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1. Remarks about combinatorics and homotopy theory

Today we will talk about translations from topology to algebra. What I like about this material is that you start out thinking like a topologist but then discover close and mysterious connections with combinatorics. Later I will convince you that there is a space X with $2n+2$ points and a map $S^n \to X$ that induces an isomorphism on all homotopy groups. Until you see the proof, that seems unbelievable. You think, "how could it possibly be true? It's just bizarre". Here is an open question of the same type:

Question 1.1 (Open)**.** *We shall see that reasonable compact spaces, such as $\mathbb{R}P^n$, can be modelled by finite spaces. How many points do you need to model $\mathbb{R}P^n$? That is, what is the minimal number of points for a finite space X of the same weak homotopy type as $\mathbb{R}P^n$?. Ideally, there should be a map $\mathbb{R}P^n \longrightarrow X$ that induces an isomorphism on homotopy groups, but there might only be a zigzag of weak homotopy equivalences connecting X to $\mathbb{R}P^n$. (See Definition 5.4 below.)*

Here is another strange question, a more philosophical one, which will make more sense once we get further into the material:

Question 1.2. *How much of combinatorics is homotopy invariant?*

However, before starting on that, we'll give some idea of what "thinking like a topologist", or at least like an algebraic topologist, means. We shall give an overview of the kind of arguments used to prove such results as the following one:

Theorem 1.3. *$\pi_3 S^2 \cong \mathbb{Z}$ and in general $\pi_q(S^2) \cong \pi_q(S^3)$ for $q \geq 3$.*

In particular, we will use special properties of the Hopf map $S^3 \xrightarrow{\eta} S^2$ which we defined last time. Before we do this, we need some tools from algebra.

2. Some tools from algebra

One of the most basic tools in all of algebra is to construct a long exact sequence which contains the groups you're interested in understanding. It will recover the idea of why, given a covering space $p : E \to B$ with fiber F, if you know the homotopy groups of E you can figure out the homotopy groups of B. This is all very vague, but by the end of today, it should make a little more sense.

2.1. **The quotient of a group.**

I will recall some basic concepts from group theory. If you are familiar with this, skip ahead. If you are not, you will not learn it in the next few paragraphs, but they will help you figure out what you need to learn to fill the gaps in your understanding.

Definition 2.1. Let G be a group and H be a subgroup, $H \leq G$. There is an equivalence relation on G given by

$$g_1 \sim g_2 \Leftrightarrow g_2^{-1} g_1 \in H.$$

The set of equivalence classes is denoted

$$G/H = \{[g] \mid g \in G\}$$

where

$$[g] = \{x \in G \mid x^{-1} g \in H\}.$$

If H is a normal subgroup, $H \trianglelefteq G$, then G/H is itself a group, with

$$[g_1][g_2] = [g_1 g_2].$$

We usually denote

$$gH := [g]$$

so as to not forget H and call it the H left coset of g. Further, G/H is called the set of *left cosets*. In conversation, mathematicians call G/H "G modulo H".

Example 2.2. One of the easiest examples. Take $G = \mathbb{Z}$. Take $H = 2\mathbb{Z}$, that is, the subgroup of even integers. Then $G/H = \mathbb{Z}/2$, the integers modulo 2. Another one, take

$$G = \mathbb{Z}^2 = \{(a, b) \mid a, b \in \mathbb{Z}\},$$

and

$$H = \{(a, 0) \mid a \in \mathbb{Z}\}.$$

What is G/H?

2.2. **Long exact sequences.**

Definition 2.3. Let A, B and C be groups. Suppose that there are maps $f : A \to B$ and $g : B \to C$. Suppose that $g \circ f = 0$ so that

$$\mathrm{im}(f) \subseteq \ker(g).$$

We write these maps in a sequence

$$A \xrightarrow{f} B \xrightarrow{g} C.$$

The *homology* of the sequence is the quotient

$$\ker(g)/\mathrm{im}(f).$$

We say that that above sequence is exact if

$$\ker(g) = \mathrm{im}(f)$$

so that its homology is zero.

Exercise 2.4. Prove that

$$B \to C \to 0$$

is exact if and only if $B \to C$ is surjective.

Exercise 2.5. Prove that

$$0 \to A \to B$$

is exact if and only if $A \to B$ is injective.

