# A FEW APPLICATIONS OF DIFFERENTIAL FORMS

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Abstract. This paper introduces the concept of differential forms by defining the tangent space of $\mathbb{R}^n$ at point $p$ with equivalence classes of curves and introducing the cotangent space as the dual of the tangent space. The first application presented is a formalization of the separation of variables technique for solving differential equations which is used in many introductory calculus classes. The next application is a proof of the Fundamental Theorem of Algebra by using a 1-form to detect the winding of the image under a polynomial of a large and small circle in the complex plane. Finally, the Gauss-Bonnet theorem is presented using more intuitive definitions rather than by formal proof.

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## 1. Introduction to Differential Forms

We begin our discussion by introducing the concept of differential forms.

**Definition 1.1.** A curve in $\mathbb{R}^n$ centered on $p$ is a smooth map $\gamma : \mathbb{R} \to \mathbb{R}^n$ such that $\gamma(0) = p$.

Suppose we have a curve in $\mathbb{R}^n$ centered on $p$, $\gamma$, such that $\gamma(t) = (\gamma_1(t), \cdots, \gamma_n(t))$. Then we can define the tangent vector to $\gamma$ at $p$ in the following way:

**Definition 1.2.** The tangent vector to $\gamma$ at $p$ is the vector $\begin{bmatrix} \left(\dfrac{\mathrm{d}\gamma_1(t)}{\mathrm{d}t}\right)_0 \\ \vdots \\ \left(\dfrac{\mathrm{d}\gamma_n(t)}{\mathrm{d}t}\right)_0 \end{bmatrix}$ with origin at $p$.

In order to define a tangent space with some generality, we will first introduce a relation $\sim$.

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**Definition 1.3.** If $\alpha$ and $\beta$ are both curves in $\mathbb{R}^n$ centered on $p$, we say $\alpha \sim \beta$ if for all $i \in \{1, \cdots, n\}$ we have $\left(\frac{\mathrm{d}\alpha_i(t)}{\mathrm{d}t}\right)_0 = \left(\frac{\mathrm{d}\beta_i(t)}{\mathrm{d}t}\right)_0$.

We can easily check that $\sim$ is reflexive, symmetric, and transitive. Thus, $\sim$ is an equivalence relation and we will now refer to curves in $\mathbb{R}^n$ centered on $p$ within their equivalence classes under $\sim$. With this, we are ready to define the tangent space of $\mathbb{R}^n$ at $p$.

**Definition 1.4.** The tangent space of $\mathbb{R}^n$ at $p$ is the set of all equivalence classes of curves in $\mathbb{R}^n$ at $p$ and is denoted by $T_p\mathbb{R}^n$.

*Remark* 1.5. $T_p\mathbb{R}^n$ is a vector space. If $[\gamma], [\delta] \in T_p\mathbb{R}^n$, then we define

$$[\gamma] + [\delta] = [\gamma + \delta],$$

where

$$(\gamma + \delta)(t) = (\gamma_1(t) + \delta_1(t), \cdots, \gamma_n(t) + \delta_n(t)).$$

It is a simple matter to check that for any $\alpha \in [\gamma]$ and $\beta \in [\delta]$ we will have $\alpha + \beta \in [\gamma + \delta]$. Similarly, scalar multiplication is defined coordinate-wise, such that for any $c \in \mathbb{R}$ we have $c[\gamma] = [c\gamma]$ where $c\gamma(t) = (c\gamma_1(t), \cdots, c\gamma_n(t))$.

Now, we introduce a bilinear map:

**Definition 1.6.** We define $D_{(\cdot)}(\cdot) : T_p\mathbb{R}^n \times C^\infty(\mathbb{R}^n) \to \mathbb{R}$ such that

$$D_{[\gamma]}(f) = (f \circ \gamma)'(0).$$

*Remark* 1.7. Notice that for fixed $[\gamma]$ and for $f, g \in C^\infty(\mathbb{R}^n)$ we have

$$D_{[\gamma]}(f * g) = f(p) * D_{[\gamma]}(g) + g(p) * D_{[\gamma]}(f).$$

**Definition 1.8.** For any fixed $p = (p_1, \cdots, p_n) \in \mathbb{R}^n$, $e_i$ will refer to the path $e_i : \mathbb{R} \to \mathbb{R}^n$ such that $e_i(t) = (p_1, \cdots, p_{i-1}, p_i + t, p_{i+1}, \cdots, p_n)$.

