CORES OF ALEXANDROFF SPACES

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Abstract. Following Kukieła, we show how to generalize some results from May's book [4] concerning cores of finite spaces to cores of Alexandroff spaces. It turns out that finite space methods can be extended under certain local finiteness assumptions; in particular, every bounded-paths space or countable finite-paths space has a core, and two bounded-paths spaces or countable finite-paths spaces are homotopy equivalent if and only if their cores are homeomorphic.

Contents

1. Introduction

An Alexandroff space is a topological space in which arbitrary intersections of open sets are open. These spaces were first introduced by P. Alexandroff in 1937 in [1] under the name of Diskrete Räume. Finite spaces are a special case of Alexandroff spaces. There is a close relationship between Alexandroff spaces, and posets. For a set X, the Alexandroff space topologies $(X, \mathscr{U})$ are in bijective correspondence with the preorders $(X, \leq)$. The topology $\mathscr{U}$ corresponding to $\leq$ is T_0 if and only if the relation $\leq$ is a partial order.

In Section 2, we review the facts from finite space theory that shall be generalized. We will set up some terminology for Alexandroff spaces and posets, and recall some results about cores of finite spaces from May's book [4]. For a finite space X, it is possible to construct a core recursively by removing beat points. As May [4] pointed out, removing beat points is a strong deformation retraction of the space X. In this case, we can remove beat points one by one until there are no more beat points left. The remaining points form a core. Thus, every finite space has a core. If X is a minimal finite space, $C(X, X)$ is the space of endomorphisms of X in the compact-open topology, and $f \in C(X, X)$ is in the same path component as

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id_X, then $f = id_X$. We will use this statement to prove that two finite spaces are homotopy equivalent if and only if their cores are homeomorphic. In the remainder of the paper, we shall generalize these results to infinite Alexandroff spaces.

In Section 3, we first introduce some classes of Alexandroff spaces, including finite-chains spaces, locally finite spaces, finite-paths spaces, and bounded-paths spaces. Next, we present Kukieła 's generalizations [3]. If an infinite Alexandroff space is sufficiently well-behaved, then we can get a core by recursively removing sets of beat points until no more beat points are left. Compared to the cores of finite spaces, we have the following results. Every bounded-paths space or countable finite-paths space has a core, and if X is a minimal finite-paths space, then the connected component of id_X in $C(X, X)$ is a singleton. Moreover, if X and Y are fp-spaces that both have cores, then X is homotopy equivalent to Y if and only if their respective cores are homeomorphic.

2. Preliminaries

Strong [6] first introduces the concept of cores of finite spaces, then May [4] and Kukieła [3] popularize his work. Before we start exploring the cores of infinite Alexandroff spaces, we will set up some terminology for Alexandroff spaces and posets, and recall a few properties and some results from May's book [4]. Everything here is from May [4] and Kukieła [3]. The reader is assumed to have some background in point set topology.

2.1. **Alexandroff Spaces and Posets.**

First we recall the definition of an Alexandroff space and then we show how to associate Alexandroff spaces with preorders and posets.

Definition 2.1. A topological space X is an *Alexandroff space* if and only if arbitrary intersections of open sets in X are open.

Lemma 2.2. *A finite space is an Alexandroff space.*

Proof. In a finite space, arbitrary intersections of open sets are still finite. Since finite intersections of open sets are open, so are arbitrary intersections. □

Definition 2.3. A *preorder* on a set X is a reflexive and transitive relation, denoted $\leq$. This means that for all $x \in X$, $x \leq x$ and that for all $x, y, z \in X$, $x \leq y$ and $y \leq z$ imply $x \leq z$. A preorder is a *partial order* if it is antisymmetric, which means that for all $x, y \in X$, $x \leq y$ and $y \leq x$ imply $x = y$. In this case, $(X, \leq)$ is called a *poset*. If $x, y \in (X, \leq)$ are *comparable*, which means $x \leq y$ or $x \geq y$, then we shall write $x \sim y$.

Now we associate Alexandroff spaces with preorders and posets, and describe bases for Alexandroff spaces.

Definition 2.4. Let X be an Alexandroff space. For $x \in X$, define U_x to be the intersection of all open sets that contain x. Define a relation $\leq$ on the set X by $x \leq y$ if $x \in U_y$, or equivalently, $U_x \subseteq U_y$. Write $x < y$ if the inclusion is proper.

