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1. Recap

Recall: a **preorder** $(X, \leqq)$ is a set X equipped with a binary relation $\leqq$ that is

(1) **reflexive**: for every $x \in X$, $x \leqq x$.
(2) **transitive**: for all $x, y, z \in X$, if $x \leqq y$ and $y \leqq z$, then $x \leqq z$.

A preorder is a **poset** if, in addition, $\leqq$ is

(3) **antisymmetric**: for every $x, y \in X$, if $x \leqq y$ and $y \leqq x$, then $x = y$.

Exercise: Suppose X is a set and R is a binary relation on X that is antisymmetric. Let $x, y \in X$ and assume that xRy and $x \neq y$. Verify $y \not R x$. This is the reason for the adjective "antisymmetric" – there is a kind of reverse symmetry across the diagonal $\{(x, x) \mid x \in X\}$.

One of the goals of this lecture will be to explain the connection between preorders/posets and topology. More generally, one of the goals of this course will be to show that certain subjects, which appear disjoint at first, are actually closely related to each other.

Recall that a topological space X is called **Alexandroff** if arbitrary intersections of open sets are open. This is unusual if we are thinking of topological spaces as generalized metric spaces; for example, the infinite intersection

$$\bigcap_{k \in \mathbb{N}} \left\{ x \in \mathbb{R}^n \,\middle|\, ||x|| < \frac{1}{k} \right\} = \{0\}$$

of open balls in $\mathbb{R}^k$ is not open in $\mathbb{R}^k$. However, every finite space is Alexandroff (because there are only finitely many closed sets), so this condition will be relevant for us going forward.

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2. Separation Axioms

Separation axioms are means of codifying the ways that points or subsets of a topological space X can be distinguished by open and/or closed sets. The stronger the separation axioms imposed, the more the open subsets of the space tend to behave like the open subsets of a metric space. We shall see, however, that only mild separation axioms can be imposed on Alexandroff spaces before they degenerate into discrete spaces.

Let X be a topological space. We say that X **is** T_0 if, for all $x, y \in X$, if $x \neq y$, then either:

(1) there is an open set $U \subset X$ such that $x \in U$ and $y \notin U$, or
(2) there is an open set $U \subset X$ such that $y \in U$ and $x \notin U$.

Succinctly, a space is T_0 if distinct points are topologically distinguishable (are not contained in the exact same open sets).[1]

We say that X is T_1 if every singleton $\{x\} \subseteq X$ is closed.

Exercise: Suppose X is a topological space. Show that X **is** T_1 if and only if for every $x, y \in X$ and $x \neq y$, there is an open subset $U \subset X$ such that $x \in U$ and $y \notin U$. Compare to the definition of T_0.

Exercise: Suppose X is Alexandroff and T_1. Prove that X is discrete, i.e. every subset of X is open and closed. (Hint: in an Alexandroff space, an arbitrary union of closed subsets is…?)

Exercise: Show that a finite topological space has at least one closed point.

Lastly, we say that X is T_2 or **Hausdorff** if, for any $x, y \in X$ such that $x \neq y$, there are open sets $U, V \subset X$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$. Succinctly: distinct points have disjoint neighborhoods.

Exercise: Suppose that X is a topological space. Prove that X is Hausdorff if and only if the diagonal

$$\Delta_X := \{(x, x) \in X \times X \,|\, x \in X\}$$

is closed in the product topology on $X \times X$.

Since Alexandroff T_1 spaces are all discrete, the spaces that will be most interesting to us will be Alexandroff and at most T_0. Some terminology: we will call Alexandroff T_0 spaces A**-spaces** and finite T_0 spaces F**-spaces**.

3. Bases and Subbases

In this section, we will discuss how to define – or decompose – the open sets in a topology in terms of operations on "simple" open sets.

3.1. **Motivating Example.** Just as a vector space like $\mathbb{R}^3$ can be "built" out of

$$(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)$$

[1] the T comes from the German word for separation, "Trennung".

by taking sums of scalings ("linear combinations") of these vectors, so too can the topology on a space X be "built" out of simple open sets. We've already seen examples of this phenomenon: if (X, d) is a metric space, then a subset $U \subseteq X$ is open in the topology associated to d if and only if for every $x \in U$, there is $\varepsilon > 0$ (depending on x) such that

$$x \in B(x, \varepsilon) := \{y \in X \mid d(x, y) < \varepsilon\} \subseteq U.$$

In this case, it follows that every open subset $U \subseteq X$ can be expressed as a union of the open balls $B(x, \varepsilon)$, as x ranges over all points in X and ε ranges over all positive real numbers.

