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1. Summary of the Homology of Posets

By now, you have realized that what algebraic topologists do is assign invariants to topological spaces which do not depend on the homotopy type. That is, we assign quantities $F(X)$ so that if $X \simeq Y$, then $F(X) \cong F(Y)$.

Further, we have the language of functors to make this more precise. What we've done so far is define functors

$$F : \mathrm{Top} \to C$$

where C is some algebraic category (abelian groups, chain complexes, ...), such that if $X \simeq Y$ then $F(X) \simeq F(Y)$ (in the case of chain complexes) or better $F(X) \cong F(Y)$ (in the case of homology).

Definition 1.1. Let $\mathcal{T}$ be some subcategory of topological spaces. A *homotopy functor* is a functor $F : \mathcal{D} \to C$ such that if $f, g : X \to Y$ are homotopic, i.e., $f \simeq g$, then $F(f) = F(g)$.

Exercise 1.2. *If F is a homotopy functor and $X \simeq Y$, then $F(X) \cong F(Y)$.*

Our first examples were

$$\pi_1 : \mathrm{Top}_* \to \mathrm{Groups},$$

and for $n \geq 2$,

$$\pi_n : \mathrm{Top}_* \to \mathrm{AbGroups}\,.$$

Now, we defined the homology of a poset, which gave functors

$$H_n^P : A\text{-spaces} \to \mathrm{Ab}\,.$$

If we think in terms the category Ab_* of sequences of abelian groups (define the maps!), we can write this as

$$H_*^P : A\text{-spaces} \to \mathrm{Ab}_*\,.$$

We also proved that

Proposition 1.3. *Let $f, g : X \to Y$ be maps of A-spaces. If $f \leq g$, then f and g induce the same map on homology, that is*

$$H_*^P(f) = H_*^P(g) : H_*^P(X) \to H_*^P(Y).$$

However, we only have the following partial result:

Theorem 1.4. *If X and Y are A-spaces and $f, g : X \to Y$ satisfy $f \leq g$, then $f \simeq g$. If X and Y are F-spaces, we have a converse. That is, in this case, if $f \simeq g$, then there are maps $f_1, \ldots, f_n$ such that,*

$$f \geq f_1 \leq f_2 \geq f_3 \leq f_4 \geq \ldots \leq f_n \geq g.$$

Therefore, we have *not* shown that H_*^P is a homotopy functor. However, we have shown that its restriction to F-spaces is such:

Proposition 1.5. *$H_*^P : F$-spaces $\to \mathrm{Ab}_*$ is a homotopy functor.*

Definition 1.6. We also defined, for $A \subset X$, the *relative homology of X with respect to A*

$$\begin{aligned} H_n^P(X, A) &= \frac{\ker\left(C_n(X)/C_n(A) \xrightarrow{d} C_{n-1}(X)/C_{n-1}(A)\right)}{\mathrm{im}\left(C_{n+1}(X)/C_{n+1}(A) \xrightarrow{d} C_n(X)/C_n(A)\right)} \\ &= n\text{'th homology of the chain complex } C_*(X)/C_*(A) \end{aligned}$$

Therefore, at least on F-spaces, we've proved that H_* has the following properties. (Property (3) is an exercise)

Proposition 1.7. *Let $H_*^P : F$-spaces $\to \mathrm{Ab}$ be the poset homology functor. Then*

(1) (Homotopy Invariance) If $f \simeq g$, then $H_*^P(f) = H_*^P(g)$
(2) (Dimension Axiom) $H_0^P(*) = \mathbb{Z}$ and $H_n^P(*) = 0$ if $n > 0$.
(3) (Additivity) If $X = \coprod X_i$, where X_i are the different path components of X, then $H_*^P(X) \cong \bigoplus_i H_*^P(X_i)$.
(4) (Exactness) If $A \subset X$, there is a long exact sequence

$$\ldots \to H_n^P(A) \to H_n^P(X) \to H_n^P(X, A) \to H_{n-1}^P(A) \to \ldots$$

We also proved that:

(a) (Mayer-Vietoris) If A and B are sieves in X and $X = A \cup B$, there is a long exact sequence

$$\ldots \to H_n^P(A \cap B) \to H_n^P(A) \oplus H_n^P(B) \to H_n^P(X) \to H_{n-1}^P(A \cap B) \to \ldots$$

(b) (Suspension isomorphism) There is an isomorphism

$$H_n^P(X) \cong \begin{cases} H_1^P(\Sigma X) \oplus \mathbb{Z} & n = 0 \\ H_{n+1}^P(\Sigma X) & n > 0 \end{cases}$$

(Note that except for (1), these all hold for A-spaces.)