Lemma 2.5. *Let X be an Alexandroff space. The set of all open sets U_x is a basis $\mathscr{B}$ for X. If $\mathscr{C}$ is another basis, then $\mathscr{B} \subseteq \mathscr{C}$, therefore $\mathscr{B}$ is the unique minimal basis for X.*

Proof. The first statement follows from Definition 2.4. For each $x \in X$, there is U_x such that $x \in U_x \in \mathscr{B}$. If $x \in B' \cap B''$ where $B', B'' \in \mathscr{B}$, then by minimality of U_x, it follows that $x \in U_x \subseteq B' \cap B''$.

Suppose $\mathscr{C}$ is another basis. Take an element U_x in $\mathscr{B}$. Note U_x is open in X, and $x \in U_x$. Since $\mathscr{C}$ is a basis, there is an open set V in $\mathscr{C}$ such that $x \in V \subseteq U_x$. By minimality of U_x, $V = U_x$. This proves $\mathscr{B} \subseteq \mathscr{C}$. □

Through the following lemmas, we can detect whether or not an Alexandroff space is T_0 in terms of its minimal basis.

Lemma 2.6. *Let X be an Alexandroff space. Two points x and y in X have the same neighborhoods if and only if $U_x = U_y$. Therefore X is T_0 if and only if $U_x = U_y$ implies $x = y$.*

Proof. If x and y in X have the same neighborhoods, then $U_x = U_y$. Conversely, if $x \in U$ where U is open, then $x \in U_x = U_y \subseteq U$. Since $y \in U_y$, then $y \in U$. A similar argument shows that if $y \in U$ where U is open, then $x \in U$. Thus, x and y have the same neighborhoods. □

Lemma 2.7. *Let X be an Alexandroff space. The relation $\leq$ on X is reflexive and transitive, so that the relation $\leq$ is a preorder. The relation $\leq$ is antisymmetric if and only if X is T_0.*

Proof. The relation $\leq$ on the set X is defined by $x \leq y$ if $U_x \subseteq U_y$. For all $x, y, z \in X$, $U_x \subseteq U_x$, and $U_x \subseteq U_y$ and $U_y \subseteq U_z$ imply $U_x \subseteq U_z$. Thus the relation $\leq$ is a preorder. The second proof follows Lemma 2.6. □

Lemma 2.8. *A preorder $(X, \leq)$ determines a topology $\mathscr{U}$ on X with basis the set of all sets $U_x = \{y \in X \,|\, y \leq x\}$. It is called the order topology on X. The space $(X, \mathscr{U})$ is an Alexandroff space. It is a T_0-space if and only if $(X, \leq)$ is a poset.*

Proof. If $x \in U_y$ and $x \in U_z$, then $x \leq y$ and $x \leq z$, hence $x \in U_x \subset U_y \cap U_z$. Thus $\{U_x\}$ is a basis for a topology. Now we show the space $(X, \mathscr{U})$ is an Alexandroff space. We claim that arbitrary intersections of open sets $U = \bigcap_{i \in I} U_i$ are still open. Suppose $x \in U$, then $U_x \subseteq U_i$ for each i. Thus $U = \bigcup_{x \in U} U_x$, hence U is open. Since $U_x = U_y$ if and only if $x \leq y$ and $y \leq x$, the Lemma 2.6 implies that $(X, \mathscr{U})$ is T_0 if and only if $(X, \leq)$ is a poset. □

We put the results of Lemma 2.7 and 2.8 together to get the following conclusion.

Proposition 2.9 ([4, Prop. 1.6.4])**.** *For a set X, the Alexandroff space topologies on X are in bijective correspondence with the preorders on X. The topology $\mathscr{U}$ corresponding to $\leq$ is T_0 if and only if the relation $\leq$ is a partial order.*

Not only are Alexandroff spaces related to preorders and posets by the proposition above, but also the continuous maps between Alexandroff spaces correspond to order-preserving functions between preorders. We have the following.

Definition 2.10. Let X and Y be preorders. A function $f : X \to Y$ is order-preserving if for every $m, n \in X$, $m \leq n$ in X implies $f(m) \leq f(n)$ in Y.