*Remark* 1.9. Notice that for any $f \in C^\infty(\mathbb{R}^n)$ we have $D_{[e_i]}(f) = \left(\frac{\partial f}{\partial x^i}\right)_p$. Furthermore, the set $\{[e_1], \cdots, [e_n]\}$ forms a basis of $T_p\mathbb{R}^n$. Often, this basis will be referenced with the notation $\left(\frac{\partial}{\partial x^i}\right)_p$ for $[e_i]$.

**Definition 1.10.** The cotangent space at $p$ is defined to be the dual space of $T_p\mathbb{R}^n$, that is the set of all linear functions from $T_p\mathbb{R}^n$ to $\mathbb{R}$. We denote the cotangent space at $p$ by $(T_p\mathbb{R}^n)^*$

*Remark* 1.11. We will write $\{(dx^1)_p, \cdots, (dx^n)_p\}$ as a basis of $(T_p\mathbb{R}^n)^*$ where $(dx^i)_p(\left(\frac{\partial}{\partial x^j}\right)_p) = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker Delta.

**Definition 1.12.** A differential 1-form $\omega$ on $\mathbb{R}^n$ is an assignment of some $\omega_p \in (T_p\mathbb{R}^n)^*$ to each $p \in \mathbb{R}^n$ that is smooth in the following sense: if

$$\omega_p = f_1(p)(dx^1)_p + \cdots + f_n(p)(dx^n)_p$$

then $f_1, \cdots, f_n$ are smooth.

Now, we define a map from $C^\infty(\mathbb{R}^n)$ to the set of 1-forms on $\mathbb{R}^n$.

**Definition 1.13.** We define $d : C^\infty(\mathbb{R}^n) \to \{\text{1-forms on } \mathbb{R}^n\}$ such that for all $f \in C^\infty(\mathbb{R}^n)$ we have $d(f) = \dfrac{\partial f}{\partial x^i} dx^i$ using Einstein summation convention to sum over indices.

*Remark* 1.14. Suppose we have $f \in C^\infty(\mathbb{R}^n)$ and a curve in $\mathbb{R}^n$ centered on $p$, $\gamma$, such that $\gamma = (\gamma_1(t), \cdots, \gamma_n(t))$. Then $D_{[\gamma]}(f) = \left(\dfrac{\partial f}{\partial x^i}\right)_p \left(\dfrac{\partial \gamma_i}{\partial t}\right)_0$. Notice that if we take $d(f)$ and evaluate it at $(p, [\gamma])$, we get

$$(d(f))_p([\gamma]) = \left(\frac{\partial f}{\partial x^i}\right)_p (dx^i)_p([\gamma]) = \left(\frac{\partial f}{\partial x^i}\right)_p \left(\frac{\partial \gamma_i}{\partial t}\right)_0 = D_{[\gamma]}(f).$$

*Remark* 1.15. We will also note for coordinate functions on $\mathbb{R}^n$, i.e. $x^i : \mathbb{R}^n \to \mathbb{R}$ such that $x^i\left(\begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{bmatrix}\right) = \alpha_i$, we have $d(x^i)$ is the 1-form that evaluates to $(dx^i)_p$ at all $p \in \mathbb{R}^n$.

**Definition 1.16.** A differential $k$-form on $\mathbb{R}^n$ is an assignment of some

$$\omega_p \in \Lambda^k((T_p\mathbb{R}^n)^*)$$

to each $p \in \mathbb{R}^n$ that is smooth in the following sense: if

$$\omega_p = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k}(p) dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

then each $f_{i_1 \cdots i_k}$ is a smooth function on $\mathbb{R}^n$.

