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1. Axioms for a homology theory and Excision

We summarize what we've shown:

Proposition 1.1. *Let*

$$H_* : \text{pairs in Top} \to \text{Ab}_*$$

be the singular homology functor. Then

(1) *(Homotopy Invariance) If $f \simeq g : (X, A) \to (Y, B)$, then $H_*(f) = H_*(g) : H_*(X, A) \to H_*(Y, B)$*
(2) *(Dimension Axiom) $H_0(*) = \mathbb{Z}$ and $H_n(*) = 0$ if $n > 0$.*
(3) *(Additivity) If $(X, A) = \coprod (X_i, A_i)$, then $H_*(X, A) \cong \bigoplus_i H_*(X_i, A_i)$.*
(4) *(Exactness) If $A \subset X$, there is a long exact sequence*

$$\ldots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \ldots$$

This exact sequence is natural. Given a map of pairs : $(X, A) \to (Y, B)$, we get an induced map of long exact sequences:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \longrightarrow & H_{n-1}(A) \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & H_n(B) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y, B) & \longrightarrow & H_{n-1}(B) \longrightarrow \cdots
\end{array}$$

Our next goal is to explain why singular homology also satisfies the following axioms:

(5) *(Excision) Suppose that $A \subset V \subset X$ is such that $\overline{A} \subset \text{int}(V)$. Then the inclusion*

$$i : (X - A, V - A) \to (X, V)$$

induces an isomorphism

$$H_*(i) : H_*(X - A, V - A) \xrightarrow{\cong} H_*(X, V).$$

(6) *(Weak Equivalence) If $f : X \to Y$ is a weak equivalence and induces a weak equivalence on subspaces $f : A \to B$, then $H_*(f) : H_*(X, A) \to H_*(Y, B)$ is an isomorphism.*

Remark 1.2. A functor

$$E_* : \text{pairs in Top} \to \text{Ab}_*$$

which satisfies (1), (3)-(6) is called a *generalized* homology theory. One can prove that if E_* is a generalized homology theory and E_* satisfies the dimension axiom (2), then $E_*(X) \cong H_*(X)$ for all X.

That (6) is desirable is clear. It implies that homology, as an invariant, does not distinguish weakly equivalence spaces. We will not prove that it holds, but we include it here for completeness.

However, (5) is a little mysterious at this point. It is important since it will be used to identify $H_n(X, A)$ in favorable cases.

Definition 1.3. Let A be a subspace of X. Let $C(i)$ be the cone on the inclusion of $A \xrightarrow{i} X$.

$$C(i) = \left(X \sqcup_{A\times\{0\}} A \times I\right)/(A \times \{1\}) = X \sqcup_A C(A)$$

The pair (X, A) is *good* if the natural map

$$C(i) = \left(X \sqcup_{A\times\{0\}} A \times I\right)/(A \times \{1\}) \to X/A$$

which collapses $A \times I$ to a point is a homotopy equivalence.

Example 1.4.

- Let A be a subspace of X. Suppose that A has a neighborhood V such that $\overline{A} \subset \text{int}(V)$ and A is a deformation retract of V. Then (X, A) is good.
- Let $M(i)$ be the mapping cylinder on A

  $$M(i) = X \sqcup_{A\times\{0\}} (A \times I)$$

  Then $(X, A \times \{1\})$ is good.

Theorem 1.5. *Suppose that A is a subspace of X and that $\overline{A} \subset \text{int}(V) \subset X$ for a neighborhood V of A such that A is a deformation retract of V. Then*

$$H_n(X, A) \cong \widetilde{H}_n(X/A)$$

Proof. We have the following commutative diagram

$$\begin{array}{ccccc}
H_n(X, A) & \xrightarrow{\cong} & H_n(X, V) & \xleftarrow{\cong} & H_n(X - A, V - A) \\
\downarrow & & \downarrow & & \downarrow \\
H_n(X/A, A/A) & \xrightarrow{\cong} & H_n(X/A, V/A) & \xleftarrow{\cong} & H_n(X/A - A/A, V/A - A/A)
\end{array}$$