Lemma 2.11. *A function $f : X \to Y$ between Alexandroff spaces is continuous if and only if it is order-preserving.*

Proof. Suppose $w, v \in X$, such that $w \leq v$.
($\Rightarrow$) Let f be continuous. Note that $U_{f(v)}$ is open in Y. By continuity of f, $f^{-1}(U_{f(v)})$ is open in X. Since U_v is the intersection of all open sets that contain v, and $w \leq v$, it follows that $w \in U_v \subseteq f^{-1}(U_{f(v)})$. Therefore $f(w) \in U_{f(v)}$, which means that $f(w) \leq f(v)$.
($\Leftarrow$) Suppose f is order-preserving, and let V be open in Y. We need to show $f^{-1}(V)$ is open in X. If $f(v) \in V$, then $f(v) \in U_{f(v)} \subset V$. Since $w \leq v$, it follows that $w \in U_v$ and $f(w) \leq f(v)$. Thus $f(w) \in U_{f(v)} \subset V$. Therefore $w \in U_v$ implies $w \in f^{-1}(V)$. Hence $f^{-1}(V) = \bigcup_{v \in f^{-1}(V)} U_v$, which is open. □

We introduce the Principle of Well-Founded Induction, which will be used many times in the rest of this paper.

Definition 2.12. A poset $(X, \leq)$ is *well-founded* if X has no infinite strictly descending sequence.

Proposition 2.13. *A poset $(X, \leq)$ is well-founded if and only if every non-empty subset of X contains a minimal element.*

Proof. ($\Rightarrow$) Let $(X, \leq)$ be well-founded a poset, and $P \subseteq X$ be a non-empty subset of X. Suppose there is no minimal element in P. Then take an element $x_0 \in P$. Since x_0 is not minimal, it follows that there exists an $x_1 \in P$ such that $x_0 > x_1$. Then we can construct an infinite strictly descending sequence $x_0 > x_1 > x_2 > ...$ through this process. This is a contradiction.
($\Leftarrow$) Assume every non-empty subset of X contains a minimal element. Let $x_0 > x_1 > x_2 > ...$ be a strictly descending chain in X. Write $P = \{x_i \,|\, i = 0, 1, 2, ...\}$, and let $x_k \in P$ be the minimal element in P. Thus $x_i = x_k$ for all $i \geq k$, which is a contradiction. □

Proposition 2.14. *(Principle of Well-Founded Induction, also called Noetherian Induction) Suppose $(X, \leq)$ is a well-founded poset, and $P(x)$ is a property of the elements of X. If for all $x \in X$, $P(y)$ is true for all $y < x$ implies $P(x)$ is true, then $P(x)$ is true for every $x \in X$.*

Proof. Suppose for all $x \in X$, $P(y)$ is true for all $y < x$ implies $P(x)$ is true, but there exists an $x \in X$ such that $P(x)$ is false. We will seek a contradition.

Suppose $F = \{x \in X \,|\, P(x) \text{ is false}\}$. F has a minimal element because by assumption $F \neq \varnothing$ and X is a well-founded poset. Let $x_0 \in F$ be a minimal element. Then:

1. $x_0 \in F$ implies $P(x_0)$ is false.
2. for all $y < x_0$, $P(y)$ is true because x_0 is minimal in F.

By our initial assumption, 2. implies $P(x_0)$ is true, a contradiction. □

2.2. **Cores of Finite Spaces.**

Now recall some results on the cores of finite spaces from May's book [4]. For the remainder of this paper, all spaces are assumed to be A-spaces (T_0-Alexandroff spaces) unless stated otherwise.

Definition 2.15. Let Y be a subspace of a space X, with the inclusion denoted by $i : Y \to X$. We say that Y is a *strong deformation retract* of X if there is a map $r : X \to Y$ such that $r \circ i$ is the identity map of Y and there is a homotopy $h : X \times I \to X$ from the identity map of X to $i \circ r$ such that $h(y, t) = y$ for all

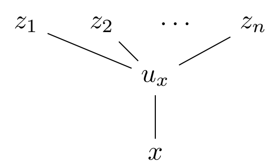
$y \in Y$ and $t \in I$. In other words, a strong deformation retraction leaves points in the subspace Y fixed throughout the homotopy.

Definition 2.16. Let X be an A-space.

(a) A point $x \in X$ is *upbeat* under u_x if there is a $u_x \in X$ such that $u_x > x$ and for every $z \in X$, $z > x$ implies $z \geq u_x$.
(b) A point $x \in X$ is *downbeat* over d_x if there is a $d_x \in X$ such that $d_x < x$ and for every $z \in X$, $z < x$ implies $z \leq d_x$.
(c) A point $x \in X$ is a *beat point* if it is either an upbeat point or a downbeat point.

X is a *minimal space* if it has no beat points. A *core* of a space X is a subspace $Y \subseteq X$ that is minimal and a strong deformation retraction (SDR) of X.