Remark 1.8. A little later, we will give an axiomatic description for functors which satisfy the above properties. (1)-(4) will be axioms for such functors. However, the list of axioms is still incomplete and (4) is rather useless since it is not clear now that $H_*^P(X, A)$ has

(1) any geometric meaning that makes it interesting to compute or
(2) is easier to compute than $H_*^P(X)$ and/or $H_*^P(A)$ so that it could be used as a tool in understanding these terms.

We need a property that allows us to identify $H_*^P(X, A)$ with $H_*^P(X/A)$ in favorable cases.

We ask ourselves:

Question 1.9. *Is there a functor $H_* : \mathrm{Top} \to \mathrm{Ab}$ which satisfies all the above properties and such that $H_*^P(X) \cong H_*(X)$?*

2. Definition of Singular Homology

To define the homology of a space, we must associate to a space X a chain complex $C_*(X)$. Then $H_*(X)$ will simply be its homology. Recall that the n-simplex Δ^n is the subspace of $\mathbb{R}^{n+1}$ given by

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1, \quad t_0 + \ldots + t_n = 1\}$$

Now, note that there are $n+1$ copies of Δ^{n-1} in Δ^n. These are the faces of Δ^n. We denote them by

$$\Delta_i^n = \{(t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) \in \Delta^n\}$$

Remark 2.1. Let $x_i = (t_0 = 0, \ldots, t_{i-1} = 0, t_i = 1, t_{i+1} = 0, \ldots, t_n = 0)$. Then $x_0, \ldots, x_n$ are the vertices of Δ^n. The face Δ_i^n is the n-simplex with vertices $x_0, \ldots, \widehat{x}_i, \ldots, x_n$.

There's a canonical homeomorphism:

$$\Delta^{n-1} \to \Delta_i^n$$

which sends $(t_0, \ldots, t_{n-1})$ to $(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n)$

Definition 2.2. A *singular n-simplex* on X is a continuous map $\sigma : \Delta_n \to X$. Let X_n be the set of singular n-simplicies:

$$X_n = \{\sigma : \Delta_n \to X\}$$

Given a singular n-simplex on X, $\sigma : \Delta_n \to X$, we can obtain $n+1$ singular $n-1$-simplices by restricting σ to the faces of Δ^n. That is,

$$\sigma|_{\Delta_i^n} : \Delta_{n-1} \cong \Delta_i^n \to X$$

Definition 2.3. The group of singular n-chains on X, denoted $C_n(X)$, is the free abelian group generated by the set of all singular n-simplices, that is

$$C_n(X) = \mathbb{Z}\{X_n\} = \left\{ \sum_{i=0}^{k} a_i \sigma_i \mid \sigma_i \in X_n, a_i \in \mathbb{Z} \right\}$$

Further, let $\partial : C_n(X) \to C_{n-1}(X)$ be defined on a generator $\sigma \in C_n(X)$ by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{\Delta_i^n}$$

One verifies that $\partial \circ \partial = 0$. The chain complex $C_*(X)$ is called the *singular chains on X*. Its homology is called the *singlular homology of X*, denoted

$$H_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$$

If $f : X \to Y$, we get a map $C_*(f) : C_*(X) \to C_*(Y)$. For a singular n-simplex $\sigma : \Delta^n \to X$, we let

$$C_n(f)(\sigma) = f \circ \sigma \in C_n(Y).$$

This induces a map

$$H_*(f) : H_*(X) \to H_*(Y).$$

Hence, singular homology is a functor:

$$H_* : \mathrm{Top} \to \mathrm{Ab}_*$$

Remark 2.4. Note that

$$H_0(X) = C_0(X)/(\mathrm{im}(\partial_1 : C_1(X) \to C_0(X))).$$

A singular 0-simplex is a function $\sigma : \Delta^0 = \{0\} \to X$, so it is just a point in X. We let

$$\sigma_x : \Delta^0 \to X$$

be the simplex with

$$\sigma_x(0) = x.$$

Therefore,

$$C_0(X) \cong \mathbb{Z}\{X\}$$

where $\mathbb{Z}\{X\}$ is the free abelian group on X. An isomorphism is given by

$$\sum_{i=0}^{k} a_i \sigma_{x_i} \mapsto \sum_{i=0}^{k} a_i \sigma_{x_i}(0) = \sum_{i=0}^{k} a_i x_i.$$

So, we can think of the elements of $C_0(X)$ as formal sums

$$\sum_{i=0}^{k} a_i x_i$$

where the points $x_i \in X$ and $a_i \in \mathbb{Z}$.