The two right horizontal maps are isomorphisms by excision. The two left horizontal maps are isomorphisms since there are homotopy equivalences of pairs $(X, A) \simeq (X, V)$ and $(X/A, A/A) \simeq (X/A, V/A)$ (here, we are using that A is a deformation retract of V). Now, note that

$$(X - A, V - A) \hookrightarrow (X/A - A/A, V/A - A/A)$$

is a homeomorphism. So the right hand vertical map is an isomorphism. The commutativity of the right hand square implies that the vertical middle map is an isomorphism, and thus the commutativity of the left hand square implies that

$$H_n(X, A) \to H_n(X/A, A/A)$$

is an isomorphism. However,

$$H_n(X/A, A/A) \cong H_n(X/A, *) \cong \widetilde{H}_n(X/A).$$

□

Exercise 1.6. *Use excision to prove that if (X, A) is a good pair, then*

$$H^*(X, A) \cong \widetilde{H}^*(X/A)$$

(Hint: $(X, A) \to (M(i), A \times I)$ is a homotopy equivalence of pairs. Further, $\widetilde{H}^*(X/A) \cong H^*(C(i), *)$, by the assumption on the pair. So it is enough to show that $H_*(M(i), I \times A) \cong H_*(C(i), *)$.)

We prove the following version of excision:

Definition 1.7. An excisive triad $(X; V, W)$ is a space X with two subspaces V, W such that X is the union of the interiors of V and W.

Theorem 1.8 (Excision)**.** *If $A \subset V \subset X$ is such that $\overline{A} \subset \operatorname{int}(V)$, then the inclusion*

$$i : (X - A, V - A) \to (X, V)$$

induces an isomorphism

$$H_*(i) : H_*(X - A, V - A) \xrightarrow{\cong} H_*(X, V).$$

Equivalently, if $(X; V, W)$ is an excisive triad, then then the inclusion

$$i : (W, W \cap V) \to (X, V)$$

induces an isomorphism

$$H_*(i) : H_*(W, W \cap V) \xrightarrow{\cong} H_*(X, V).$$

Proof Sketch. First, to deduce the first formulation from the second, we let $W = X - A$ an apply excision to the triad $(X; V, W)$. In this case,

$$W \cap V = V - A$$

and the claim follows. To go back, let $A = X - W$.

(i) Given V and W as in the claim, we define a chain complex

$$C_n(V + W) = \{\sigma : \Delta^n \to X \mid \sigma(\Delta^n) \subset V \text{ or } \sigma(\Delta^n) \subset W\}.$$

These are the simplicies whose image is either entirely in V or entirely in W.

(ii) This step is the hard part. One must show that the natural inclusion

$$C_*(V + W) \xrightarrow{i} C_*(X)$$

is a chain homotopy equivalence, so that, in particular, it induces an isomorphism on homology. A brief idea of how this goes. Take $\sigma : \Delta^n \to X$. It may not lie in either V nor W. However, if we can subdivide Δ^n into arbitrarily small sub-simplices $\Delta^n = \bigcup_i (\Delta^n)_i$ so that the restriction of $\sigma_i = \sigma|_{(\Delta^n)_i}$ lies in V or in W (it is in this part that we use $\operatorname{int}(V) \cup \operatorname{int}(W) = X$). Then, σ represents the same homology class as a sum of the σ_i's with appropriate signs.

(iii) We use what we did in the previous step to prove that
$$C_n(V+W)/C_n(V) \to C_n(X)/C_n(V)$$
and
$$C_n(W)/C_n(V\cap W) \to C_n(V+W)/C_n(V)$$
induce isomorphisms on homology. Note that the second isomorphism is clear since the chain complexes are isomorphic: chains that are in W which do not lie in $V\cap W$ are the same as chains that are either entirely in V or entirely in W but are not in V.

(iv) We conclude that the map
$$C_n(V)/C_n(V\cap W) \to C_n(X)/C_n(V)$$
is an isomorphism on homology.

□

Now, to justify the fact that Mayer-Vietoris is not an axiom, we derive it from excision.