Remark 2.17. The following is part of a Hasse diagram of an upbeat x under u_x.



From this picture, we can see that for an upbeat x, there exists a smallest element u_x among all the elements bigger than x. If we turn this picture upside down, it is exactly how part of a Hasse diagram of a downbeat point x looks. The key point of this picture is that for any upbeat or downbeat point, there is exactly one edge connecting to it from above or connecting to it from below.

Theorem 2.18. [4, Thm. 2.4.4] *Any finite space X has a core.*

Proof. (sketch) Construct a strong deformation retraction from X to X minus one beat point. After finitely many such deformation retractions, there are no more beat points left. The remaining points form a core. □

Definition 2.19. If X, Y are topological spaces, then $C(X,Y)$ denotes the space of continuous maps $X \to Y$ in the compact-open topology.

Proposition 2.20. *For a finite space X and $f, g \in C(X,X)$, the following are equivalent:*

1. *f and g are in the same path component of $C(X,X)$,*
2. *there is a sequence of maps $\{f = f_1, f_2, ..., f_q = g\}$ such that $f_i \sim f_{i+1}$ for $i = 1, ..., q-1$.*

A proof can be found following Proposition 2.2.12 in May's book [4].

Remark 2.21. Kukieła [3] studies the behavior of the compact open topology on $C(X,Y)$, where X and Y are Alexandroff spaces. He proves that the compact open topology on $C(X,Y)$ is Alexandroff if X is finite and Y is any Alexandroff space. Also he shows that the space $C(X,X)$ is never an Alexandroff space if X is infinite. But generally speaking, the compact open topology on $C(X,Y)$ is weaker than the Alexandroff topology induced by the order on $C(X,Y)$.

Theorem 2.22. *If X is a minimal finite space, and $f \in C(X,X)$ is in the same path component as id_X, then $f = id_X$.*

Proof. Let $f : X \to X$ be a continuous map in $C(X, X)$. Since $f \in C(X, X)$ is in the same path component as id_X, according to the above proposition, without loss of generality, assume $f \leq id_X$. We will show that $f = id_X$.

Since X is a finite space, X contains no strictly decreasing infinite sequence, which means that X is well-founded. So we can use Noetherian induction. Take $y \in X$ and suppose $f(x) = x$ for all $x < y$. We will show that if $f(y) < y$, then y is a downbeat point over $f(y)$, contradicting minimality of X. Hence, we must have $f(y) = y$, and by Noetherian induction, $f = id_X$. So, suppose $f(y) < y$. For any $x < y$, $x = f(x) \leq f(y) < y$ by induction and monotonicity. This means y is a downbeat point over $f(y)$, contradiction. By the previous remarks, it follows $f = id_X$.

A similar argument shows that if $f \geq id_X$, then $f = id_X$.

Therefore, for the sequence of maps $\{f = f_1, f_2, ..., f_q = id_X\}$ where $f_i \sim f_{i+1}$ for $i = 1, ..., q-1$, $f_i = id_X$ for each i. □

We will give the proof for a generalization later in this paper. See the proof of Theorem 3.17 and Remark 3.18 for details.

Corollary 2.23. *Finite spaces X and Y are homotopy equivalent if and only if they have homeomorphic cores X^C and Y^C, respectively. In particular, the core of X is unique up to homeomorphism.*

Proof. ($\Leftarrow$) If X^C is homeomorphic to Y^C, then X^C is homotopy equivalent to Y^C. Since X, Y are homotopy equivalent to their cores X^C and Y^C, this implies X is homotopy equivalent to Y.
($\Rightarrow$) We use Theorem 2.22 to prove this direction. Suppose X is homotopy equivalent to Y. Since the cores X^C and Y^C are strong deformation retracts of X and Y, X^C is homotopy equivalent to Y^C. Therefore there exist continuous maps $f : X^C \to Y^C$ and $g : Y^C \to X^C$, such that $g \circ f \simeq id_{X^C}$, and $f \circ g \simeq id_{Y^C}$. It follows that $g \circ f$ and $f \circ g$ are in the same path components of id_{X^C} and id_{Y^C} respectively. By the theorem above, $g \circ f = id_{X^C}$, and $f \circ g = id_{Y^C}$. Therefore, $g = f^{-1}$, and $X^C \cong Y^C$.