Further, suppose that two points $x_0, x_1 \in X$ are in the same path component, that is, that there is a path $\sigma : I \to X$ with $\sigma(0) = x_0$ and $\sigma(1) = x_1$. Since $I = \Delta^1$, we have $\sigma \in C_1(X)$. Further,

$$\begin{aligned} \partial_1(\sigma) &= \sigma|_{\Delta^1_0} - \sigma|_{\Delta^1_1} \\ &= \sigma_{x_1} - \sigma_{x_0} \\ &\mapsto x_1 - x_0. \end{aligned}$$

Therefore, in $C_0(X)/\mathrm{im}(\partial_1)$, we have the relation that $x_1 = x_0$ for any two generators which are connected by a path. Therefore, any two generators for $C_0(X)$ which are in the same path components are identified when we pass to the quotient $H_0(X) = C_0(X)/\mathrm{im}(\partial_1)$. Further, these are the only relations, so $H_0(X)$ is a free abelian group with one generator for each path component of X. This proves:

Proposition 2.5. *If $X = \coprod_{\alpha \in I} X_i$ where X_i are the distinct path components of X, then*

$$H_0(X) \cong \bigoplus_{\alpha \in I} \mathbb{Z}.$$

An explicit isomorphism is induced by the map

$$C_0(X) \to \bigoplus_{\alpha \in I} \mathbb{Z}$$

given by

$$\sum_{i=0}^{k} a_i x_i \mapsto \left(\sum_{x_i \in X_\alpha} a_i \right)_{\alpha \in I}$$

Proposition 2.6 (Dimension Axiom)**.** *$H_0(*) \cong \mathbb{Z}$ and $H_n(*) = 0$ if $n > 0$.*

Proof. We have $C_n(*) = \mathbb{Z}$ since there is only one map $\sigma_n : \Delta^n \to *$, namely, the constant map at $*$. Further, since

$$\sigma_n|_{\Delta^n_i} = \sigma_{n-1}$$

we have for $n > 0$

$$\begin{aligned}\partial_n(\sigma) &= \sum_{i=0}^{n}(-1)^i \sigma|_{\Delta^n_i} \\ &= \sum_{i=0}^{n}(-1)^i \sigma_{n-1} \\ &= \begin{cases} 0 & n \text{ is odd} \\ \sigma_{n-1} & n \text{ is even} \end{cases}\end{aligned}$$

The chain complex thus looks like

$$\ldots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \to \ldots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

□

Remark 2.7. We can define a related functor

$$\widetilde{H}_* : \mathrm{Top}_* \to \mathrm{Ab}$$

called the *reduced* homology as follows. For a based space X, there are maps

$$* \xrightarrow{i} X \xrightarrow{p} *$$

These induce maps on homology

$$\begin{array}{ccccc} H_*(*) & \xrightarrow{i} & H_*(X) & \xrightarrow{p} & H_*(*) \\ \| & & \| & & \| \\ \mathbb{Z} & \longrightarrow & H_*(X) & \longrightarrow & \mathbb{Z} \end{array}$$

This implies that $H_0(X)$ has a direct summand isomorphic to $\mathbb{Z}$. We defined $\widetilde{H}_*(X)$ to be the other summand,

$$H_*(X) \cong \mathbb{Z} \oplus \widetilde{H}_*(X).$$

Therefore,

$$\widetilde{H}_*(X) \cong H_*(X)/H_*(*).$$

If X is path connected, the reduced homology is the same as the homology of X in positive degrees. All we did is remove a copy of $\mathbb{Z}$ in degree zero:

$$H_n(X) = \begin{cases} \widetilde{H}_0(X) \oplus \mathbb{Z} & n = 0 \\ \widetilde{H}_n(X) & n > 0. \end{cases}$$

3. The Hurewicz homomorphism

Let X be a path connected based space with base point $*$. Note that a based map $f : S^1 \to X$ can be viewed as a map $\sigma_f : \Delta^1 \to X$ where $\sigma(0) = \sigma(1) = *$. Further, since $\sigma_f(0) = \sigma_f(1)$,

$$\partial_1(\sigma_f) = \sigma_f(1) - \sigma_f(0) = \sigma_* - \sigma_* = 0.$$

Therefore, $\sigma_f \in C_1(X)$ is a cycle, so it represents an element in $H_1(X)$. Further, after today, you should be able to do the following exercise:

Exercise 3.1. *Prove that if $f \simeq g : S^1 \to X$ are homotopic, then $[\sigma_f] = [\sigma_g] \in H_1(X)$. (Hint: consider the maps $H_*(f), H_*(g) : H_*(S^1) \to H_*(X)$.)*

Once we've done that exercise, we get a map

$$h : \pi_1(X) \to H_1(X)$$

This map is called the *Hurewicz homomorphism*. We won't prove it right now, but

$$\pi_1(X)/[\pi_1(X), \pi_1(X)] \cong H_1(X),$$

that is $H_1(X)$ is the abelianization of $\pi_1(X)$.

Remark 3.2. For any group G, $[G, G]$ is the normal subgroup of G generated by the elements of the form $ghg^{-1}h^{-1}$, for $g, h \in G$. Note that, in $G/[G, G]$, the residue class of $ghg^{-1}h^{-1}$ is equal to that of the identity e. This implies that gh and hg are equivalent modulo $[G, G]$ for all $g, h \in G$. Therefore, $G/[G, G]$ is abelian.

