_0)}$$

Therefore, h_* and $(h^{-1})_*$ are inverses, and h_* is an isomorphism. □

Thus, we find that the fundamental group of a space is a topological invariant. Homeomorphic spaces have isomorphic fundamental groups. In other words, spaces that are homeomorphic have "the same" fundamental group.

We care about isomorphisms because they are one-to-one correspondences between two groups that preserve the group operation. This lets us, in a way, treat seemingly different groups interchangeably. We will do this later when we represent the fundamental group of the circle as the group of integers with the addition operation.

5. Covering Spaces

Covering spaces and path liftings are useful tools for understanding the fundamental group of a space Y. We take a covering map $p : X \to Y$ of the space Y. A path f in Y then lifts with respect to p to a unique path $\tilde{f} : I \to X$ if we fix an initial point for $\tilde{f}$. Path homotopies also lift. Thus, entire homotopy classes lift. In particular, we note that elements of the fundamental group lift to classes of paths in the covering space. We can use this to obtain information about the fundamental group.

Definition 5.1. Let $p : X \to Y$ be continuous and surjective. Let $U \subset Y$ be an open set in Y. We say $U \subset Y$ is **evenly covered** by p if $p^{-1}(U)$ is a union of an arbitrary number of disjoint open sets V_α such that for all α, the function $p\,|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism.

Definition 5.2. Let $p : X \to Y$ be continuous and surjective. We call p a **covering map** if for all $y \in Y$, there exists an open neighborhood U such that $y \in U$ and U is evenly covered by p. The space X is called a **covering space** of Y.

Definition 5.3. Let $p : X \to Y$ be a function. Let $f : A \to Y$ be a continuous function. A **lift** or **lifting** of f is a function $\tilde{f} : A \to X$ such that $p \circ \tilde{f} = f$.

Theorem 5.4. *Let $p : X \to Y$ be a covering map. Let $p(x_0) = y_0$. Let $f : [0, 1] \to Y$ be a path such that $f(0) = y_0$. There exists exactly one lifting $\tilde{f} : [0, 1] \to X$ such that $\tilde{f}(0) = x_0$.*

Proof. p is a covering map, so we can cover Y with a collection $\{U_\alpha\}_{\alpha\in A}$ of open neighborhoods U that are evenly covered by p. Because f is continuous, $f^{-1}(U_\alpha)$ is open for all U_α. The collection of preimages $f^{-1}(U_\alpha)$ is thus an open cover of I. That open cover can be refined to a cover comprised of open intervals. As I is compact, this cover has a finite open subcover. That finite open cover can then be refined to a finite closed cover consisting of closed intervals. Each of these closed intervals is a subset of the preimage of a set U_α. Thus the image of each of these closed intervals is a subset of some evenly covered set $U_\alpha \in Y$.

Now we construct $\tilde{f}$. Let $f([0,s_1]) \subset U_0$. We know $p^{-1}(f(0))$ is spread across disjoint open sets $V_\beta \subset Y$ that evenly cover U_0. Recall $x_0 \in p^{-1}(f(0))$. As the sets V_β are disjoint, only one such set, which we denote V_0, contains x_0. Let $\tilde{f}(0) = x_0$, as intended. Because we want $f = p \circ \tilde{f}$, we have $\tilde{f}([0,s_1]) \subset p^{-1}(f([0,s_1])) \subset p^{-1}(U_0) = \bigcup V_\beta$. We want $\tilde{f}$ to be continuous, so it must map the connected set $[0,s_1]$ to a connected set in Y. Then $\tilde{f}([0,s_1]) \subset V_0$. We have $\tilde{f}(s) = (p\mid_{V_0})^{-1}(f(s))$, which is well-defined because $(p\mid_{V_0})$ is a homeomorphism. Therefore $\tilde{f} = (p\mid_{V_0})^{-1} \circ f$ over $[0,s_1]$. This function is continuous.