*Remark* 1.17. Notice that $\{dx^{i_1} \wedge \cdots \wedge dx^{i_k} \mid i_1 < \cdots < i_k;\ i_1, \cdots i_k \in \{1, \cdots, n\}\}$ forms a basis of $\Lambda^k((T_p\mathbb{R}^n)^*)$

Now, we will extend the $d$ operator in the following way, keeping with our original definition by referring to functions in $C^\infty(\mathbb{R}^n)$ as 0-forms:

**Definition 1.18.** Now, we define $d : \{k\text{-forms on } \mathbb{R}^n\} \to \{(k+1)\text{-forms on } \mathbb{R}^n\}$ such that

$$d\Big(\sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}\Big) = \sum_{j=1}^{n} \sum_{i_1 < \cdots < i_k} \frac{\partial f_{i_1 \cdots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Finally, we will establish how these forms transition between vector spaces. We will assume we have a smooth map $F : \mathbb{R}^n \to \mathbb{R}^m$.

**Definition 1.19.** The pushforward of vectors under $F$ at a fixed point $p \in \mathbb{R}^n$ is a map $F_* : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ such that $F_*([\gamma]) = [F \circ \gamma]$.

The pullback of a differential form on $\mathbb{R}^m$ under $F$ is a differential form on $\mathbb{R}^n$. We will define the pullback of a 1-form, but it can be extended to $k$-forms.

**Definition 1.20.** Let $\omega = f_1(y^1, \cdots y^m)dy^1 + \cdots + f_m(y^1, \cdots y^m)dy^m$ be a differential 1-form on $\mathbb{R}^m$. Then the pullback of $\omega$ under $F$ is a 1-form on $\mathbb{R}^n$, $F^*\omega = \displaystyle\sum_{i=1}^{m}(f_i \circ F) * (\sum_{j=1}^{n} \frac{\partial y^i}{\partial x^j} dx^j)$.

*Remark* 1.21. For fixed $p \in \mathbb{R}^n$ and a fixed curve in $\mathbb{R}^n$ centered on $p$, $\gamma$, and a fixed 1-form on $\mathbb{R}^m$, $\omega = f_1(y^1, \cdots y^m)dy^1 + \cdots + f_m(y^1, \cdots y^m)dy^m$, we have

$$(F^*\omega)_p([\gamma]) = \sum_{i=1}^{m}(f_i \circ F)_p * (\sum_{j=1}^{n} \frac{\partial y^i}{\partial x^j}(dx^j)_p([\gamma]))$$

which simplifies to

$$(F^*\omega)_p([\gamma]) = \sum_{i=1}^{m}(f_i \circ F)_p * (dy^i)_p(F_*[\gamma])$$

and further simplifies to

$$(F^*\omega)_p([\gamma]) = \omega_{F(p)}(F_*[\gamma]).$$

## 2. Using Differential Forms to Solve Differential Equations

First, we will introduce a few classifications of differential forms.

**Definition 2.1.** A differential 1-form $\omega$ is exact if there exists $f \in C^\infty(\mathbb{R}^n)$ such that $df = \omega$.

**Definition 2.2.** A differential 1-form $\omega$ is closed if $d\omega = 0$.

Now, a few useful facts:

**Lemma 2.3.** *If a differential* 1*-form is exact, it is closed.*

*Proof.* Suppose $\omega$ is an exact differential 1-form. Then there exists $f \in C^\infty(\mathbb{R}^n)$ such that $df = \omega$. Therefore,

$$\omega = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$$

Then, we have

$$d\omega = \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i$$

However,

$$\frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i = -\frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j$$

It follows that $d\omega = 0$ and $\omega$ is closed. ☐

The following lemma is presented without proof as it is beyond the scope of this paper.

**Lemma 2.4** (Poincaré's Lemma for 1-forms)**.** *A* 1*-form* $\omega$ *defined on an open set* $U \subset \mathbb{R}^n$ *is closed if and only if for all* $p \in U$ *there exists a neighborhood* $V \subset U$ *containing* $p$ *with a differentiable function* $f : V \to \mathbb{R}$ *such that* $df = \omega|_V$*.*

Now, we will consider differential equations.

**Definition 2.5.** A first order differential equation is an equation of the form $\phi'(x) = f(x, \phi(x))$, which is often written in the form $y' = f(x, y)$ or $\frac{dy}{dx} = f(x, y)$.

