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"There are so many things I want to tell you and there is so little time. Usually in undergraduate courses you expect to see a proof of everything stated. I am not going to do that. I want to show ideas and be clear enough that you can later explore them. The main emphasis will be towards combinatorics and the connection with Alexandroff spaces."

1. Quotient Topology

Let $f : X \to Y$. Suppose that X is a topological space. We can give a topology to Y by letting the open sets be:

$$\mathcal{T}_Y = \{U \subset Y \mid f^{-1}(U) \text{ is open in } X\}$$

Exercise 1.1. *With this topology on Y, the map $f : X \to Y$ is continuous.*

One of the most common examples of this construction is the *quotient topology.*

Definition 1.2. Given a topological space X and an equivalence $\sim$ relation on X, *the quotient of X by $\sim$* is the set of equivalences classes

$$Y = \{[x] \mid x \in X\}.$$

Here,

$$[x] = \{a \in X \mid a \sim x\}.$$

There is a natural map

$$p : X \to Y,$$

where

$$p(x) = [x].$$

We give Y the quotient topology, so that U is open in Y if and only if $p^{-1}(U)$ is open in X. A typical name for Y is

$$Y = X/(\sim),$$

to suggest that Y is the quotient of X by the relation $\sim$.

Example 1.3. One can construct the torus as a quotient of the square. Let

$$I^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}.$$

We take the smallest equivalence relation on I^2 such that, for all $0 \leq x_1 \leq 1$,

$$(x_1, 0) \sim (x_1, 1)$$

and, for all $0 \leq x_2 \leq 1$,

$$(0, x_2) \sim (1, x_2).$$

Then

$$I^2/(\sim) = T$$

where T is a torus. Note that any two points which are equivalent under $\sim$ are glued together.

Exercise 1.4. *Explore the open sets on the torus.*

Exercise 1.5. *Construct the Möbius strip and a cylinder as a the quotient of*

$$I^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}.$$

Now construct something homeomorphic to a two sphere S^2 as the quotient of I^2.

Example 1.6. If X is topological space and $A \subset X$, then X/A usually denotes the quotient of X by the relation that

$$a \sim a'$$

for all $a, a' \in A$. That is, we collapse all of the points in A to a single point.

2. Gluing spaces and pushout diagrams

Another example is a *pushout*:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g} & & \downarrow \\ Y & \longrightarrow & ? \end{array}$$

We want to fill the bottom corner with the most natural space which would make the diagram commute.

Remark 2.1. *A* diagram commutes *if, when you pick a point and you go around the diagram via the maps, it does not matter which path you take, the output will be the same any way you choose.*

The way to do this, is to glue X and Y along the image of A:

$$X \cup_A Y = X \sqcup Y/(f(a) \sim g(a)).$$

That is, $X \cup_A Y$ is the quotient of the disjoint union $X \sqcup Y$ by the relation

$$f(a) \sim g(a)$$

for $a \in A$. There are natural maps

$$i_X : X \to X \cup_A Y$$

and

$$i_Y : X \to X \cup_A Y.$$

We can describe i_X as the composite of the inclusion $i : X \to X \sqcup Y$ followed by the quotient map $p : X \sqcup Y \to X \cup_A Y$,

$$i_X : X \xrightarrow{i} X \sqcup Y \xrightarrow{p} X \cup_A Y$$

There is a similar description for i_Y.

Exercise 2.2. *The diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle i_X} \\ Y & \xrightarrow{i_Y} & X \cup_A Y \end{array}$$

commutes.

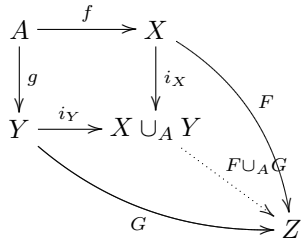
This kind of diagram is called a *pushout diagram*. It has the following *universal property*. If there is a topological space Z together with maps $F : X \to Z$ and $G : Y \to Z$ such that

$$F(f(a)) = G(g(a)),$$

the we can create a map

$$F \cup_A G : X \cup_A Y \to Z$$

by just gluing the maps F and G. This is expressed in the following commutative diagram:



Exercise 2.3. *Try to understand how this gives you the right construction for gluing topological spaces together. Use this construction to glue two copies of the disc $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ in such a way that the resulting space is the two sphere S^2.*

Exercise 2.4. *Can you generalize the construction to one which allows you to glue three spaces together? What about an arbitrary number of spaces?*

3. Homotopy Groups

Last time, we gave a cryptic definition:

$$\pi_n(X) := \pi_0 \Omega^n X,$$

where $\Omega^n(X) = \Omega(\Omega^{n-1}(X))$, $\Omega^0(X) = X$ and $\Omega(X) = \mathrm{Map}_*(S^1, X)$.

Recall that $\mathrm{Map}(X, Y)$ is the topological space whose points are continuous functions from X to Y, and whose open sets are described as follows. Fix K a compact set in X and U an open set in Y. Then

$$C(K, U) \subset \mathrm{Map}(X, Y),$$

is the subset

$$C(K, U) = \{f \mid f(K) \subset U\}$$

The open sets in Y are arbitrary unions and finite intersections of sets of this form.