Now we consider $\tilde{f}([s_1,s_2])$, where $f([s_1,s_2]) \subset U_1$, and U_1 is an evenly covered neighborhood. Let U_1 be evenly covered by disjoint sets W_β. As $\tilde{f}(s_1)$ is defined already, $\tilde{f}(s_1)$ is contained in a predetermined set W_β, which we denote W_0. Then, as by previous reasoning, $\tilde{f} = (p\mid_{W_0})^{-1} \circ f$ for all $s \in [s_1,s_2]$. Now, $\tilde{f}$ is defined over $[0,s_2]$, and it is continuous by the Pasting Lemma.

We can continue this for the finitely many closed intervals $[s_i,s_{i+1}]$ that are a closed cover of I. By induction, we have that $\tilde{f}$ is uniquely defined and continuous of all $[0,1]$. □

Theorem 5.5. *Let $p : X \to Y$ be a covering map. Let $p(x_0) = y_0$. Let $F : I \times I \to Y$ be continuous such that $F(0,0) = y_0$. There exists exactly one lifting $\tilde{F} : I \times I \to X$ such that $\tilde{F}(0,0) = x_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.*

Proof. This proof is similar to the previous one, so this proof is more of a sketch. We can construct a finite closed cover $\{[s_i,s_{i+1}] \times [t_k,t_{k+1}]\}_{i\in[n],k\in[m]}$ of $I \times I$ such that for all i,k there exists an evenly covered neighborhood $U_\alpha \subset Y$ such that $F([s_i,s_{i+1}] \times [t_k,t_{k+1}]) \subset U_\alpha$. This construction is similar to the one from the previous proof.

Let $F([0,s_1] \times [0,t_1]) \in U_0$. Each point of $p^{-1}(F(0,0))$ is contained in a different set V_α in the collection of disjoint open sets $\{V_\alpha\}_{\alpha\in A}$ that evenly covers U_0. $\tilde{F}(0,0) = x_0 \in p^{-1}(F(0,0))$. Then $\tilde{F}(0,0)$ is contained in one of those sets V_α, which we denote V_0. In order for $\tilde{F}$ to be continuous, the image of a connected set must be connected. Thus $\tilde{F}([0,s_1] \times [0,t_1])$ is contained entirely in V_0. Then $\tilde{F}(s,t) = (p\mid_{V_0})^{-1}(F(s,t))$ for all $(s,t) \in [0,s_1] \times [0,t_1]$.

Now consider $([s_1,s_2] \times [0,t_1])$, which is the next rectangular section of $I \times I$, the one to the right of the rectangle we previously defined. There exists an evenly covered open neighborhood U_1 such that $F([s_1,s_2] \times [0,t_1]) \subset U$. Each point of $p^{-1}(F(s_1,t_1))$ is contained in a different one of the disjoint open sets W_β that evenly cover U_1. Then $\tilde{F}(s_1,t_1)$ is contained in exactly one of those sets W_β, which we denote W_0. Then, $\tilde{F}(s,t) = (p\mid_{W_0})^{-1}(F(s,t))$ for all $(s,t) \in [s_1, s_2] \times [0, t_1]$.

We can continue defining $\tilde{F}$ this way until $\tilde{F}$ is defined for the entire "bottom row" $[0,1] \times [0,t_1] \subset I \times I$. We can also define $\tilde{F}$ one rectangle at a time vertically from the bottom row, until $\tilde{F}$ is defined for all of $I \times I$.

Now, suppose F is a path homotopy. $\tilde{F}(s,0)$ and $\tilde{F}(s,1)$ are paths. $\tilde{F}(0,0) = x_0 \in V_0$, and the path $\tilde{F}(0,t)$ with s fixed at 0 is continuous. $\tilde{F}(0 \times [0,1]) \in p^{-1}(F(0 \times [0,1])) = p^{-1}(y_0)$. Because p is a covering map, the set $p^{-1}(y_0)$ is discrete. For $\tilde{F}$ to map the connected set $[0] \times [0,1]$ continuously into $p^{-1}(y_0)$, $\tilde{F}$ must be a constant map. Therefore, $\tilde{F}(0,[0,1]) = \{x_0\}$. We can show by similar reasoning $\tilde{F}([1] \times [0,1])$ maps to a single point. Then $\tilde{F}$ is a path homotopy. $\square$