Proposition 1.9 (Mayer-Vietoris)**.** *Given an excisive triad $(X;V,W)$, there is a long exact sequence*
$$\ldots \to H_n(V\cap W) \xrightarrow{i} H_n(V)\oplus H_n(W) \xrightarrow{p} H_n(X) \xrightarrow{\Delta} H_{n-1}(V\cap W) \to \ldots$$

Proof. The maps i and p are defined in terms of the inclusions. The map Δ is the composite:
$$H_n(X) \to H_n(X,V) \cong H_n(W, W\cap V) \xrightarrow{\partial} H_{n-1}(V\cap W)$$
where $\partial : H_n(W, W\cap V) \to H_{n-1}(V\cap W)$ is the map from the exactness sequence in axiom (4) and the isomorphism $H_n(X,V) \cong H_n(W, W\cap V)$ comes from excision. Then one can check by hand that the sequence is exact. □

Recall that Mayer-Vietoris implied the suspension isomorphism, which we write here in a slightly different form.

Proposition 1.10 (Suspension Isomorphism)**.** *There are isomorphisms*
$$\widetilde{H}_{n+1}(\Sigma X) \cong \widetilde{H}_n(X)$$

Proof. Exercise. □

Proposition 1.11. *If $X = \bigvee_{i\in I} X_i$ where the inclusion of the base point x_i in each X_i is good, then*
$$\widetilde{H}_*(X) \cong \bigoplus_{i\in I} \widetilde{H}_*(X_i).$$

Proof. The pair $(\coprod_{i\in I} X_i, \{x_i\}_{i\in I})$ and the pairs (X_i, x_i) are good, hence,
$$H_*(\coprod_{i\in I} X_i, \{x_i\}_{i\in I}) \cong \widetilde{H}_*(\coprod_{i\in I} X_i/\{x_i\}_{i\in I}) \cong \widetilde{H}_*(X)$$
and
$$H_*(X_i, x_i) \cong \widetilde{H}_*(X_i/x_i) \cong \widetilde{H}_*(X_i).$$
But by additivity,
$$H_*(\coprod_{i\in I} X_i, \{x_i\}_{i\in I}) \cong \bigoplus_{i\in I} H_*(X_i, \{x_i\}_{i\in I})$$

The result follows by comparing these isomorphisms. □

2. Axioms for a Reduced homology theory

If we are dealing with based spaces, it is more natural to consider the reduced homology theory

$$\widetilde{H}_* : \mathrm{Top}_* \to \mathrm{Ab}_*$$

as opposed to the homology of pairs. In this case, the axioms are a little modified:

Proposition 2.1. *Let* Top_* *denote based spaces such that* $(X, *)$ *is a good pair. Let*

$$\widetilde{H}_* : \mathrm{Top}_* \to \mathrm{Ab}_*$$

be the reduced singular homology functor. Then

1. *(Homotopy Invariance) If* $f \simeq g : X \to Y$*, then* $\widetilde{H}_*(f) = \widetilde{H}_*(g) : \widetilde{H}_*(X) \to \widetilde{H}_*(Y)$
2. *(Dimension Axiom)* $\widetilde{H}_*(*) = 0$
3. *(Additivity) If* $X = \bigvee_i X_i$*, then* $\widetilde{H}_*(X) \cong \bigoplus_i \widetilde{H}_*(X_i)$*.*
4. *(Exactness) If* (A, X) *is a good pair, there is an exact sequence*
$$\widetilde{H}_n(A) \to \widetilde{H}_n(X) \to \widetilde{H}_n(X/A)$$
5. *(Suspension) There are natural isomorphisms*
$$\widetilde{H}_{n+1}(\Sigma X) \cong \widetilde{H}_n(X)$$
6. *(Weak Equivalence) If* $f : X \to Y$ *is a weak equivalence, then* $\widetilde{H}_*(f) : \widetilde{H}_*(X) \to \widetilde{H}_*(Y)$ *is an isomorphism.*

One can deduce a long exact sequence from (4) and (5) (see Concise for details).