Suppose X_0^C and X_1^C are two cores of a finite space X. Take $X = Y$. Then X homotopy equivalent to X implies that X_0^C is homeomorphic to X_1^C. □

3. Cores of Alexandroff Spaces

3.1. **Some Classes of Alexandroff Spaces.**

In this part, we introduce some classes of Alexandroff spaces that will be needed later. These spaces satisfy finiteness conditions that will allow us to construct cores. Following Kukieła's paper [3], we have the following definitions.

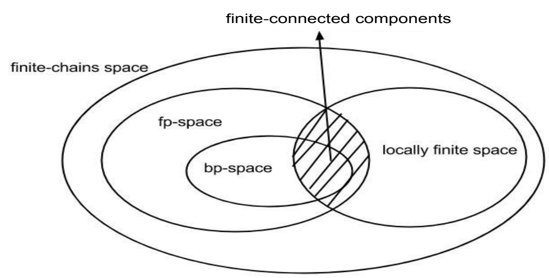
Definition 3.1. Let X be an Alexandroff space. A (finite or infinite) sequence (x_n) of elements of X is an *s-path* if $x_i \neq x_j$ for $i \neq j$ and $x_{i-1} \sim x_i$ for all $i > 0$. Given a finite s-path $k = (x_0, \ldots, x_m)$, we say m is the *length* of k and call k an *s-path from x_0 to x_m*.

Definition 3.2. An Alexandroff space X is:

1. a *finite-chains space* if every chain in X is finite,
2. a *locally finite space* if for every $x \in X$, the set $\{y \in X : y \sim x\}$ is finite,
3. a *finite-paths space (fp-space)* if every s-path of elements of X is finite,

4. a *bounded-paths space (bp-space)* if there exists an $n \in \mathbb{N}$ such that every s-path of elements of X has less than n elements.

Remark 3.3. Bp-spaces form a strict subclass of fp-spaces and both fp-spaces and locally finite spaces are strict subclasses of finite-chains spaces. Moreover, the connected components of the spaces, which are both fp-spaces and locally finite, are finite. This can be proved using Proposition 3.4 in Kukieła [3]. The following picture shows the relationship between different classes of Alexandroff spaces.



3.2. **Cores of Infinite Alexandroff Spaces.**

In this part, we generalize results about cores of finite spaces to the infinite case. All these results originate from Kukieła [3].

Recall the definition of beat points. We have the following.

Definition 3.4. Let X be an Alexandroff space (with distinguished point p). A (basepoint-fixing) retraction $r : X \to r(X)$ is called:

1. a *comparative retraction* if $r(x) \sim x$ for every $x \in X$.
2. an *up-retraction* if $r(x) \geq x$ for every $x \in X$.
3. a *down-retraction* if $r(x) \leq x$ for every $x \in X$.
4. a *retraction removing a beat point*, if there exists an $x \in X$ being an upbeat point under some $u_x \in X$ or a downbeat over some $d_x \in X$ such that $r(x) = u_x$ or $r(x) = d_x$, and $r(y) = y$ for all $y \neq x$.

Remark 3.5. Every comparative retraction can be written as a composition of an up-retraction and a down-retraction. If $r : X \to A$ is a comparative retraction, then $r = r_d \circ r_u$, where

$$r_u(x) = \begin{cases} r(x) & \text{if } r(x) \geq x \\ x & \text{if } r(x) \leq x \end{cases}$$

and

$$r_d(x) = \begin{cases} r(x) & \text{if } r(x) \leq x \\ x & \text{if } r(x) \geq x \end{cases} .$$

Definition 3.6. Let X be an Alexandroff space (with distinguished point p). Let $\mathcal{C}$ be the class of all comparative retractions and $\mathcal{I}$ be the class of {retractions removing a beat point} $\cup$ {identity maps}. The space X is called a *$\mathcal{C}$-minimal space* (or an *$\mathcal{I}$-minimal space*) if there is no retraction $r : X \to r(X)$ in $\mathcal{C}$ (or $\mathcal{I}$) other than id_X. The space X is called a *$\mathcal{C}$-core* (or an *$\mathcal{I}$-core*) if X is a $\mathcal{C}$-minimal subspace (or an $\mathcal{I}$-minimal subspace) that is a strong deformation retraction of X.

Proposition 3.7. *A space X is $\mathcal{I}$-minimal if and only if it has no beat points.*

Proof. ($\Rightarrow$) This direction follows from the definition above. Since in the class of $\mathcal{I}$, there is no retraction of removing a beat point other than id_X, it follows that there are no beat points in X.
($\Leftarrow$) If X has no beat points, then the retractions of removing a beat point are the same as the identity maps. This means id_X is the only retraction in $\mathcal{I}$, which implies X is $\mathcal{I}$-minimal. $\square$

Remark 3.8. In Kukieła 's paper, a $\mathcal{C}$-minimal space (or an $\mathcal{I}$-minimal space) is called a $\mathcal{C}$-core (or an $\mathcal{I}$-core). We are changing the terminology because in May's book [4], cores are required to be strong deformation retracts of the space they are contained in.