Definition 2.6. A short exact sequence is a sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

such that every three term subsequence is exact. That is, $A \to B$ is injective, $B \to C$ is surjective and $\mathrm{im}(f) = \ker(g)$. In this case, by the first isomorphism theorem,

$$C \cong B/A.$$

Definition 2.7. A *chain complex* of abelian groups is a sequence of abelian groups $\{A_n\}_{n\geq 0}$ with homomorphisms $d_n : A_{n+1} \to A_n$ such that the composite of two successive maps is zero, that is $d_n \circ d_{n+1} = 0$. The maps d_n are called the *differentials*.

Note 2.8. This remark is about mathematicians' laziness when it comes to chain complexes. We often write a chain complex as a tower:

$$\ldots \xrightarrow{d_{n+1}} A_{n+1} \xrightarrow{d_n} A_n \xrightarrow{d_{n-1}} A_{n-1} \to \ldots \to A_0$$

Further, we tend to be lazy and forget the indexing on the maps d_n. If we write $d : A_n \to A_{n-1}$, we mean d_{n-1} since that's the only index that makes sense! Therefore, we can write $d \circ d = 0$ and understand this to mean that the composite of successive differentials is zero.

We also write chain complexes in very compact notation as (A_*, d) or even just A_* or, still shorter, sometimes just A. The $*$ in A_* is thought of as varying over the natural numbers, and it is used in the notation A_* to remind us of the sequence of abelian groups A_n. The notation does not show the differential. So if one says "A_* is a chain complex", you can deduce that it comes with differentials $d : A_n \to A_{n-1}$.

Definition 2.9. A chain complex

$$\ldots \xrightarrow{d_{n+1}} A_{n+1} \xrightarrow{d_n} A_n \xrightarrow{d_{n-1}} A_{n-1} \to \ldots \to A_0$$

is exact if it is exact at every stage. That is,

$$A_{n+1} \to A_n \to A_{n-1}$$

is exact for every n. In this case, we call the chain complex a *long exact sequence*.

Definition 2.10. The homology of a chain complex is

$$H_n(A_*) = \ker(d_{n-1})/\,\mathrm{im}(d_n).$$

Note 2.11. We often gather all the homology groups and write $H_*(A_*)$. Here, the $*$ in A_* is part of the name of the chain complex and can even be omitted, but the $*$ in H_* is never omitted: we are thinking of the sequence of groups $H_n(A_*)$. We admit that the ideas can be confusing at first.

Exercise 2.12. A chain complex A_* is exact if and only if $H_n(A_*) = 0$ for all n. That is, the homology measures the deviation from exactness.

Example 2.13. Consider the sequence

$$0 \to \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

Here, we mean that

$$A_n = \begin{cases} \mathbb{Z}/2 & n = 1 \\ \mathbb{Z}/4 & n = 2 \\ \mathbb{Z}/4 & n = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Further, the differentials are the obvious ones, except that $d : \mathbb{Z}/4 \to \mathbb{Z}/4$ is multiplication by 2 and $\mathbb{Z}/4 \to \mathbb{Z}/2$ is the natural projection, that is, reduction modulo 2. Then, one computes that

$$H_n(A_*) = \begin{cases} \mathbb{Z}/2 & n = 3 \\ 0 & \text{otherwise.} \end{cases}$$

To understand why $H_2(A_*) = 0$, notice that the kernel at that stage is $2\mathbb{Z}/4$, but the image of the previous differential is exactly that. Therefore, the sequence is exact at that spot.

3. Covering Spaces and the long exact sequence on homotopy groups

Definition 3.1. For any map $p : E \to B$, the *fiber* of f at $b \in B$ is the set $f^{-1}(b) \subseteq E$. That is, the fiber at a point is the preimage of that point.

Recall that a cover, $p : E \to B$ is a continuous map such that there exists a discrete topological space F (just a disjoint union of points with no interesting topology on it) such that, for every $b \in B$, there exists an open set U in B such that $b \in U$ and $p^{-1}(U) \cong U \times F$.

The *covering space* E looks like a cartesian product locally, but globally it is not a cartesian product.

Exercise 3.2. If $p : E \to B$ is a covering space, then for any $b \in B$, the fiber of p at b is precisely F as above.

Remark 3.3. Suppose that $p : E \to B$ is a covering map of based topological spaces. Choose a point e as the base point of E and take $b = p(e)$ to be the base point of B. Since $p^{-1}(b) \cong F$ is a subset of E we get an inclusion

$$F = p^{-1}(b) \xrightarrow{i} E.$$

Therefore, given a cover, we often write it as a sequence

$$F \xrightarrow{i} E \xrightarrow{p} B.$$

When written like that, you think of F as the common fiber, but you are really fixing a choice of b and letting $F = p^{-1}(b)$. Despite the ambiguity coming from the choice, we often write covering spaces just as sequences $F \xrightarrow{i} E \xrightarrow{p} B$.