Example 3.3.

- We've seen that $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$ the free group on two generators. This implies that $H_1(S^1 \vee S^1) = \mathbb{Z} \oplus \mathbb{Z}$, the free *abelian* group on two generators.
- If $\pi_1(X)$ is already abelian, then $h : \pi_1(X) \to H_1(X)$ is an isomorphism.

In fact, we can do this for higher homotopy groups also:

Definition 3.4 (Hurewicz homomorphism)**.** Let X be a based topological space. Let $f : S^n \to X$ be a continuous map and $\sigma_f : \Delta^n \to X$ be the singular n-simplex obtained from f by identifying

$$S^n \cong \Delta^n / \partial(\Delta^n).$$

There is a map

$$h : \pi_n(X) \to H_n(X)$$

defined by

$$h([f]) = [\sigma_f]$$

which is a group homomorphism for all n such that $n \geq 1$.

Theorem 3.5 (Hurewicz)**.** *Let X be a based space. Suppose that $\pi_k(X) = 0$ when $0 \leq k \leq n-1$. Then*

$$h : \pi_n(X) \to H_n(X)$$

is an isomorphism if $n > 1$ and the abelianization if $n = 1$.

Remember that homotopy groups are hard to compute, but homology is not. This theorem is important since it identifies something hard, $\pi_n(X)$, with something easy to compute, $H_n(X)$.

4. Some Properties of Singular Homology

Definition 4.1. A pair (X, A) of topological spaces is a space X with a subspace $A \subset X$. A map $f\colon (X, A) \to (Y, B)$ of pairs is a map $f : X \to Y$ such that $f(A) \subset B$. Such a map is a (weak) homotopy equivalence if it induces (weak) homotopy equivalences $X \to Y$ and $A \to B$. Maps of pairs $f, g : (X, A) \to (Y, B)$ are homotopic if there is a map of pairs $F : (X \times I, A \times I) \to (Y, B)$ with $F(0) = f$ and $F(1) = g$.

As for poset homology, we can define relative homology groups for singular homology:

Definition 4.2. Let (X, A) be a pair. The *relative singular homology of X with respect to A*, denoted $H_n(X, A)$, is defined as follows. Let

$$C_n(X, A) = C_n(X)/C_n(A)$$

with differential induced by that on $C_*(X)$. Then

$$H_n(X, A) = \frac{\ker(\partial : C_n(X, A) \to C_{n-1}(X, A))}{\mathrm{im}(\partial : C_{n+1}(X, A) \to C_n(X, A))}.$$

Remark 4.3. It will be convenient to study singular homology as a functor

$$H_* : \text{pairs in Top} \to \mathrm{Ab}_*$$

We note that

$$H_*(X) = H_*(X, \emptyset)$$

since $C_*(\emptyset) = 0$ so that

$$C_*(X, \emptyset) = C_*(X)/C_*(\emptyset) = C_*(X).$$

Therefore, we lose nothing by considering pairs.

Proposition 4.4 (Exactness)**.** *There is a long exact sequence*

$$\ldots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \ldots$$

This exact sequence is natural. Given a map of pairs : $(X, A) \to (Y, B)$, we get an induced map of long exact sequences.

Proof. By definition, there is an exact sequence of chain complexes:

$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$$

Then we apply the snake lemma. □

Exercise 4.5. *If X is a based space, then*

$$\widetilde{H}_*(X) \cong H_*(X, *)$$

Proposition 4.6 (Additivity)**.** *If $X = \coprod (X_i, A_i)$, then*

$$H_*(X, A) \cong \bigoplus_i H_*(X_i, A_i).$$

Proof. Exercise. □

Definition 4.7. A homotopy between maps of chain complexes $f_*, g_* : A_* \to B_*$ is a sequence of homomorphisms $P_n : A_n \to B_{n+1}$ such that

$$f_n - g_n = \partial P_n + P_{n-1}\partial$$

Theorem 4.8. *If $f \simeq g : (X, A) \to (Y, B)$, then $H_*(f) = H_*(g)$.*

Proof. We sketch the argument when $A, B = \emptyset$. Given a homotopy $F : X \times I \to Y$ from f to g, we use it to construct a homotopy $P_n : C_n(X) \to C_{n+1}(Y)$ where

$$C_n(f) - C_n(g) = \partial P_n + P_{n-1}\partial$$

in a way very similar to that Inna used to construct P when she was proving that if $f \leq g$, then $H_*^P(f) = H_*^P(g)$. The key point is to split $\Delta^n \times I$ into $n+1$-simplices and this is done exactly like Inna's proof for posets.

This argument can also be made to work for other choices of A and B. □