Corollary 5.6. *Let $p : X \to Y$ be a covering map. Let $p(x_0) = y_0$. Let f and g be path homotopic paths in Y from y_0 to y_1. Let $\tilde{f}$ and $\tilde{g}$ be lifts of f and g respectively that start at x_0. Then $\tilde{f}$ and $\tilde{g}$ are path homotopic.*

Proof. There exists a path homotopy F from f to g. Let $\tilde{F}$ be the lift of F where $\tilde{F}(0,0) = x_0$. Then $\tilde{F}$ is a path homotopy. The restriction $\tilde{F}\mid_{[0,1]\times[0]}$ is a lifting of $F\mid_{[0,1]\times[0]}$ where $\tilde{F}\mid_{[0,1]\times[0]}(0,0) = x_0$. Thus $F\mid_{[0,1]\times[0]}$ is interchangeable with the path f. Recall that $F\mid_{[0,1]\times[0]}$ and f have unique lifts that start at x_0. These are $\tilde{F}\mid_{[0,1]\times[0]}$ and $\tilde{f}$ respectively. Thus, $\tilde{F}\mid_{[0,1]\times[0]}(s,0) = \tilde{f}(s)$.

The same is true for $\tilde{F}\mid_{[0,1]\times[1]}$ and $\tilde{g}(s)$. Therefore $\tilde{F}$ is a path homotopy from $\tilde{f}$ to $\tilde{g}$. Thus $\tilde{f} \simeq_p \tilde{g}$. $\square$

Proposition 5.7. *Let S^1 denote the circle centered at the origin in $\mathbb{R}^2$ with radius 1. Let $p : \mathbb{R} \to S^1$ be the map $p(x) = (\cos 2\pi x, \sin 2\pi x)$. Then p is a covering map.*

Proof. We can imagine the covering map p as a function that takes the real number line and wraps it into a coil such that a complete revolution is completed after every unit interval. The coil is then projected onto the unit circle.

Let $y \in S^1$. If $y \neq (1,0)$, then $y \in S^1 \setminus \{(1,0)\}$, which is an open neighborhood of S^1. This open neighborhood is evenly covered. $p^{-1}(S^1 \setminus \{(1,0)\}) = \bigcup_{n\in\mathbb{Z}} (n, n+1)$. The preimage of the open neighborhood $S^1 \setminus \{(1,0)\}$ is the union of disjoint open sets $(n, n+1)$. For each set, the $p\mid_{(n,n+1)}: (n,n+1) \to S^1 \setminus \{(1,0)\}$ is a homeomorphism. From the properties of the sine and cosine functions, we know $\sin 2\pi x$ and $\cos 2\pi x$ have a cycle of 1, so $p\mid_{(n,n+1)}$ is surjective and injective. The restriction $p\mid_{(n,n+1)}$ is continuous because $\sin 2\pi x$ and $\cos 2\pi x$ are continuous. Because $p\mid_{(n,n+1)}$ is a bijection, $(p\mid_{(n,n+1)})^{-1}$ is a function. Let $(a,b) \subset (n, n+1)$ be a basis set of $(n, n+1)$. Then $((p\mid_{(n,n+1)})^{-1})^{-1}((a,b)) = p\mid_{(n,n+1)})(a,b) = U$, where U is an open set of $S^1 \setminus \{0\}$. Thus, $p\mid_{(n,n+1)})^{-1}$ is continuous. Therefore $p\mid_{(n,n+1)}$ is a homeomorphism.

If $y = (1,0)$, we can choose $S^1 \setminus \{-1, 0\}$ as an evenly covered neighborhood containing y. We find that $p\mid_{(n+\frac{1}{2}, n+\frac{3}{2})}$ is the corresponding homeomorphism.