Corollary 3.9. *Suppose X is a finite-chains space. Then X is $\mathcal{C}$-minimal if and only if X is $\mathcal{I}$-minimal.*

Proof. ($\Leftarrow$) Suppose X is $\mathcal{I}$-minimal and that $r : X \to r(X)$ is a $\mathcal{C}$-retraction. Factor r as $r_d \circ r_u$ with r_u an up-retraction, r_d a down-retraction. Then $r_d \leq id_X$, $r_u \geq id_X$ and proceed as Theorem 2.22 to show $r_d = r_u = id_X$. This will prove $\mathcal{C}$-minimality.
($\Rightarrow$) A retraction removing a beat point is also a comparative retraction. So if X is $\mathcal{C}$-minimal, then there is no comparative retraction, and hence no $\mathcal{I}$-retraction, other than id_X. Therefore X is $\mathcal{I}$-minimal. $\square$

Definition 3.10 ([3, Defn. 5.9]; cf. Exercise 24 in Chapter 4 of [5])**.** Let γ be an ordinal and X be an Alexandroff space (with the distinguished point p). Let $\{r_\alpha : X_\alpha \to X_{\alpha+1}\}_{\alpha<\gamma}$ be a family of of (basepoint-fixing) retractions from $\mathcal{C}$ (or $\mathcal{I}$) such that $X_0 = X$, $X_{\alpha+1} = r_\alpha(X_\alpha)$ for all $\alpha < \gamma$ and $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$ for limit ordinals $\alpha < \gamma$. By transfinite recursion, we define a family of retractions $\{R_\alpha : X \to X_\alpha\}_{\alpha\leq\gamma}$ such that:

1. $R_0 = id_X$,
2. $R_{\alpha+1} = \gamma_\alpha \circ R_\alpha$,
3. for a limit ordinal α and an $x \in X$, if there exists $\beta_0 < \alpha$ such that $R_\beta(x) = R_{\beta_0}(x)$ for all $\beta_0 \leq \beta < \alpha$, then $R_\alpha(x) = R_{\beta_0}(x)$, and if not, we leave $R_\alpha(x)$ undefined.

The recursion ends when R_γ is defined or when R_α cannot be totally defined for some limit ordinal α. In the first case we say the family $\{r_\alpha\}_{\alpha<\gamma}$ is *infinitely composable* and X is $\mathcal{C}$-*dismantlable* (or $\mathcal{I}$-*dismantlable*) to X_γ (*in γ steps*). In the second case we say the family $\{r_\alpha\}_{\alpha<\gamma}$ is *not infinitely composable.*

Definition 3.11. Let X be a finite-chains space. Let $u_X : X \to X$ be given by:

$$u_X(x) = \begin{cases} u_x & \text{if } x \text{ is upbeat under } u_x \\ x & \text{otherwise} \end{cases}$$

Since $u_X(x) \geq x$ for every $x \in X$ and X is a finite-chains space, it follows that for every $x \in X$ there exists an $N_x \in \mathbb{N}$ such that $(u_X)^n(x) = (u_X)^{N_x}(x)$ for every $n \geq N_x$. Let $U_X : X \to U_X(X)$ be an up-retraction given by $U_X(x) = (u_X)^{N_x}(x)$. Similarly, we define the down-retraction $D_X : X \to D_X(X)$.

Remark 3.12. It is easy to check that that u_X and U_X are order-preserving, as well as d_X and D_X. Given $x, y \in X$ such that $x < y$, we will show $u_X(x) \leq u_X(y)$. Note that we can assume $x < y$ here, because if $x = y$, then $u_X(x) = u_X(y)$.

(a) If neither x nor y is upbeat point, then $u_X(x) = x < y = u_X(y)$.
(b) If x is an upbeat point under u_x and y is not an upbeat point, then $u_X(x) = u_x \leq y = u_X(y)$.
(c) If y is an upbeat point under u_y and x is not an upbeat point, then $u_X(x) = x < y < u_y = u_X(y)$.
(d) If both x and y are upbeat points, then $u_X(x) = u_x \leq y < u_y = u_X(y)$.