3. Simplicial Homology

Now that we've introduced singular homology, one would think that we would go and try to compute it for some spaces. However, the singular chain complex $C_*(X)$ is so big that it is virtually impossible to compute with it. The advantage of singular homology is that it is defined for all spaces and that the definition is general and flexible enough to allow one to prove all the properties we want from homology. Singular homology is the homology we use to build theory. To prove theorems, we need more hands-on definitions, ones that have the flavor of poset homology. We will define another homology theory called *simplicial homology*, denoted H_*^Δ. We will prove that when it is defined,

$$H_*^\Delta(X) \cong H_*(X)$$

Finally, we will show how $H_*^P(X)$ for a poset X is just an instance of simplicial homology. This will tell us that, if X is a poset (and hence an A-space),

$$H_*^P(X) \cong H_*(X).$$

Definition 3.1. An *ordered simplicial complex* X is the following data:

- a poset X^0, whose elements are called the vertices of X
- a set

$$\mathcal{S} = \{\sigma \subset X^0 \mid \sigma \text{ is totally ordered with respect to the order on } X^0\},$$

whose elements are called the *simplices* of X, which satisfies

– $\{x\} \in \mathcal{S}$ for all $x \in X^0$
– if $\sigma \in \mathcal{S}$, and $\tau \subset \sigma$, then $\tau \in \mathcal{S}$

We can construct a topological space from the simplicial complex X. (Details will be given later.) Indeed, we think of the ordered subset $\sigma = [x_0 < \ldots < x_n]$ as an n-simplex with vertices $x_0, \ldots, x_n$. If $\tau = [y_0 < \ldots < y_k] = \sigma_1 \cap \sigma_2$, then σ_1 and σ_2 are glued along the k-simplex τ.

A map between simplicial sets is a map of posets $f : X \to Y$ such that if $\sigma = [x_0 < \ldots < x_n]$ is a simplex in X, then $[f(x_0) \leq \ldots \leq f(x_n)]$ with repetitions omitted is a simplex in Y.

Given a simplicial complex X, realized as a space (in a manner to be explained later), each n-simplex in X determines a map $\sigma : \Delta^n \to X$. Let

$$C_n^\Delta(X) = \mathbb{Z}\{\sigma : \Delta^n \to X \mid \sigma \text{ is a simplex in } X\}$$

Note that there is a natural inclusion

$$C_n^\Delta(X) \to C_n(X).$$

The boundary map in $C_*^\Delta(X)$ is defined exactly as for that of $C_*(X)$. If $f : X \to Y$ is a map of simplicial complexes, then

$$C_*^\Delta(f)([x_0 < \ldots < x_n]) = \begin{cases} [f(x_0) < \ldots < f(x_n)] & f(x_i) \neq f(x_{i+1}),\ 0 \leq i \leq n-1 \\ 0 & \text{otherwise} \end{cases}$$

Definition 3.2. If X is a simplicial complex, the simplicial homology of X is the homology of $H_*^\Delta(X)$.

Exercise 3.3. *Compute the homology of $\mathbb{R}P^2$, of the Klein bottle K and of S^2 using this definition. Compute the homology of S^1 using the smallest model of S^1 you can find. What is the smallest model for S^2 you can find? What about S^n? Compute their homology.*

Definition 3.4. If $A \subset X$ is a sub-complex, then we let

$$C_*^\Delta(X, A) = C_*^\Delta(X)/C_*^\Delta(A).$$

Then, $H_*^\Delta(X, A)$ is the homology of $C_*^\Delta(X, A)$.

Remark 3.5. We automatically obtain a long exact sequence

$$\ldots \to H_n^\Delta(A) \to H_n^\Delta(X) \to H_n^\Delta(X, A) \to H_{n-1}^\Delta(A) \to \ldots$$

4. Equivalence of singular and simplicial homology

Theorem 4.1. *The natural map $H_*^\Delta(X) \to H_*(X)$ is an isomorphism.*

We will need the following lemma.

Lemma 4.2 (Five lemma)**.** *Consider the commutative diagram of abelian groups:*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

If the rows are exact and the left two and right two vertical arrows are isomorphisms, then the middle vertical arrow is also an isomorphism.