Lemma 3.4. *If F is a based discrete space, then $\pi_0 F \cong F$ and $\pi_q F = 0$ for $q \geq 1$.*

Proof. All maps $p : S^q \to F$ for $q \geq 1$ must send S^q to the base point of F, otherwise it would not be continuous. $\square$

Exercise 3.5. When is the map $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ given by $f(x) = x^n$ a cover? What is the fiber F? Does the answer change if you consider the map $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$? What if you keep the point $\{0\}$?

Theorem 3.6. *Suppose that E is path connected so that $\pi_0(E, e) = \{e\}$. Let $p : E \to B$ be a covering map, and suppose that $p(e) = b$. Let $i : F \to E$ be the inclusion $F = p^{-1}(b) \to E$. This gives a map on homotopy groups*

$$\pi_n(F, e) \xrightarrow{i_*} \pi_n(E, e).$$

We also have a map

$$\pi_n(E, e) \xrightarrow{p_*} \pi_n(B, b).$$

One can define certain maps

$$\pi_q(B, b) \xrightarrow{\partial} \pi_{q-1}(F, e).$$

With these maps, there is a long exact sequence

$$\cdots \xrightarrow{\partial} \pi_q(F, e) \xrightarrow{i_*} \pi_q(E, e) \xrightarrow{p_*} \pi_q(B, b) \xrightarrow{\partial} \pi_{q-1}(F, e) \xrightarrow{i_*} \cdots$$

$$\cdots \longrightarrow \pi_1(F, e) \xrightarrow{i_*} \pi_1(E, e) \xrightarrow{p_*} \pi_1(B, b) \xrightarrow{\partial} \pi_0(F, e) \xrightarrow{i_*} \pi_0(E, e) = \{e\}$$

Remark 3.7. Note that $\pi_0(F, e)$ is just a set, not a group. By saying that the sequence is exact at the last stage, we just mean that the map $\pi_1(B, b) \to \pi_0(F, e)$ is a surjective map of sets and that the image of $\pi_1(E, e) \to \pi_1(B, b)$ maps to $e \in \pi_0(F, e)$.

Exercise 3.8. Realize that surjectivity and injectivity of a group homomorphism is tested on the underlying sets. That is, a group homomorphism $f : G \to H$ is surjective if and only if it is a surjection as a map of sets (forget all the good properties coming from the fact that it's a group homomorphism). The same holds for injectivity.

Note 3.9. We will not define ∂. However, we want you to realize that just knowing that it exists and fits into the long exact sequence above is a powerful tool. We will see this through the next examples.

Theorem 3.10. *If $p : E \to B$ is a covering space, then $\pi_q(E, e) \cong \pi_q(B, b)$ if $q \geq 2$. Further, there is an exact sequence*

$$0 \to \pi_1 E \to \pi_1 B \to \pi_0 F \to 0.$$

Proof. Since F is a discrete topological space, $\pi_q(F, e) = 0$ for all $q \geq 1$. Therefore, if $q \geq 1$ the long exact sequence on homotopy groups of Theorem 3.6 breaks up into short exact sequences:

$$0 = \pi_{q+1}(F, e) \to \pi_{q+1}(E, e) \to \pi_{q+1}(B, b) \to \pi_q(F, e) = 0.$$

Hence, $\pi_{q+1}(E, e) \cong \pi_{q+1}(B, b)$ when $q \geq 1$. Now, if $q = 0$, all we get is

$$0 = \pi_1(F, e) \to \pi_1(E, e) \to \pi_1(B, b) \to \pi_0(F, e) \to 0.$$

$\square$

Corollary 3.11. *$\pi_q(S^1) = 0$ for $q \geq 2$.*

Proof. We saw that there is a covering map $p : \mathbb{R} \to S^1$ with fiber $F = \mathbb{Z}$. Beware, here F is just a set, not a group. Nonetheless, it follows from the previous result that, for $q \geq 2$, $\pi_q(S^1) \cong \pi_q(\mathbb{R})$. But $\mathbb{R}$ is contractible, so $\pi_q(\mathbb{R}) = 0$. $\square$

Remark 3.12. Note that we also get an exact sequence

$$0 \to \pi_1(\mathbb{R}) \to \pi_1(S^1) \to \pi_0(F) \to 0$$

This gives us that $\pi_1(S^1) \cong \pi_0(F)$ as sets.