Therefore p is a covering map. $\square$

6. The Fundamental Group of a Circle

Theorem 6.1. *The fundamental group of S^1, regardless of base point, is isomorphic to the additive group of integers. In other words, there exists a bijective function $F : \pi_1(S^1, b_0) \to (\mathbb{Z}, +)$ such that $F([f] * [g]) = F([f]) + F([g])$ for all $[f], [g] \in \pi_1(S^1, b_0)$.*

The fundamental group of a circle can be thought of as the different numbers of revolutions that can be made around it. A double revolution loop cannot be homotoped to a single revolution loop around a circle. If two loops wrap around a circle a different number of times, then the loops cannot be transformed continuously to each other. Thus each element of $\pi_1(S^1)$ corresponds to an integer. The direction the loops travels, clockwise or counterclockwise, determines the sign of the corresponding integer. We use our previous findings about path liftings to show this.

Proof. Let $p : \mathbb{R} \to S^1$ be a function defined $p(s) = (\cos 2\pi s, \sin 2\pi s)$. We have already shown p is a covering map of S^1. Let $b_0 \in S^1$ denote the point $(1, 0)$. Let $[f]$ be an element of $\pi_1(S^1, b_0)$. Then f is a loop based at b_0. We know $p(0) = b_0$. Then there is a unique lifting $\tilde{f}$ such that $f = p \circ \tilde{f}$ and $\tilde{f}(0) = 0$. Because f has the endpoint b_0, $\tilde{f}(1) = n$ for some integer n.

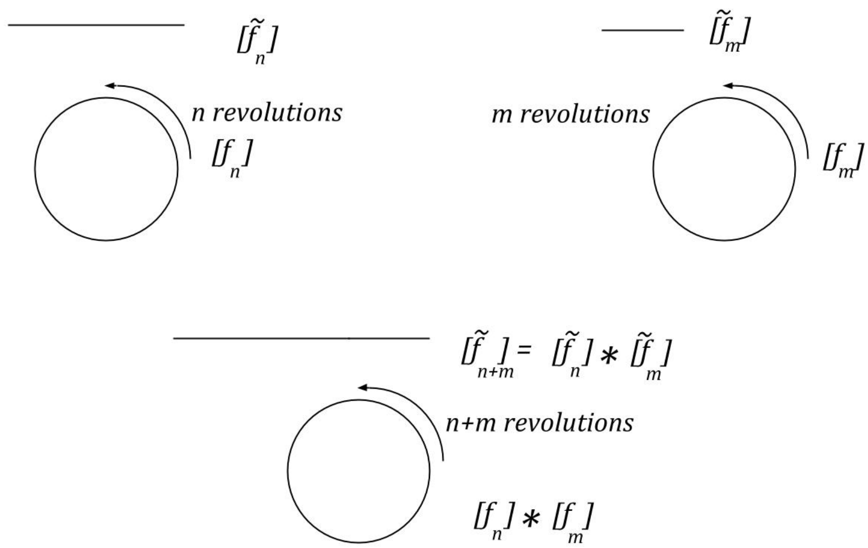
We know from the previous theorem that if g is path homotopic to f then $\tilde{g}$ is path homotopic to $\tilde{f}$. Then $\tilde{f}$ and $\tilde{g}$ are paths with the same endpoint, which is $\tilde{f}(1) = \tilde{g}(1) = n$, where $n \in \mathbb{Z}$. Thus, all loops in $[f]$ lift to unique paths starting at 0 and ending at some integer n. We can thus make a well-defined function $\psi : \pi_1(S^1, b_0) \to \mathbb{Z}$ where $\psi([f])$ is defined as the endpoint of the lifted path of f that begins at 0.

Now we check that ψ is surjective. Let $n \in \mathbb{Z}$, and let $\tilde{f_n} : [0, 1] \to \mathbb{R}$ be a path from 0 to n. Let $f_n = p \circ \tilde{f_n}$. We have that $f_n : [0, 1] \to S^1$ is a loop based at b_0. Thus, $[f_n] \in \pi_1(S^1, b_0)$, and $\psi([f_n]) = \tilde{f_n}(1) = n$.

Now we check that ψ is injective. Let $\psi([f]) = \psi([g]) = n$. The loop f lifts to a path $\tilde{f}$ from 0 to n. The loop g also lifts to a path $\tilde{g}$ from 0 to n. There thus exists the straight line homotopy F between $\tilde{f}$ and $\tilde{g}$. Then $p \circ F$ is a homotopy between f and g. Thus, $[f] = [g]$.