Proof. Exercise. □

Proof Sketch of Theorem 4.1. We will prove it for finite dimensional simplicial complexes, that is, simplicial complexes whose simplices have dimension at most k for some $k \in \mathbb{N}$. We do this by induction on k.

If $k = 0$, note that since $X = X^0$, the set of vertices, the inclusion

$$H_n^\Delta(X^0) \to H_n(X^0)$$

is the identity for all n.

Now, suppose that if X has simplices of dimension at most $k-1$, then

$$H_n^\Delta(X) \to H_n(X)$$

is an isomorphism for all n.

Let X have simplices of dimension at most k. Let $X^{k-1} \subset X$ be the sub-complex of X consisting of all the simplices of X that have dimension at most $k-1$. Then there is a map of long exact sequence

$$\begin{array}{ccccccccc}
H_{n+1}^\Delta(X, X^{k-1}) & \longrightarrow & H_n^\Delta(X^{k-1}) & \longrightarrow & H_n^\Delta(X) & \longrightarrow & H_n^\Delta(X, X^{k-1}) & \longrightarrow & H_{n-1}^\Delta(X^{k-1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n+1}(X, X^{k-1}) & \longrightarrow & H_n(X^{k-1}) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, X^{k-1}) & \longrightarrow & H_{n-1}(X^{k-1})
\end{array}$$

By induction,

$$H_*^\Delta(X^{k-1}) \to H_*(X^{k-1})$$

is an isomorphism, so the second and last vertical arrows are isomorphisms. We will show that

$$H_*^\Delta(X, X^{k-1}) \to H_n(X, X^{k-1})$$

is an isomorphism. This will imply that the first and fourth vertical maps in the diagram are also isomorphisms. We will then conclude by the five lemma that the middle map is an isomorphism.

The pair (X, X^{k-1}) is a good pair, hence

$$H_n(X, X^{k-1}) \cong \widetilde{H}_n(X/X^{k-1}).$$

However,

$$X/X^{k-1} \cong \bigvee_\alpha \Delta_\alpha^k/\partial\Delta_\alpha^k$$

where α runs over all the k-simplices of X. Hence,

$$H_n(X, X^{k-1}) \cong \widetilde{H}_n(X/X^{k-1}) \cong \bigoplus_\alpha \widetilde{H}_*(\Delta_\alpha^k/\partial\Delta_\alpha^k)$$

since

$$\Delta_\alpha^k/\partial\Delta_\alpha^k \cong S^k,$$

this is

$$H_n(X, X^{k-1}) = \begin{cases} 0 & n \neq k \\ \bigoplus_\alpha \mathbb{Z}\{\sigma_\alpha\} & n = k \end{cases}$$

where $\sigma_\alpha : \Delta_\alpha^k \to X$ is the singular n-simplex defined by the inclusion $\sigma_\alpha : \Delta_\alpha^k \to X$.

The complex $C_n^\Delta(X, X^{k-1})$ is zero unless $n = k$ in which case it has one generator for each $\sigma_\alpha : \Delta_\alpha^k \to X$. So

$$H_n^\Delta(X, X^{k-1}) = \begin{cases} 0 & n \neq k \\ \bigoplus_\alpha \mathbb{Z}\{\sigma_\alpha\} & n = k. \end{cases}$$

The map

$$C_k^\Delta(X, X^{k-1}) \to C_k(X, X^{k-1})$$

sends σ_α to it self, so induces an isomorphism

$$H_n^\Delta(X, X^{k-1}) \xrightarrow{\cong} H_n(X, X^{k-1}).$$

□

5. The simplicial complex of a poset

We define a simplicial complex $\mathcal{K}(X)$ associated to a poset X as follows.

- The set of zero simplices (vertices) is $X^0 = X$.
- The set of n-simplices X^n is the set of totally ordered subsets

$$x_0 < x_1 < \ldots < x_n$$

  of size $n + 1$. Call that simplex $\Delta^n_{x_0<\ldots<x_n}$.