**Definition 2.6.** A first order separable differential equation is an equation of the form $\phi'(x) = g(x)h(\phi(x))$, which again may be written as $y' = g(x)h(y)$ or $\frac{dy}{dx} = g(x)h(y)$.

Suppose we have a first order separable differential equation, $y' = g(x)h(y)$. We choose a differential 1-form $\omega = dy - g(x)h(y)dx$ and we need to find all paths $\phi$ such that $\phi^*\omega = 0$.

If for some function $f(x, y)$ we have $\frac{\omega}{f(x,y)}$ is closed, then there exists a function $F(x, y)$ such that $dF = \frac{\omega}{f(x,y)}$. Curves of the form $\phi(t)$ such that $F(t, \phi(t)) = c$ where $c$ is some constant are desired. The reason for this is if $\frac{\omega}{f(x,y)} = dF$ then

$$F_y = \frac{1}{f(x,y)}$$

and

$$F_x = -\frac{g(x)h(y)}{f(x,y)}.$$

If $(t, \phi(t))$ satisfies $F(t, \phi(t)) = c$ for all $t$, then

$$dF(t, \phi(t)) = 0.$$

It follows that

$$F_x dt + F_y * \phi'(t)dt = 0.$$

Therefore,

$$\frac{\phi'(t)}{f(t,\phi(t))}dt - \frac{g(t)h(\phi(t))}{f(t,\phi(t))}dt = 0.$$

It follows that $\phi'(t) = g(t)h(\phi(t))$, so $\phi(t)$ is a solution to the differential equation.

*Remark* 2.7. The integration of 1-forms is equivalent to a line integral in multivariate calculus. There are more formal ways to specify this, but it is not necessary for this application.

We first assume $dF = \frac{\omega}{f(x,y)} = \frac{dy - g(x)h(y)dx}{f(x,y)}$. For simplicity, we take $f(x, y) = h(y)$. If we integrate $dF$ from some initial condition $p = (b, a)$ along an arbitrary path to $(x, y)$ we will find a solution to the differential equation. Thus,

$$F = \int_0^F dF = \int_a^y \frac{dy}{h(y)} - \int_b^x g(x)dx.$$

If these expressions are integrable, we can set it equal to an arbitrary constant and solve for $y$.

Let us consider an example of a separable differential equation, $y' = xy$. Our first step is to set $\omega = dy - xydy$. In this case, we choose $f(x, y) = y$ so

$$\frac{\omega}{f(x,y)} = \frac{dy}{y} - xdx$$

leaves us with separated variables. Thus, we have

$$\int_0^F dF = \int_a^y \frac{dy}{y} - \int_b^x xdx$$

which evaluates to

$$F(x, y) = ln(y) - ln(a) - \frac{x^2}{2} + \frac{b^2}{2}$$

We now set it equal to an arbitrary constant and combine constants to find

$$ln(y) - \frac{x^2}{2} = c$$

which we can solve for $y$ and find

$$y = ke^{x^2/2}$$

with $k = e^c$.

## 3. A Proof of the Fundamental Theorem of Algebra

In order to prove the Fundamental Theorem of Algebra, we first present another result concerning closed 1-forms.

*Remark* 3.1. We stated earlier that integrals of 1-forms could be considered to be regular line integrals. This makes use of a more general fact: if $f$ is a map from $A$ to $B$ and $\omega$ is a 1-form on $B$ and we let $\gamma$ be a path in $A$, the integral of $\omega$ over the path $f \circ \gamma$ is equivalent to the integral of $f^*\omega$ over $\gamma$.