Now we check U_X is order-preserving. Note that for any pair $x \leq y$, there is some $N \gg 0$ such that $U_X(x) = u_X^N(x)$ and $U_Y(y) = u_X^N(y)$. Since u_X is monotone, $U_X(x) \leq U_X(y)$ by induction.

Similarly, we can check d_X and D_X are order-preserving as well.

Definition 3.13. Given an ordinal γ and a finite-chains space X, we define a sequence of retractions $\{r_\alpha : X_\alpha \to X_{\alpha+1}\}_{\alpha<\gamma}$ by transfinite recursion.

Let $X_0 = X$, $X_{\alpha+1} = r_\alpha(X)$ and $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$ if α is a limit ordinal. For $\alpha = 0$ or α a limit ordinal and n a finite ordinal, let

$$r_{\alpha+n} = \begin{cases} D_{X_{\alpha+n}} & \text{if n is even} \\ U_{X_{\alpha+n}} & \text{if n is odd} \end{cases}$$

We call this sequence of retractions $\{r_\alpha : X_\alpha \to X_{\alpha+1}\}_{\alpha<\gamma}$ *the standard sequence of X (of length γ).*

Theorem 3.14 ([3, Thm. 4.18])**.** *Let X, Y be Alexandroff spaces and $\{f_\alpha : X \to Y\}_{\alpha\leq\gamma}$, where γ is a countable ordinal, be a family of continuous map such that:*

1. *if $\alpha = \beta + 1$, then $f_\alpha \sim f_\beta$,*
2. *if α is a limit ordinal, then for every $x \in X$ exists $\beta_x^\alpha < \alpha$ such that $f_\beta(x) \leq f_\alpha(x)$ for all $\beta_x^\alpha \leq \beta \leq \alpha$.*

Then f_0 is homotopic to f_γ.

Definition 3.15. An Alexandroff space X (with distinguished point p) is *countably $\mathcal{C}$-dismantlable* (or *$\mathcal{I}$-dismantlable*) to $X' \subseteq X$ if it is $\mathcal{C}$-dismantlable (or $\mathcal{I}$-dismantlable) to X' in γ steps, where γ is a countable ordinal.

The above theorem and definition imply that when an Alexandroff space X is countably $\mathcal{C}$-dismantlable (or $\mathcal{I}$-dismantlable) to a $\mathcal{C}$-minimal subspace (or an $\mathcal{I}$-minimal subspace), we can build a strong deformation retraction from X. By Corollary 3.9 above, a $\mathcal{C}$-core of X is the same as an $\mathcal{I}$-core. For convenience, we call it a core of X in the rest of paper.

We now present the main theorems on cores from Kukieła's paper [3].

Theorem 3.16. *Every bp-space or countable fp-space X has a core. Moreover, if X is a bp-space with path length bounded by some $n \in \mathbb{N}$, then X can be $\mathcal{C}$-dismantled to a core in fewer than $2n + 2$ steps.*

Recall that in the finite case, we can construct a core by removing beat points one by one until we obtain a minimal space. Since removing a beat point is a SDR, this produces a core. However, in the infinite case, we use the standard sequence to remove many beat points at a time, and repeat. After countably many steps, X is $\mathcal{C}$-dismantled to a core. The following is the sketch of the proof, and details can be found in Theorom 5.14 [3].

Proof. (Sketch) Assume X is an infinite Alexandroff space. Let Ω be the first ordinal of cardinality greater that X. Let $\{r_\alpha : X_\alpha \to X_{\alpha+1}\}_{\alpha<\gamma}$ be the standard sequence of X of length Ω.

First, we claim that if X is an fp-space, then the standard sequence is infinitely composable. If not, then for some limit ordinal α, r_α could not be totally defined and we could construct an infinite s-path in X, using a point that moves infinitely often. This would contradict that X is an fp-space. Since the standard sequence of X is infinitely composable, it will be constant beginning with some $\alpha_0 < \Omega$. If not, then X would have cardinality at least Ω, which is a contradiction. Thus we obtain an $\mathcal{I}$-minimal space at α_0. If X is countable, then $\Omega = \omega_1$, the first uncountable ordinal. Therefore $\alpha_0 < \omega_1$ is countable, and we can construct a SDR to X_{α_0} by Theorem 3.14. Thus X_{α_0} is a core of X.