Lemma 3.13. *If $p : E \to B$ is a covering space, then $\pi_q(E) \cong \pi_q(B)$ if $q \geq 2$. Further, there is an exact sequence*

$$0 \to \pi_1 E \to \pi_1 B \to \pi_0 F \to 0$$

Exercise 3.14. Use the long exact on homotopy groups to prove that $\pi_1 \mathbb{R}P^2 \cong \mathbb{Z}/2$ as groups. You may have to go back and review the definition of $\mathbb{R}P^2$ to do this exercise.

4. Fiber Bundles

There is another kind of maps that has the same nice properties as the covering maps. These are called *fiber bundles.*

Definition 4.1. A fiber bundle $p : E \to B$ with fiber F is a continuous map such that, for every $b \in B$, there exists U open in B with $b \in U$ such that

$$p^{-1}(b) \cong U \times F.$$

The whole idea, again, is that locally the space E in a fiber bundle $p : E \to B$ looks like it's a cartesian product, but globally, E is not just a product.

Example 4.2. The Hopf map

$$\eta : S^3 \to S^2$$

which we described last time is a fiber bundle. Recall how we defined it. We considered

$$S^3 \subseteq \mathbb{C}^2 = \mathbb{R}^4.$$

We put an equivalence relation on S^3 by letting

$$(z_1, z_2) \sim \lambda(z_1, z_2)$$

for $\lambda \in \mathbb{C}$, $\lambda \neq 0$. But note that if this is the case, $||\lambda|| = 1$, so that

$$\lambda \in S^1 \subseteq \mathbb{C} = \mathbb{R}^2.$$

Hence, the fiber over a point $b = (z_1, z_2) \in S^2 = S^3/(\sim)$ is a copy of S^1. Indeed, if

$$\eta((z_1, z_2)) = \eta((w_1, w_2)),$$

then there exists $\lambda \in S^1$ such that

$$(z_1, z_2) \sim \lambda(w_1, w_2).$$

Further, for any $\lambda \in S^1$, we get a different element of the fiber $\eta^{-1}(b)$, so that

$$\eta^{-1}(b) \cong S^1.$$

For this reason, we write the Hopf fibration as a sequence:

$$S^1 \to S^3 \xrightarrow{\eta} S^2$$

Exercise 4.3. Split S^2 into two open hemisphere D such that $\eta^{-1}(D) \cong D \times S^1$. (To illustrate what we mean, one choice of such hemispheres is given by a "fat" upper hemisphere D_u and a "fat" lower hemisphere D_l. That is, if

$$S^2 = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\},$$

we let

$$D_u = \{(x_1, x_2, x_3) \in S^2 \mid x_1 > -0.5\}$$

and

$$D_l = \{(x_1, x_2, x_3) \in S^2 \mid x_1 < 0.5\}.$$

Now, note that

$$S^2 = D_u \cup D_l$$

and that these are both open sets.)

Exercise 4.4. Note that there is a strong analogy between the facts that $\mathbb{R}P^1 \simeq S^1$ and $\mathbb{C}P^2 \simeq S^2$. Understand the relationship between these facts.

Exercise 4.5. Prove that the Möbius band is a fiber bundle over S^1. What is the fiber F?

Theorem 4.6. *Given a fiber bundle, there is a long exact sequence on homotopy groups. That is, Theorem 3.6 holds word for word.*

Theorem 4.7. $\pi_2(S^2) \cong \mathbb{Z}$ *and* $\pi_q(S^3) \cong \pi_q(S^2)$ *for* $q \geq 3$

Proof. The Hopf fibration $S^1 \to S^3 \to S^2$ is a fiber bundle. Therefore, there is a long exact sequence on homotopy groups. Since $\pi_q(S^1) = 0$ for $q \geq 2$, for $q \geq 2$ we have exact sequences

$$0 = \pi_{q+1}(S^1) \to \pi_{q+1}(S^3) \to \pi_{q+1}(S^2) \to \pi_q(S^1) = 0$$

Therefore, $\pi_q(S^2) \cong \pi_q(S^3)$ for $q \geq 3$. Now, $\pi_1(S^3) = \pi_2(S^3) = 0$ since $\pi_q S^n = 0$ when $q < n$, so we also have an exact sequence

$$0 = \pi_2(S^1) \to \pi_2(S^3) = 0 \to \pi_2(S^2) \to \pi_1(S^1) \to \pi_1(S^3) = 0$$

This gives $\pi_2(S_2) \cong \pi_1 S^1 \cong \mathbb{Z}$. $\square$

We won't give the proof, but in fact $\pi_3(S^3) \cong \pi_3(S^2) \cong \mathbb{Z}$.