Define $\phi : \mathbb{Z} \to \pi_1(S^1, b_0)$ where $\phi(n) = [f_n]$ is the inverse of ψ. We have that ϕ is a homomorphism as $\phi(n + m) = [f_{n+m}] = [f_n] * [f_m] = \phi(n) * \phi(m)$.

The function ψ is bijective and its inverse ϕ is a homomorphism, so ψ is an isomorphism between $\pi_1(S^1, b_0)$ and $\mathbb{Z}$. In other words, we can treat the fundamental group of S^1 as if they were integers that we can add.



□

7. Fundamental Theorem of Algebra

Theorem 7.1. *Let $f : X \to Y$ be continuous, closed, and surjective. Let $g : X \to Z$ be a function such that g is constant for all sets A where $f(A)$ is a single point. Then, g induces a function $h : Y \to Z$ such that $h \circ f = g$. If g is continuous then h is continuous.*

Proof. We check that h is defined for all $y \in Y$: As f is surjective, for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Therefore, for all $y \in Y$, $h(y) = h(f(x)) = g(x)$. The function g is constant for all $x \in f^{-1}(y)$, so $h(y) = h(f(x)) = g(x)$ is well-defined.

Suppose g is continuous. Let U be a closed subset of Z. Then, $f^{-1}(h^{-1}(U)) = g^{-1}(U)$ is closed. Then $f(f^{-1}(h^{-1}(U))) = h^{-1}(U)$ is closed. Then h is continuous. □

Lemma 7.2. *Let $h : S^1 \to X$ be a continuous map. The following are equivalent:*

(1) h is nulhomotopic
(2) h can be extended to a continuous map of B^2, the unit ball in $\mathbb{R}^2$, into X.
(3) h_* sends all elements of the domain to the identity element. Or in other words, h_* is the trivial homomorphism.

Proof. We will show (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).
(1) $\implies$ (2): Assume that h is nulhomotopic. There exists a homotopy F from h to e_x. There exists a map $q : S^1 \times I \to B^2$ such that $q(s,t) = (1-t)s$. This map is continuous and surjective and closed. The function q is constant only on singleton sets and $S^1 \times \{1\}$. F is constant on those sets as well, so by Theorem 7.1,

F induces a continuous function $k : B^2 \to X$ where $k \circ q = F$. The function k is an extension of h to domain B^2.
(2) $\implies$ (3): Assume that h can be extended continuously to function k with domain B^2. Let $j : S^1 \to B^2$ be the inclusion map. Then, $h = k \circ j$. Then, $h_* = (k \circ j)_* = k_* \circ j_*$. The fundamental group of B^2 is trivial because it is simply connected. Then j_* is the trivial homomorphism. Then, h_* is the trivial homomorphism.
(3) $\implies$ (1): Let h_* be the trivial homomorphism. Let $p : I \to S^1$ be the loop $p(s) = (\cos 2\pi s, \sin 2\pi s)$. Let $h(p(0)) = x_0$. Then $[h \circ p] = [e_{x_0}]$. Then $h \circ p$ is path homotopic to e_{x_0}. There exists a path homotopy $F : I \times I \to X$ between $h \circ p$ and e_{x_0}. There also exists a map $Q : I \times I \to S^1 \times I$, where $Q(s,t) = (p(s), t)$. This map is continuous, surjective, and closed. The homotopy F induces a continuous map $H : S^1 \times I \to X$ such that $F = H \circ Q$. The function H is a path homotopy between h and the constant map e_{x_0}. □

Theorem 7.3. *Every non-constant single-variable polynomial with complex coefficients has at least one complex root.*

Proof. Let $f(x) = a_n x^n + \cdots + a_1 x^1 + a_0$ be a nonconstant complex polynomial. There exists large enough $c \in \mathbb{C}$ such that $\left|\frac{a_{n-1}}{a_n c^1}\right| + \cdots + \left|\frac{a_0}{a_n c^n}\right| < 1$. Now consider the polynomial $h(x) = x^n + \frac{a_{n-1}}{a_n c^1} x^{n-1} + \cdots + \frac{a_0}{a_n c^n}$. We have $f(x) = c^n a_n h(\frac{x}{c})$. If $c^n a_n h(\frac{x}{c})$ has a complex root, then $f(x)$ has a complex root. It will thus suffice to show that $h(x)$ has a root. To simplify notation, from now on let $h(x) = x^n + \alpha_{n-1} x^{n-1} + \cdots + \alpha_0$. We construct h like this so that $h\ |_{S^1}$ does not touch the origin.