If X is a bp-space with path length bounded by some $n \in \mathbb{N}$, one can show that the standard sequence is constant after $2n + 2$ steps. For if not, then X would contain an s-path of length greater than n, which is a contradiction. □

Recall $C(X, X)$ denotes the space of all continuous maps $X \to X$ in the compact-open topology. We have the following theorem.

Theorem 3.17. *If X is a $\mathcal{I}$-minimal fp-space, then the connected component of id_X in $C(X, X)$ is a singleton.*

To be consistent with May's notation in Definition 2.2.2 [4], we shall write $W(K, U) = \{f : X \to Y \mid f(K) \subseteq U\}$ for the canonical subbasis elements of $C(X, Y)$. Details can be found in Theorom 5.16 [3].

Proof. (Sketch) One first shows that for every $x \in X$, there exists a subspace $x \in A_x \subseteq X$ such that:

1. A_x is finite,
2. if $y \in A_x$ is not maximal in X, then $|A_x \cap max\{z \in X \mid z < y\}| \geq 2$,
3. if $y \in A_x$ is not minimal in X, then $|A_x \cap min\{z \in X \mid z > y\}| \geq 2$.

A_x can be thought of as the image of a tree (but the order on the tree is not the same as the order on X). If A_x is not finite, we could construct a tree A_x, where at each node, there are at most 4 new branches. König's Lemma[1] would imply that if A_x is infinite, then X has an infinite s-path, which contradicts that X is an fp-space.

Since for all $y \in A_x \subseteq X$, $id_X(y) = y \leq y$, it follows that $id_X \in \bigcap_{y \in A_x} W(\{y\}, U_y)$, which is an open neighborhood of id_X. We can show that this $\bigcap_{y \in A_x} W(\{y\}, U_y)$ is also closed. Thus $\bigcap_{y \in A_x} W(\{y\}, U_y)$ is a clopen set containing id_X. From point set topology, the connected component of id_X is a subset of the intersection of all clopen sets $\bigcap_{y \in A_x} W(\{y\}, U_y)$ containing id_X, therefore the component of id_X is contained in $\bigcap_{x \in X} \bigcap_{y \in A_x} W(\{y\}, U_y)$.

Next, one can show that for every $x \in X$, if $f \in \bigcap_{y \in A_x} W(\{y\}, U_y)$, then $f|_{A_x} = id_{A_x}$. If not, then one may inductively construct an infinite, strictly decreasing sequence in A_x, which is a contradiction as well. Thus the connected component of id_X is contained in $\bigcap_{x \in X} \bigcap_{y \in A_x} W(\{y\}, U_y) = \{id_X\}$, and hence the connected component of id_X is exactly $\{id_X\}$. □

[1]König's Lemma: Let P be a well-founded poset, and $S(x) = min\{y \in P \mid y > x\}$ be the set of immediate successors of x. If for all $x \in P$, $S(x)$ is finite, and there exists an $x \in P$ such that the set $\{y | y \geq x\}$ is infinite, then there exists an infinite ascending chain in P.

Remark 3.18. We can get Theorem 2.22 from Theorem 3.17 directly: if X is finite, then it is an fp-space. If X is also minimal, then it satisfies the conditions of Theorem 3.17. If f and id_X are in the same path component, then they are in the same connected component, hence equal by Theorem 3.17.

Corollary 3.19. *Suppose X and Y are fp-spaces, and suppose that they both* have *cores X^C and Y^C. Then X is homotopy equivalent to Y if and only if X^C is homeomorphic to Y^C.*

The proof of the corollary above is identical to the proof of Corollary 2.23.

Lastly, we introduce the concept of chain-complete posets. Although they do not belong to one of those classes of infinite Alexandroff spaces considered in Definition 3.4, we still have similar results.

Definition 3.20. A poset P is called *chain-complete* if every chain in P has both a supremum and an infimum in P.

Definition 3.21. An *antichain* in a poset P is a subset $A \subseteq P$ such that no two elements in A are comparable.

Theorem 3.22 ([3, Thm. 5.8]; cf. Thm 6.11 [2])**.** *Every chain-complete poset X with no infinite antichains has a finite core.*

Remark 3.23. In Corollary 3.19, instead of requiring X and Y to be fp-spaces, we only need X^C and Y^C to be fp-spaces. Also note that if X^C is a finite core, then it is an $\mathcal{I}$-minimal fp-space, so we can use Theorem 3.17 above. In this case, it is straightforward to prove that if any two chain-complete posets X, Y without infinite antichains have finite cores X^C and Y^C, respectively, then X is homotopy equivalent to Y if and only if X^C is homeomorphic to Y^C.

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