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1. Connectedness of A-spaces

Theorem 1.1. *An A-space X is path connected if and only if it is connected.*

First, let's prove that U_x is connected.

Lemma 1.2. *For $x \in X$, where X is an A-space, U_x is connected.*

Proof. Suppose that $U_x = U \sqcup V$, where U and V are open. Then $x \in U$ or $x \in V$. If $x \in U$, then $U_x \subset U$, so $U_x = U$ and $V = \emptyset$. □

Now, assume that X is connected. Let x and y be points in X. If there is a finite zigzag:

$$x \leq z_1 \geq z_2 \leq z_3 \ldots \leq z_n \geq y$$

we will write $x \sim y$. Check that this is an equivalence relation.

Lemma 1.3. *Let X be a connected A-space. Let*

$$A_x = \{y \mid x \sim y\}.$$

Then $A_x = X$.

Proof. First, we show that A_x is open. Let $y \in A_x$. Then there is a zigzag

$$x \leq z_1 \geq z_2 \leq z_3 \ldots \leq z_n \geq y.$$

If $w \leq y$, then w is connected by a finite zigzag to x:

$$x \leq z_1 \geq z_2 \leq z_3 \ldots \leq z_n \geq y = z_{n+1} \geq w$$

So, $w \sim x$ and hence $w \in A_x$. Therefore, $U_y \subset A_x$, so that

$$A_x = \bigcup_{y \in A_x} U_y.$$

So A_x is open since it is the union of a collection of open sets. To finish the proof, it suffices to show that the complement of A_x is also open, which we leave as an exercise. □

Exercise 1.4. *Let X be a connected A-space. Show that the complement of A_x is open.*

Lemma 1.5. *Let $I = [0,1]$ with its metric topology. Let $x, y \in X$ with $x \leq y$. Let $p : I \to X$ be defined by*

$$p([0,1)) = x$$

and

$$p(1) = y.$$

Then p is a continuous path from x to y.

Proof. Clearly $p(0) = x$ and $p(1) = 1$. To prove that p is continuous, we prove that $p^{-1}(V)$ is open for any open set $V \subset X$. If $x, y \notin V$, then $p^{-1}(V) = \emptyset$ so $p^{-1}(V)$ is open. If $x \in V$ and $y \notin V$, then $p^{-1}(V) = [0,1)$ which is open. If $y \in V$, then $U_y \subset V$. Since, $x \leq y$, $x \in U_y$, and hence $x \in V$. Therefore, $p^{-1}(V) = [0,1]$, which is open. □

Together, Lemma 1.3 and Lemma 1.5 prove the following result:

Lemma 1.6. *If X is a connected A-space, then it is path connected.*

Remember that the converse, path connected implies connected, is always true.

2. The mapping space

Let $\mathrm{Map}(X, Y)$ be the set of continuous functions from X to Y with the compact open topology. We claim that, under very mild hypotheses, $\pi_0 \mathrm{Map}(X, Y)$ is the set of homotopy classes of maps from X to Y. A path between two maps $f, g : X \to Y$ is a homotopy

$$\widetilde{h} : I \to \mathrm{Map}(X, Y)$$

with $\widetilde{h}(0) = f$ and $\widetilde{h}(1) = g$. Given such a homotopy, we get a function

$$h : X \times I \to Y$$

with

$$h(x, t) = \widetilde{h}(t)(x).$$

But the topology might conceivably be off: h might not be continuous.

Definition 2.1. *A topological space is first-countable if every point $x \in X$ has a* countable neighborhood basis. *That is, for every $x \in X$ there is a* **countable** *set of open sets*

$$\mathcal{B}_x = \{B \subset X \mid B \text{ is open and } x \in B\},$$

such that for every open U with $x \in U$, there exists $B \in \mathcal{B}_x$ such that $B \subset U$.

Remark 2.2. *Metric spaces and Alexandroff spaces are both first countable.*

When X is first countable, h is continuous and thus f and g are homotopic.

Definition 2.3. *Suppose X and Y are Alexandroff spaces. Fix two continuous maps $f, g : X \to Y$. We say that $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.*

Remark 2.4. *We would like to say that $f \simeq g$ if and only if there exists a zigzag*

$$f \leq z_1 \geq z_2 \leq \ldots \geq z_n \leq g$$

That's not quite true, and Peter doesn't understand exactly when it holds and when it fails. However, if X and Y are finite, then this holds. Here's the idea. Convince yourself that this holds when the compact-open topology on $\mathrm{Map}(X, Y)$ is the same as the pre-order topology. Then, do the exercise:

Exercise 2.5. *If X and Y are both finite, then the compact open topology and the order topology on* $\mathrm{Map}(X, Y)$ *are the same.*

Note that the compact open topology is not the same as the pre-order topology on $\mathrm{Map}(X, Y)$ if X and Y are not finite.

Question 2.6. *If the topologies agree, are X and Y necessarily finite?*

3. Cones

Definition 3.1. *A space is contractible if is is homotopy equivalent to a point.*

Exercise 3.2. *A totally ordered finite space is contractible.*

Question 3.3. *Is a totally ordered, but not necessarily finite space, contractible?*

Definition 3.4. *The cone on X is*

$$C(X) = X \times I/(X \times \{1\}).$$

Definition 3.5. *The non-Hausdorff cone on X is $C_{nH}(X) = X \sqcup \{*\}$ with topology such that the proper open subsets of $C_{nH}(X)$ are exactly the open subsets of X.*

Exercise 3.6. *The non-Hausdorff cone is contractible.*

Lemma 3.7. *Let X be a topological space and $y \in X$. Supppose that the only open set that contains y is X. Then X is contractible to y.*

Proof. Let

$$i : \{y\} \to X$$

be the inclusion and

$$r : X \to \{y\}$$

the only possible map. Then $r \circ i = \mathrm{id}$ and $i \circ r \simeq \mathrm{id}$ where the homotopy

$$h : X \times I \to X$$

is defined by

$$h(x, t) = \begin{cases} x & t < 1 \\ y & t = 1 \end{cases}$$

□

Exercise 3.8. *Check that h is continuous.*

Lemma 3.9. *For any $x \in X$, U_x is contractible.*

Proof. The only open set in U_x that contains x is U_x itself. □

Here the time was running short and Peter got a bit sketchy.

Exercise 3.10. *There is a continuous function*

$$C(X) \cup_X C(X) \to C_{nH}(X) \cup_X C_{nH}(X)$$

That sends cone points to conepoints and restricts to the identity on X.

It turns out this map is always a weak homotopy equivalence.

This is an example of a more general phenomenon: given a space which is obtained by gluing together two spaces, we can understand weak homotopy equivalences by understanding weak homotopy equivalences on the restrictions to the different pieces.

Remark 3.11. *There is a finite space with $2n+2$ points which has the same homotopy groups as a sphere S^n. Indeed, you can construct it inductively from S^0 by gluing non-Hausdorff cones.*

Here is a conundrum: take two four point circles, then

$$\pi_0 \operatorname{Map}(S^1, S^1)$$

is finite. However, for honest circles,

$$\pi_1(S^1) = \pi_0 \operatorname{Map}(S^1, S^1) = \mathbb{Z}$$

The problem here is that, even if two spaces X and Y are weakly homotopy equivalent to spaces X' and Y', it does **not** follow that $\operatorname{Map}(X, Y)$ is weakly equivalent to $\operatorname{Map}(X', Y')$!