Further, a map of posets $f : X \to Y$ induces a map of simplicial complexes

$$\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$$

I will not prove the following proposition (see the Finite Space book).

Proposition 5.1. *There are weak equivalences $\psi_X : \mathcal{K}(X) \to X$ with the property that, given a map of posets $f : X \to Y$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{K}(X) & \xrightarrow{\mathcal{K}(f)} & \mathcal{K}(Y) \\ \downarrow{\scriptstyle \psi_X} & & \downarrow{\scriptstyle \psi_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

Proof Sketch. We construct the map but do not verify the properties. If u is an interior point for some simplex $\Delta^n_{x_0<\ldots<x_n}$, then $\psi_X(u) = x_0$. The rest of the proof is in Proposition 5.1.2 of the Finite Space book. □

Proposition 5.2. *The poset homology $H_*^P(X)$ is isomorphic to the simplicial homology $H_*^\Delta(\mathcal{K}(X))$. Further we can choose the isomorphism*

$$\Psi_X : H_*^\Delta(\mathcal{K}(X)) \to H_*^P(X)$$

in such a way that, given a map $f : X \to Y$, the following diagram commutes:

$$\begin{array}{ccc} H_*^\Delta(\mathcal{K}(X)) & \xrightarrow{H_*^\Delta(\mathcal{K}(f))} & H_*^\Delta(\mathcal{K}(Y)) \\ \downarrow{\scriptstyle \Psi_X} & & \downarrow{\scriptstyle \Psi_Y} \\ H_*^P(X) & \xrightarrow{H_*^P(f)} & H_*^P(Y) \end{array}$$

Proof. We will use the definition of $C_*^P(X)$ which uses strict inequalities. That is, $C_*^P(X)$ is the free abelian group generated by symbols of the form

$$[x_0 < \ldots < x_n].$$

Note that in this case, if $f : X \to Y$ is a map of posets, then

$$C_*^P(f)([x_0 < \ldots < x_n]) = \begin{cases} [f(x_0) < \ldots < f(x_n)] & f(x_i) \neq f(x_{i+1}),\ 0 \leq i \leq n-1 \\ 0 & \text{otherwise} \end{cases}$$

Constructing an isomorphism

$$C_*^\Delta(\mathcal{K}(X)) \to C_*^P(X)$$

which makes the following diagrams commute

$$\begin{array}{ccc} C_*^\Delta(\mathcal{K}(X)) & \longrightarrow & C_*^P(X) \\ {\scriptstyle C_*^\Delta(\mathcal{K}(f))}\downarrow & & \downarrow{\scriptstyle C_*^P(f)} \\ C_*^\Delta(\mathcal{K}(Y)) & \longrightarrow & C_*^P(Y) \end{array}$$

amounts to realizing that the definition of these chain complexes is essentially the same.

□

Corollary 5.3. *The singular homology $H_*(X)$ is isomorphic to $H_*^P(X)$.*

Proof. Since $\psi_X : \mathcal{K}(X) \to X$ is a weak equivalence, $H_*(\psi_X)$ is an isomorphism on singular homology, hence, we have a sequence of isomorphisms:

$$\begin{aligned} H_*(X) &\cong H_*(\mathcal{K}(X)) \\ &\cong H_*^\Delta(\mathcal{K}(X)) \\ &\cong H_*^P(X) \end{aligned}$$

□

6. Deriving more properties for H_*^P

Proposition 6.1. *A weak equivalence $\psi : X \to Y$ between posets induces an isomorphism $H_*^P(X) \to H_*^P(Y)$.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} H_*^P(X) & \xrightarrow{H_*^P(f)} & H_*^P(Y) \\ {\scriptstyle \Psi_X^{-1}}\downarrow & & \downarrow{\scriptstyle \Psi_Y^{-1}} \\ H_*^\Delta(\mathcal{K}(X)) & \xrightarrow{H_*^\Delta(\mathcal{K}(f))} & H_*^\Delta(\mathcal{K}(Y)) \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ H_*(\mathcal{K}(X)) & \xrightarrow{H_*(\mathcal{K}(f))} & H_*(\mathcal{K}(Y)) \\ {\scriptstyle H_*(\psi_X)}\downarrow & & \downarrow{\scriptstyle H_*(\psi_Y)} \\ H_*(X) & \xrightarrow{H_*(f)} & H_*(Y) \end{array}$$