**Lemma 3.2.** *If $\omega$ is a closed 1-form on $\mathbb{R}^n$ and $\gamma_1, \gamma_2$ are closed paths in $\mathbb{R}^n$ such that there exists a smooth map $F : S^1 \times [0,1] \to \mathbb{R}^n$ such that $F|_{S^1 \times \{0\}} = \gamma_1$ and $F|_{S^1 \times \{1\}} = \gamma_2$ (where $S^1$ is the unit disk) then $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$.*

*Proof.* Observe that

$$\int_{\gamma_1} \omega = \int_{S^1 \times \{0\}} F^*\omega$$

Similarly,

$$\int_{\gamma_2} \omega = \int_{S^1 \times \{1\}} F^*\omega$$

Now, we notice that

$$\int_{S^1 \times \{0\}} F^*\omega - \int_{S^1 \times \{1\}} F^*\omega = \int_{\partial(S^1 \times \{0\})} F^*\omega$$

By Stoke's Theorem, we have

$$\int_{\partial(S^1 \times \{0\})} F^*\omega = \int_{S^1 \times \{0\}} d(F^*\omega)$$

This can be simplified, as

$$\int_{S^1 \times \{0\}} d(F^*\omega) = \int_{S^1 \times \{0\}} F^*d\omega = \int_{S^1 \times \{0\}} F^*0 = 0$$

Therefore, we have

$$\int_{S^1 \times \{0\}} F^*\omega - \int_{S^1 \times \{1\}} F^*\omega = 0$$

and we have the desired result

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$$

□

**Theorem 3.3** (Fundamental Theorem of Algebra)**.** *Every complex polynomial has at least one complex root.*

*Proof.* Consider a polynomial of the form $a_n z^n + \cdots + a_1 z + a_0$. We assume $a_0 \neq 0$ else $z = 0$ is a trivial root. Furthermore, we assume $a_n = 1$ because we can divide through by a constant without changing the roots of the polynomial. We are left with the polynomial $z^n + \cdots + a_1 z + a_0$.

Consider the 1-form $\omega = \dfrac{dz}{z}$ on $\mathbb{C} \setminus \{0\}$. It is easy to verify that $d\omega = 0$ so $\omega$ is closed. We suppose that $z^n + \cdots + a_1 z + a_0$ has no complex roots, and thus we can define a continuous map $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ such that $f(z) = z^n + \cdots + a_1 z + a_0$. The pullback of $\omega$ is thus a closed form on $\mathbb{C}$. It can be easily verified that

$$f^*\omega = \frac{(nz^{n-1} + a_{n-1}(n-1)z^{n-2} + \cdots + a_2(2)z + a_1)dz}{z^n + \cdots + a_1 z + a_0}$$

Consider $\gamma_1 : [0, 2\pi] \to \mathbb{C}$ such that $\gamma_1(t) = re^{it}$ and $\gamma_2 : [0, 2\pi] \to \mathbb{C}$ such that $\gamma_2(t) = Re^{it}$. These paths satisfy the conditions of Lemma 3.2. However, when we integrate, we will notice a discrepancy.

$$\int_{\gamma_1} f^*\omega = \int_0^{2\pi} (re^{it})^* f^*\omega$$

To take this pullback, we recognize $z = re^{it}$ and make the necessary substitutions from the formulas above

$$\int_{\gamma_1} f^*\omega = \int_0^{2\pi} ire^{it} \frac{(n(re^it)^{n-1} + a_{n-1}(n-1)(re^it)^{n-2} + \cdots + a_2(2)(re^it) + a_1)dt}{(re^it)^n + \cdots + a_1(re^it) + a_0}$$

$$= \int_0^{2\pi} i \frac{(n(re^it)^n + a_{n-1}(n-1)(re^it)^{n-1} + \cdots + a_2(2)(re^it)^2 + a_1(re^it))dt}{(re^it)^n + \cdots + a_1(re^it) + a_0}$$

We are specifically interested in the case of a small circle around the origin, so we take the limit as $r$ approaches 0:

$$\lim_{r \to 0} \int_{\gamma_1} f^*\omega = \int_0^{2\pi} 0dt = 0$$

Now we consider our other, larger curve.

$$\int_{\gamma_2} f^*\omega = \int_0^{2\pi} (Re^{it})^* f^*\omega$$

Pulling back in the same manner, we obtain

$$\int_{\gamma_2} = \int_0^{2\pi} iRe^{it} \frac{(n(Re^it)^{n-1} + a_{n-1}(n-1)(Re^it)^{n-2} + \cdots + a_2(2)(Re^it) + a_1)dt}{(Re^it)^n + \cdots + a_1(Re^it) + a_0}$$

$$= \int_0^{2\pi} i \frac{(n(Re^it)^n + a_{n-1}(n-1)(Re^it)^{n-1} + \cdots + a_2(2)(Re^it)^2 + a_1(Re^it))dt}{(Re^it)^n + \cdots + a_1(Re^it) + a_0}$$