Exercise: Prove this claim.

Warning: while the expression

$$(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$$

of a vector in $\mathbb{R}^3$ is unique, the expression $U = \bigcup_{i \in I} B(x_i, \varepsilon_i)$ typically is not. The analogy is not perfect.

3.2. **Bases for a Topology.** With the above example in mind, suppose that X is a topological space and $\mathscr{B}$ is some set of open subsets of X. We do *not* assume that $\mathscr{B}$ contains every open subset of X. We say that $\mathscr{B}$ is a **basis** for the topology on X if every open subset $U \subseteq X$ is a union of elements in $\mathscr{B}$:

$$U = \bigcup_{i \in I} B_i$$

for some set I and $B_i \in \mathscr{B}$. By convention, the empty union ($I = \varnothing$) is taken to be $\varnothing$.

Exercise: Keeping the above notation, show that $\mathscr{B}$ is a basis for the topology on X if and only if the following property holds: for any open $U \subseteq X$ and $x \in U$, there exists $B \in \mathscr{B}$ (depending on x and U) such that $x \in B \subseteq U$.

Exercise: Specialize the above to the case of the topology induced by a metric on a set X.

The above exercise gives us a way of recognizing bases when we see them.

In the other direction, suppose that X is a set (without any given topology) and $\mathscr{B}$ is a set of subsets of X satisfying the following properties:

(1) for every $x \in X$, there is some $B \in \mathscr{B}$ such that $x \in B$.
(2) for every $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, there is some $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Then the set of all arbitrary (possibly empty) unions of sets in $\mathscr{B}$ is a topology on X, and $\mathscr{B}$ is a basis for that topology.

Exercise: Check this claim.

Exercise: Check that the set τ of all unions of sets in $\mathscr{B}$ is the weakest/coarsest topology on X that makes each $B \in \mathscr{B}$ open. In other words, show that if σ is any

topology on X and each $B \in \mathscr{B}$ is open in this topology, then each $U \in \tau$ is open as well. This means τ is the topology "generated" by the basis $\mathscr{B}$.

Exercise: Check that if the set τ is a topology for X, i.e. $\mathscr{B}$ is a basis for some topology on X, then $\mathscr{B}$ must satisfy properties (1) and (2) above. Thus, (1) and (2) are axioms for a basis of open sets on X.

3.3. **Subbases for a Topology.** Subbases provide another useful way to "decompose" the open subsets of a space into simpler subsets. As above, suppose first that X is a topological space and that $\mathscr{S}$ is some set of open subsets of X. Again, we do *not* assume $\mathscr{S}$ contains every open subset of X. We say that $\mathscr{S}$ is a **subbasis** for the topology on X if every open subset $U \subseteqq X$ can be expressed as a (possibly infinite) union of finite intersections of the elements of $\mathscr{S}$:

$$U = \bigcup_{i \in I} \left[\bigcap_{j=0}^{N_i} S_{i,j} \right]$$

for some index set I, natural numbers $N_i > 0$, and $S_{i,j} \in \mathscr{S}$. By convention, the empty union ($I = \varnothing$) is taken to be $\varnothing$.

Exercise: Suppose X and Y are topological spaces. Prove that the subsets of $X \times Y$ of the form $U \times Y$ and $X \times V$ for U open in X and V open in Y form a subbasis for the product topology on $X \times Y$.

Exercise: Suppose $\mathscr{S}$ is a set of open subsets in a topological space X and let $\mathscr{B}$ be the set of all finite intersections of elements of $\mathscr{S}$. Prove that $\mathscr{S}$ is a subbasis for the topology on X if and only if $\mathscr{B}$ is a basis for the topology on X.

As before, we can define topologies on a set in terms of subbases. Suppose X is a set (without any given topology) and that $\mathscr{S}$ is a set of subsets of X that cover X, i.e. $X = \bigcup_{S \in \mathscr{S}} S$. Then the set of all unions of finite intersections of sets in $\mathscr{S}$ is a topology on X, and $\mathscr{S}$ is a subbasis for this topology. For example, when we defined the mapping space $\mathrm{Map}(X, Y)$, we took the subsets

$$B(K, U) := \{ f : X \to Y \text{ continuous} \,|\, f(K) \subseteqq U \}$$

as a defining subbasis for the topology on $\mathrm{Map}(X, Y)$.