Exercise 4.8. There are two other "Hopf bundles"

$$S^3 \to S^7 \to S^4 \ \ \text{and} S^7 \to S^{15} \to S^8$$

Deduce all you can about the homotopy groups of these spheres from the associated long exact sequences on homotopy groups.

We won't prove this theorem; it dates from the 1950's but is still amazing:

Theorem 4.9 (Serre)**.** $\pi_n S^n = \mathbb{Z}$, $\pi_{4n-1} S^{2n} = \mathbb{Z} \oplus$ *finite and* $\pi_q(S^n)$ *is finite otherwise.*

5. Where we are going

We will make clear how to translate concepts about homotopy into statements about partially ordered sets, and hence, into combinatorics. We will do this through the study of Alexandroff spaces. Let me wave my hands at where we are going. I actually do have some kind of a plan.

Definition 5.1. A set $(X, \leq)$ is a preorder if it has a relation $\leq$ that is

- reflexive: $x \leq x$
- transitive: if $x \leq y$ and $y \leq z$, then $x \leq z$

We say that $(X, \leq)$ is a partial order or poset the relation $\leq$ is also

- anti-symmetric: if $x \leq y$ and $y \leq x$, then $x = y$

There are separation axioms on topological spaces and we need one now.

Definition 5.2. A space is T_0 if for points $x \in U$ if and only if $y \in U$ for all open U, then $x = y$. (T comes from the german word *Trennung*, which means seperation)

Remember that an Alexandroff space is a topological space such that any intersection of open sets is open. We will prove the following theorem next time.

Theorem 5.3. *There is a topology on a preorder $(X, \leq)$ which makes it an Alexandroff space. Conversely, given an Alexandroff topology on X, there is an associated preorder. That is, Alexandroff spaces and preorders are exactly the same notion. Similarly, posets and Alexandroff spaces which are T_0 are exactly the same notion.*

Classically, up to the thirties, a good space was a simplicial complex, something built out of simplices. We will see how to build these and how to go from Alexandroff spaces to simplicial complexes. We will also see how to go backwards, and see what kind of information these correspondences preserve. There are some really strange phenonema. Under this correspondence, one can ask to what notion the notions of homotopy equivalence and weak homotopy equivalence correspond.

There is a notion in simplicial complexes of *simple homotopy equivalences*. These are homotopy equivalences which are obtained by collapsing simplices (faces, segments, etc). Weak homotopy equivalences between Alexandroff spaces will correspond to actual homotopy equivalences between their asssociated simplicial complexes, whereas actual homotopy equivalences between Alexandroff spaces will correspond to simple homotopy equivalences between their associated simplical complexes.

There's an old problem in combinatorics that goes as follows. Consider spaces with at most n points. You can count how many such spaces there are, and in particular, how many there are which are non-homeomorphic T_0-spaces. Here's a table with some of those numbers; all cases $n \leq 10$ are tabulated in the book in progress on the web page.

n=1	1
n=2	2
n=3	5
n=4	16
...	...
n=10	2,567,284

You can also ask how many non-homotopy equivalent finite T_0-spaces there are. Two undergraduate participants proved in their REU project here that, asymptotically, the answer is the same as the number of non-homeomorphic finite T_0-spaces.

Definition 5.4. Two spaces have the same *weak homotopy type*, or are *weakly homotopy equivalent*, if there is a sequence of maps, or a zizag,

$$\begin{tikzcd}
 & X_2 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & X_{n-1} \arrow[dl] \arrow[dr] & \\
X_1 & & X_3 & & \cdots & & X_{n-2} & & X_n
\end{tikzcd}$$

where each map induces an isomorphism on homotopy groups (and we are sloppy about whether n might be even or odd). This means that X_1 and X_n have the same homotopy groups, but that there might not be a direct weak homotopy equivalence between them.

We shall make the following question more precise later.

Question 5.5 (Open)**.** *Explain combinatorially what it means for two finite T_0-spacse to have the same weak homotopy type.*

As a final remark, it has recently been proved that "anything you can do with algebraic topology, you can do with posets". A paper generalizing even that remarkable statement (to the algebraic topology of spaces with actions by some given discrete group G) has been proven just this spring, at Chicago, by J.P.M, Marc Stephan, and Inna Zakharevich.