The vertical composites are isomorphisms and the bottom map is an isomorphism. Check that the five lemma implies that the top horizontal map $H_*^P(f)$ is an isomorphism. □

Proposition 6.2. *Homotopic maps $f, g : X \to Y$ between posets induce the same map on homology $H_*^P(f) = H_*^P(g)$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
H_*^P(X) & \xrightarrow{H_*^P(f)-H_*^P(g)} & H_*^P(Y) \\
\downarrow{\scriptstyle \Psi_X^{-1}} & & \downarrow{\scriptstyle \Psi_Y^{-1}} \\
H_*^\Delta(\mathcal{K}(X)) & \xrightarrow{H_*^\Delta(\mathcal{K}(f))-H_*^\Delta(\mathcal{K}(g))} & H_*^\Delta(\mathcal{K}(Y)) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
H_*(\mathcal{K}(X)) & \xrightarrow{H_*(\mathcal{K}(f))-H_*(\mathcal{K}(g))} & H_*(\mathcal{K}(Y)) \\
\downarrow{\scriptstyle H_*(\psi_X)} & & \downarrow{\scriptstyle H_*(\psi_Y)} \\
H_*(X) & \xrightarrow{H_*(f)-H_*(g)} & H_*(Y)
\end{array}$$

The vertical composites are isomorphisms and the bottom map is zero. Check that the five lemma implies that the top horizontal map $H_*^P(f) - H_*^P(g)$ is zero. □

Note the following result:

Theorem 6.3 (Thibault)**.** *If $f, g : X \to Y$ are homotopic maps of posets, where X is an F-space and Y is an A-space, then $\mathcal{K}(f) \simeq \mathcal{K}(g)$.*

From this theorem, we have

Corollary 6.4. *If f and g are as above, then $H_*^P(f) = H_*^P(g)$.*

Proof. Since $H_*^P(f) = H_*^\Delta(\mathcal{K}(f))$ and $H_*^P(g) = H_*^\Delta(\mathcal{K}(g))$, this follows from the similar result for simplicial homology. □

However, Proposition 6.2 does not have any restriction on X.

Proposition 6.5. *Let $(X; V, W)$ be an excisive triad of A-spaces. Then*

$$H_*^P(X, V) \cong H_*^P(W, W \cap V).$$

Therefore, if (X, A) is a good pair, then

$$H_*^P(X, A) \cong \widetilde{H}_*^P(X/A).$$

Proof. Once we have proved the results of this section for pairs, the result follows as before from the diagram:

$$\begin{array}{ccc}
H_*^P(W, W\cap V) & \xrightarrow{H_*^P(i)} & H_*^P(X,V) \\
\downarrow{\scriptstyle \Psi^{-1}_{(W,W\cap V)}} & & \downarrow{\scriptstyle \Psi^{-1}_{(X,V)}} \\
H_*^\Delta(\mathcal{K}(W),\mathcal{K}(W\cap V)) & \xrightarrow{H_*^\Delta(\mathcal{K}(i))} & H_*^\Delta(\mathcal{K}(X),\mathcal{K}(V)) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
H_*(\mathcal{K}(W),\mathcal{K}(W\cap V)) & \xrightarrow{H_*(\mathcal{K}(i))} & H_*(\mathcal{K}(X),\mathcal{K}(V)) \\
\downarrow{\scriptstyle H_*(\psi_{(W,W\cap V)})} & & \downarrow{\scriptstyle H_*(\psi_{(X,V)})} \\
H_*(W, W\cap V) & \xrightarrow{H_*(i)} & H_*(X,V)
\end{array}$$

□

Question 6.6. *What is a "good pair" (X, A) in A-spaces? What is an excisive triad $(X; V, W)$ of A-spaces? More precisely, can we give insightful conditions on the posets which are sufficient for (X, A) to be good or $(X; V, W)$ to be excisive?*