Exercise: Check that the set τ of all unions of finite intersections of sets in $\mathscr{S}$ is a topology on X.

Exercise: Check that the set τ is the weakest/coarsest topology on X that makes each $S \in \mathscr{S}$ open. In other words, show that if σ is any topology on X and each $S \in \mathscr{S}$ is open in this topology, then each $U \in \tau$ is open in this topology as well. This means τ is the topology "generated" by the subbasis $\mathscr{S}$.

4. The Equivalence Between A-Spaces and Posets

We shall now sketch how A-spaces and posets are essentially interchangeable objects. Fix a set X.

Suppose first that X has an Alexandroff T_0 topology τ. We shall construct a partial order $\leqq_\tau$ on X as follows: first, for each $x \in X$, define

$$U_x := \bigcap_{\substack{V \text{ open} \\ x \in V}} V.$$

Exercise: $\mathscr{B} = \{U_x \mid x \in X\}$ is a basis for the topology on X. Moreover, it is a minimal basis: if $\mathscr{C}$ is another basis for the topology on X, then $\mathscr{B} \subseteqq \mathscr{C}$.

Next, define the relation $\leqq_\tau$ on X by

$$x \leqq_\tau y \overset{\text{Def}}{\iff} U_x \subseteqq U_y.$$

Exercise: For a general Alexandroff space, $\leqq_\tau$ is a preorder on X. If X is T_0, i.e. an A-space, then $\leqq_\tau$ is a partial order.

So, A-space topologies τ on X give rise to partial orders $\leqq_\tau$ on X.

In the other direction, suppose that $\leqq$ is a partial order on X, and define a topology $\tau_\leqq$ as follows: first, take

$$U_x := \{y \in X \mid y \leqq x\}.$$

The sets U_x satisfy the axioms for a basis of open sets; let $\tau_\leqq$ be the topology generated by them. Said differently, the topology $\tau_\leqq$ has as its open sets the unions of the basic open sets U_x (as x ranges over all points of X).

Exercise: Check that $\tau_\leqq$ is an A-space topology on X.

So, partial orders on X give rise to A-space topologies on X. Moreover,

Exercise: Check that these two constructions are inverse: $\leqq_{\tau_\leqq} = \leqq$ and $\tau_{\leqq_\tau} = \tau$.

This means that a partial order on X is interchangeable with an A-space topology via the two constructions just outlined.

Importantly, there is also a way of relating the structure-preserving functions between posets and A-spaces. For A-spaces, these are just the continuous maps $f : X \to Y$. For posets $(X, \leqq)$, $(Y, \leqq)$ a structure-preserving map $f : (X, \leqq) \to (Y, \leqq)$ is a set function $f : X \to Y$ such that for every $x, x' \in X$, if $x \leqq x'$, then $f(x) \leqq f(x')$. Such maps are sometimes called **monotone** or **order-preserving**.

Proposition. *Suppose $(X, \leqq_X)$ and $(Y, \leqq_Y)$ are posets and $f : X \to Y$ is a set map. Then $f : (X, \leqq_X) \to (Y, \leqq_Y)$ is an order-preserving map if and only if $f : (X, \tau_{\leqq_X}) \to (Y, \tau_{\leqq_Y})$ is continuous.*

Proof. First, suppose that f is continuous. We must show that f is order preserving. Suppose $w \leqq x$ in X. Then $U_w \subseteqq U_x$, by definition. Since $U_{f(x)} \ni f(x)$ is open in Y, $f^{-1}[U_{f(x)}] \ni x$ is open in X, hence $U_x \subseteqq f^{-1}U_{f(x)}$. Therefore $w \in U_w \subseteqq f^{-1}U_{f(x)}$, therefore $f(w) \in U_{f(x)}$, therefore $U_{f(w)} \subseteqq U_{f(x)}$. Thus, $f(w) \leqq f(x)$.

Converse: exercise. □

5. Introductory Category Theory

At this point, we shall introduce some basic category theory to make sense of the equivalence just discussed (and some of the other constructions we've seen in this course).