This time, in the interest of examining a large curve, we let $R$ approach $\infty$:

$$\lim_{R \to \infty} \int_{\gamma_2} = \int_0^{2\pi} indt = 2in\pi$$

This therefore results in a contradiction, and $z^n + \cdots + a_1 z + a_0$ must have at least one complex root. $\square$

## 4. The Gauss-Bonnet Theorem

First, we will develop the concept of the Euler Characteristic for closed surfaces. It is easiest to think of in terms of polyhedra.

**Definition 4.1.** The Euler Characteristic $\chi$ for a polyhedron is given by

$$\chi = V - E + F$$

where $V$ is the number of vertices, $E$ is the number of edges, and $F$ is the number of faces.

The Euler Characteristic is also defined for more general closed surfaces; the simplest way to understand the definition is to note that if a surface can be deformed into another without breaking the surface, making new connections, or intersecting itself, then the two surfaces share the same Euler Characteristic. For example, a cube has 8 vertices, 12 edges, and 6 faces. $\chi = 8 - 12 + 6 = 2$. A cube can be deformed in a sphere without breaking, gluing, or self intersecting, so for a sphere, $\chi = 2$.

Next, we develop a seemingly unrelated property of surfaces, the Gaussian curvature. First, we recall the curvature of a curve in space.

**Definition 4.2.** Let $\gamma$ be a curve in $\mathbb{R}^3$ with arclength parameterization given by $\gamma(s)$. The curvature $\kappa_\gamma$ of $\gamma$ at point $p$ is given by $\kappa_\gamma(p) = |(\gamma''(s))_p|$.

**Definition 4.3.** The Gaussian curvature $K$ of a surface $M \subset \mathbb{R}^3$ at point $p$ is given by $\kappa_1\kappa_2$ where $\kappa_1$ is the maximum curvature of a geodesic in $M$ at $p$ and $\kappa_2$ is the minimum curvature of a geodesic in $M$ at $p$.

Now, we consider a special map from a surface in $\mathbb{R}^3$ to the unit sphere, $S^2$.

**Definition 4.4.** For a surface $M \subset \mathbb{R}^3$, the Gauss map, $G : M \to S^2$ is defined such that every $p \in M$ maps to its unit normal vector, that is a point on $S^2$.

We consider the differential 2-form

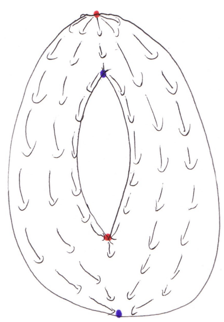
$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

defined on $\mathbb{R}^3 \setminus \{0\}$. This is related to the form we used to investigate the Fundamental Theorem of algebra. It shares a property of counting a winding number - the previous form counted how many radians around the origin a curve wrapped when integrated, and this form counts a similar wrapping for surfaces. For instance, $\int_{S^2} \omega = 4\pi$.

An important property of $\omega$ is that $G^*\omega = K d\sigma$ where $d\sigma$ is the area element of $M$. In fact, there is also an important relation between $\int_M G^*\omega$ and $\int_{S^2} \omega$ which allows us to quickly integrate the Guassian curvature. To quantify this relation, we must first examine the idea of the degree of $G$. An informal explanation of $\deg(G)$ would be the number of preimages of the majority of points in $S^2$.

To formally examine $\deg(G)$, we start by considering the height function $h$ for $M$ such that $h(x^1, x^2, x^3)$ returns the height ($x^3$ coordinate) of $M$ at that location. We find the zeroes of $-\nabla h$, the negative gradient of the surface (this can be visualized if you think of pouring water over the surface - the points with no net flow are zeroes of this vector field). We can divide these zeroes into two classes: those with upward unit normal vectors and those with downward unit normal vectors. We now further split the zeroes into two more classes: those with positive $K$ (extrema) and those with negative $K$ (saddle points). The degree of $G$ can be given by the number of zeroes with upward (or downward) normal vectors and positive $K$ less the number of zeroes with upward (or downward) normal vector and negative $K$. Note that the calculation can use either upward or downward normal vector zeroes, but not a mix of the two.