Assume that $h(x)$ has no complex roots. We shall prove this assumption false by deriving a contradiction.

Assuming that $h(x)$ has no complex roots, h maps to the space $\mathbb{R}^2 \setminus \{(0,0)\}$. As h is a polynomial, $h\ |_{S^1}$ and $h\ |_{B^2}$ are continuous maps to the space $\mathbb{R}^2 \setminus \{(0,0)\}$. By Lemma 7.2, $h\ |_{S^1}$ is nulhomotopic.

Now consider the function $g : S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ where the domain and target space are in the complex plane and $g(x) = x^n$. Let $k : S^1 \to S^1$ be the function $k(x) = x^n$. Let $j : S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ be the inclusion map. Then $g = j \circ k$. Consequently $g_* = j_* \circ k_*$. Let $p : I \to S^1$ be covering map $p(s) = (\cos 2\pi s, \sin 2\pi s)$. The homotopy class $[p]$ corresponds to the integer 1 in the group isomorphism ψ constructed in Theorem 6.1. As 1 is the generator of $\mathbb{Z}$, $[p]$ is the generator of the fundamental group of S^1. In other words, all elements of the group $\pi_1(S^1, b_0)$ are multiples of $[p]$. Then $k \circ p$ is a loop that travels n times as fast as $[p]$. In other words, $k \circ p = p * p * \cdots * p$ n times. Having shown that $\pi_1(S^1, b_0)$ is isomorphic to $(\mathbb{Z}, +)$, we have that k_* is injective, akin to a function that multiplies integers by n.

There exists a function $l : \mathbb{R}^2 \setminus \{(0,0)\} \to S^1$ such that $l(x) = \frac{x}{\|x\|}$. Then $l \circ j$ is the identity map on S^1. Then $(l \circ j)_* = l_* \circ j_*$ is the identity homomorphism.

Therefore j_* is injective. Because k_* and j_* are injective, g_* is injective. The domain of g_* is nontrivial, so g_* is nontrivial. Then g is not nulhomotopic.

However, we can construct a homotopy $F : S^1 \times I \to \mathbb{R}^2 \setminus \{(0,0)\}$ between g and h. Let $F(x,t) = x^n + t(\alpha_{n-1}x^{n-1} + \cdots + \alpha_0)$. For all $(x,t) \in S^1 \times I$, by the Triangle Inequality we have

$$\begin{aligned} |F(x,t)| &\geq |x^n| - |t(\alpha_{n-1}x^{n-1} + \cdots + \alpha_0)| \\ &\geq |x^n| - t(|\alpha_{n-1}x^{n-1}| + \cdots + |\alpha_0)|) \end{aligned}$$

Note that for all $x \in S^1$ and all $k \in \mathbb{Z}$, we have $|x^k| = 1$. Then

$$\begin{aligned} |x^n| - |t(\alpha_{n-1}x^{n-1} + \cdots + \alpha_0)| &= 1 - t(|\alpha_{n-1}| + \cdots + |\alpha_0)|) \\ &> 1 - t \cdot 1 \\ &\geq 0 \end{aligned}$$

Thus, for all $(x,t) \in S^1 \times I$, $F(x,t) \neq 0$. Therefore, g and $h\,|_{S^1}$ are homotopic.

If $h\,|_{S^1}$ is nulhomotopic, then g is nulhomotopic. This contradicts that g is not nulhomotopic. Thus, our assumption was false. The polynomial $h(x)$ has a complex root, and therefore $f(x)$ has a complex root. $\square$

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