There are (at least) two distinct ways of viewing categories. On the one hand, category theory can be viewed as supplying a language for describing general mathematical phenomena, and by extension, categories become organizational objects that collect relevant data in one place. On the other hand, categories can also be regarded as algebraic structures that can and should be studied in their own right. These perspectives are very different, and jumping between them can take some getting used to.

Let's start with some language. The terminology itself was borrowed from Kant.[2] A **category** $\mathscr{C}$ consists of the following data:

(1) a "collection"[3] of objects $\mathrm{Ob}(\mathscr{C})$
(2) for each pair of objects $X, Y \in \mathrm{Ob}(\mathscr{C})$, a set $\mathscr{C}(X,Y)$ thought of as the set of maps (also called **morphisms**) $f : X \to Y$.
(3) for each $X \in \mathrm{Ob}(\mathscr{C})$, a distinguished identity element $\mathrm{id}_X \in \mathscr{C}(X,X)$
(4) for all objects $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, a composition map

$$\circ = \circ_{XYZ} : \mathscr{C}(Y,Z) \times \mathscr{C}(X,Y) \to \mathscr{C}(X,Z)$$

such that the following axioms hold

Associativity: For any triple $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$: $h \circ (g \circ f) = (h \circ g) \circ f$.

Unit: For any $X \xrightarrow{f} Y$, $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$.

Here are some examples:

Name	Objects ($X, Y \in \mathrm{Ob}(\mathscr{C})$)	Morphisms ($\mathscr{C}(X,Y)$)
Set	Sets	functions $X \to Y$
Top	Topological Spaces	continuous maps $X \to Y$
Grp	Groups	group homomorphisms $X \to Y$
Ab	Abelian Groups	group homomorphisms $X \to Y$

The definition of a category is elementary, and examples abound. It is necessary, however, to define categories so that we can make sense of what a structure-preserving map between categories is. These are called **functors**. A functor $F : \mathscr{C} \to \mathscr{D}$ between the categories $\mathscr{C}$ and $\mathscr{D}$ consists of the following data:

(1) a function $F : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D})$,
(2) for each pair of objects $X, Y \in \mathrm{Ob}(\mathscr{C})$, a function

$$F = F_{XY} : \mathscr{C}(X,Y) \to \mathscr{D}(FX, FY)$$

such that the following axioms hold:

Preservation of $\circ$: for any pair of arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$F_{XZ}(g \circ f) = F_{YZ}(g) \circ F_{XY}(f).$$

Preservation of id's: for any $X \in \mathscr{C}$, $F_{XX}(\mathrm{id}_X) = \mathrm{id}_{FX}$.

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[2] see the introduction to Mac Lane's *Categories for the Working Mathematician*.

[3] This terminology is deliberately vague. We need to think about large "sets" in general.

One functor that we've studied in this course is the fundamental group. Define the category $\mathbf{Top}_*$ of pointed spaces to be the category whose objects are ordered pairs (X, x), where X is a topological space and $x \in X$, and whose morphisms $f : (X, x) \to (Y, y)$ are the continuous maps $f : X \to Y$ that satisfy $f(x) = y$. This is a category, and the fundamental group construction defines a functor

$$\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}.$$

$\pi_1(X, x)$ is the fundamental group of X based at x, and for any $f : (X, x) \to (Y, y)$, $\pi_1(f)$ is the induced map $f_* : \pi_1(X, x) \to (Y, y)$ sending $[\gamma] \mapsto [f \circ \gamma]$.

Thus, category theory gives us a language for describing the constructions of algebraic topology. But, again, categories are also objects of study in their own right. For example – a group can be regarded as a category.

To see this, first recall that a **monoid** is a set X equipped with a binary associative multiplication $\cdot : X \times X \to X$ and a two-sided identity element $e \in X$. In other words, a monoid is a group without inverses. A monoid is equivalent to a category with a single object, as follows: given a category $\mathscr{C}$ with one object $*$, composition $\circ$ and $\mathrm{id} \in \mathscr{C}(*, *)$ define an associative, unital multiplication on the set $\mathscr{C}(*, *)$. Thus, $\mathscr{C}(*, *)$ is a monoid. Conversely, given a monoid $(X, \cdot, e)$, define a category with a single object $*$ by setting $X(*, *) := X$, letting composition be the monoid multiplication $\cdot$, and letting $e = \mathrm{id}_* \in X(*, *)$.