FIGURE 1. Negative Gradient of Height Function on Torus



In Figure 1, we show an example of the gradient sketched on a torus. The orange dots are zeroes with upward pointing normal vectors. The purple dots are zeroes with downward pointing normal vectors. For both, there is on extremum and one saddle point, so the degree of the Gauss map on a torus is 0.

With a firm way to calculate $\deg(G)$, we turn back to our application, and present that $\int_M G^*\omega = \deg(G)\int_{S^2}\omega$. This relation is not unexpected, because both $\omega$ and the degree of $G$ are related to the way our surface wraps around the origin.

Another important result is the Poincare-Hopf theorem, which allows us to relate the degree of this map back to the Euler characteristic. Using this theorem, we obtain the relation $\chi(M) = 2\deg(G)$. This final piece allows us to string together

$$\int_M K d\sigma = \int_M G^*\omega = \deg(G)\int_{S^2}\omega = 4\pi\deg(G) = 2\pi\chi(M)$$

This final result,

$$\int_M K d\sigma = 2\pi\chi(M)$$

is known as the Gauss-Bonnet theorem.

Now, we present a brief application of the Gauss-Bonnet theorem.

**Corollary 4.5.** *Let $f$ be a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that for some compact $B \subset \mathbb{R}^2$ $f$ is identically zero outside of $B$. Then the integral of the gaussian curvature of the graph of $f$ in $\mathbb{R}^3$ is 0.*

Let $A'$ be the graph of $f$ in $\mathbb{R}^3$. We note that the Gauss map $G : A' \to S^2$ can be expressed such that

$$G(u, v, f(u, v)) = \left(\frac{-f_u}{\sqrt{f_u^2 + f_v^2 + 1}}, \frac{-f_v}{\sqrt{f_u^2 + f_v^2 + 1}}, \frac{1}{\sqrt{f_u^2 + f_v^2 + 1}}\right).$$

This can be verified by directly calculating the unit normal vectors to this surface. Further, we present that

$$K d\sigma = G^*\omega = \frac{f_{uu}f_{vv} - f_{uv}f_{vu}}{(f_u^2 + f_v^2 + 1)^{3/2}} du \wedge dv.$$

This too can be verified through directly pulling back $\omega$ under the above map, however the calculation is tedious. We observe that it should be possible to directly integrate this expression for any function of the specified type. However, in practice the integration will usually not be possible without numerical methods. Instead, we present a much simpler proof using the Gauss-Bonnet theorem.

*Proof.* Let $A$ be the graph of $f|_B$ in $\mathbb{R}^3$. It is possible to construct a smooth surface $S \subset \mathbb{R}^3$ such $S \cup A$ is a smooth surface in $\mathbb{R}^3$ with no holes or self intersections, i.e. $S \cup A$ can be smoothly deformed into a sphere. Such a surface $S$ is hard to express explicitly, but an example would be the lower hemisphere of a sphere with the edges tapered to smoothly connect with $A$. Because $A$ must smoothly join with the identically zero region of the graph of $f$ outside of $B$, the same $S$ should be usable for any choice of $f$.

We recognize that since $S \cup A$ can be smoothly deformed into a sphere,

$$\chi(S \cup A) = 2.$$

We consider the Gauss-Bonnet theorem in regards to $S \cup A$:

$$\int_{S\cup A} K d\sigma = 2\pi\chi(S \cup A) = 4\pi.$$

We observe that we can split the integral into the two subsections of the surface:

$$\int_{S\cup A} K d\sigma = \int_S K d\sigma + \int_A K d\sigma = 4\pi.$$

Now, we make the observation that $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = 0$ satisfies these conditions. We note that the curvature of the graph of this function is identically zero. This can be verified by calculation with the formula presented above. When we put this result into our calculation with the Gauss-Bonnet theorem, we see that

$$\int_S K d\sigma = 4\pi.$$

Therefore, for any $f$ that satisfies the above conditions, we see that

$$\int_A K d\sigma = 0$$

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