Exercise: Check that these two constructions are "inverse up to isomorphism" on objects, i.e. doing one after the other yields an object isomorphic (but not necessarily equal) to the object you started with. Once we have a bit more language, we will make this situation more precise.

To describe groups we need one more definition. Given any category $\mathscr{C}$, a morphism $f : X \to Y$ in $\mathscr{C}$ is said to be an **isomorphism** (or **invertible**) if there is a morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In such a case, we usually write $g = f^{-1}$ and say that g is the inverse of f.

Exercise: Show that if g and h are both inverses of f, then $g = h$. More generally, show that if g is a left inverse for f ($g \circ f = \mathrm{id}_X$) and h is a right inverse for f ($f \circ h = \mathrm{id}_Y$), then $g = h$ and g is a two-sided inverse for f. This justifies the terminology "the inverse".

The difference between monoids and groups is that every element of a group must have a two-sided inverse. Given that we are identifying composition of morphisms $\circ$ with the monoid multiplication $\cdot$, this means that a group is equivalent to a category with one object *such that* every morphism has an inverse. More generally, a category $\mathscr{C}$ is called a **groupoid** if and only if every morphism of $\mathscr{C}$ is invertible.

Exercise: Suppose X is a topological space. Define a category ΠX as follows: let the objects of ΠX be the points of X, and for each pair of points $x, y \in \Pi X$, let $\Pi X(x, y)$ be the set of homotopy classes of paths $\gamma : x \to y$, where we require that homotopies fix endpoints. Define the identity $\mathrm{id}_x \in \Pi X(x, x)$ to be the class of the constant path at x. Show that ΠX is a groupoid under path concatenation.

It is called the **fundamental groupoid** of the space X.[4]

We see that categories are in some sense generalized monoids – or monoids with many objects – and therefore categories have a certain algebraic quality.

On the other hand, categories can also be regarded as generalized preorders – or preorders with many arrows – and therefore categories have a certain combinatorial quality. Specifically, we claim that a preorder is equivalent to a small category $\mathscr{C}$ such that for any $x, y \in \mathscr{C}$, there is at most one morphism $f : x \to y$. Here a category $\mathscr{C}$ is called **small** if $\mathrm{Ob}(\mathscr{C})$ is a set. This equivalence is effected as follows: if $\mathscr{C}$ is such a category, we can define a preorder on $\mathrm{Ob}(\mathscr{C})$ by taking

$$x \leqq y \overset{\text{Def}}{\iff} x \overset{\exists}{\to} y.$$

The identities in $\mathscr{C}$ guarantee $\leqq$ is reflexive, and transitivity follows from the existence of composition. In the other direction, given a preorder $(X, \leqq)$, we can define a small category $\mathscr{C}$ by taking $\mathrm{Ob}(\mathscr{C}) := X$ and putting in a single morphism $f : x \to y$ if and only if $x \leqq y$.

Exercise: Check that these constructions are "inverse up to isomorphism" on objects.

Under this correspondence, a preorder P (regarded as a small category) is a poset if and only if there are no non-identity isomorphisms in P.

Exercise: Show that a preorder category P is a poset if and only if for each pair of objects $x, y \in P$, the set $P(x, y) \cup P(y, x)$ has at most one element. Succinctly (and sneakily): a preorder is a poset if there is at most one morphism between any pair of objects in P; we must understand the "morphisms between x and y" as referring to morphisms in either direction.

To summarize what we've done so far: we saw that categories like **Top** could be regarded as organizational objects that gather all topological spaces together, and we also saw that categories generalize algebraic and combinatorial objects such as groups and posets. We introduced functors to describe how structure could pass between two categories, and in light of the algebraic/combinatorial character of categories, functors can also be regarded as generalized monoid homomorphisms or monotone maps. The next logical step is to organize all of this data... into a new category. Let **Cat** be the category of all (small) categories and functors between them.[5] We have a notion of isomorphism in **Cat**, namely, a functor with a two-sided inverse. In this new setting, the constructions we described between A-spaces (Alexandroff spaces) and posets (preorders) may be summarized as follows: the category of all A-spaces (Alexandroff spaces) is isomorphic to the category of all posets (preorders).

Exercise: Fill in the details of this last claim. What are the categories? What are the functors? Check they are inverse.

[4] See *Concise* for further discussion.

[5] With strong enough foundations, the size restriction can be improved.