Exercise 3.1. *If X is compact and Y is a metric space, then this is the same as the metric topology on* $\mathrm{Map}(X,Y)$. *Recall that the metric on* $\mathrm{Map}(X,Y)$ *is given by:*

$$d(f,g) = \sup\{d(f(x),g(x)) \mid x \in X\}$$

Lemma 3.2. *As topological spaces,*

$$\mathrm{Map}(X,\mathrm{Map}(Y,Z)) \cong \mathrm{Map}(X \times Y, Z)$$

Proof. If we ignore the topology, this bijection of sets is trivial, indeed,

$$f(x)(y) = g(x,y). \tag{3.3}$$

The topology has been arranged to make this a homeomorphism. □

We are interested in based maps, that is, maps which preserve chosen basepoints. Then how does this change the previous result? That is, what topological space P can replace the cartesian product $X \times Y$ to make the following true:

$$\mathrm{Map}_*(X,\mathrm{Map}_*(Y,Z)) \cong \mathrm{Map}_*(P,Z)$$

Exercise 3.4. *Consider a map $f : X \to \mathrm{Map}_*(Y,Z)$. Since*

$$\mathrm{Map}_*(Y,Z) \subset \mathrm{Map}(Y,Z),$$

f gives a map from X to $\mathrm{Map}(Y,Z)$. *Consider the map $g : X \times Y \to Z$ corresponding to f as in (3.3). Which points $(x,y) \in X \times Y$ are forced to satisfy $g(x,y) = *_Z$ by the fact that f is based point preserving?*

Definition 3.5. We define the *wedge* of X and Y,

$$X \vee Y = X \sqcup Y / (*_X \sim *_Y).$$

The wedge sits inside the cartesian product,

$$X \vee Y \subset X \times Y$$

as

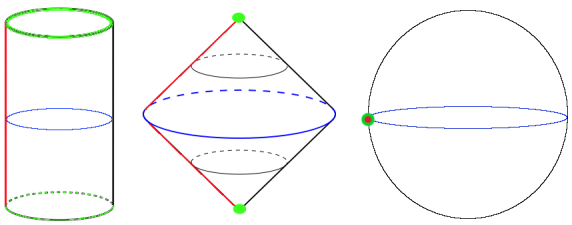
$$X \times \{*_Y\} \cup \{*_X\} \times Y.$$

Definition 3.6. The *smash* product is

$$X \wedge Y = X \times Y / (X \vee Y).$$

Definition 3.7. The *suspension* is $X \wedge S^1$, or

$$\Sigma X := \frac{[0,1] \times X}{(\{0\} \times X) \cup (\{1\} \times X) \cup ([0,1] \times *_X)}$$



Exercise 3.8. *From the depiction of $S^1 \times S^1$ as a torus, convince yourself that*

$$S^1 \times S^1/(S^1 \vee S^1) \simeq S^2.$$

Lemma 3.9. *There is a homeomorphism*

$$\mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z)) \cong \mathrm{Map}_*(X \wedge Y, Z).$$

Proof. If you do Exercise 3.4, you will at least understand why there is a bijection. □

Recall that by definition,

$$\Omega(X) = \mathrm{Map}_*(S^1, X).$$

Exercise 3.10.

$$S^n \wedge S^m \cong S^{n+m}.$$

Lemma 3.11.

$$\Omega^n(X) \cong \mathrm{Map}_*(S^n, X)$$

Proof. Exercise. □

For now on, we assume that our spaces are nice enough that path components agree with connected components. Let's look at

$$\pi_0 \mathrm{Map}_*(X, Y)$$

Let K_+ be the union of a space K and a disjoint basepoint and let

$$I_+ \xrightarrow{h} \mathrm{Map}_*(X, Y)$$

be such that $h(0) = f : X \to Y$ and $h(1) = g : X \to Y$. Then this is the same as an element

$$h \in \mathrm{Map}_*(I_+ \wedge X, Y).$$

What is $I_+ \wedge X$?

$$I_+ \wedge X = I \times X/(I \times \{*_X\})$$

So h is a homotopy between f and g which at all times t preserves the base point.

Definition 3.12. Define $[X, Y]$ to be the set of homotopy classes of basepoint preserving maps, where the homotopies are basepoint preserving in this sense.

Exercise 3.13.

$$[X, Y] = \pi_0 \mathrm{Map}_*(X, Y)$$

Example 3.14.

$$[S^1, X] = \pi_0 \mathrm{Map}_*(S^1, *_X) = \pi_1(X, *_X)$$

Exercise 3.15.

$$\begin{aligned} \pi_n(X) &= \pi_0 \Omega^n(X) \\ &= [S^n, X] \\ &= [S^1, \Omega^{n-1} X] \\ &= \pi_1(\Omega^{n-1} X) \end{aligned}$$

So this is a group. We will show it is commutative when $n > 1$ in a second.

Example 3.16. Let's look at $\pi_2(X) = [S^2, X]$. We think of $S^2 = I^2/\partial I^2$, where ∂I^2 is the boundary of the square I^2. A collection of pictures convinced us that the composition of maps is commutative. The pictures we used are an example of the *Eckman-Hilton trick*.

Lemma 3.17. *$\pi_n X$ is abelian for all $n \geq 2$.*

Remark 3.18. *Historically, in the development of algebraic topology, one needed a language to talk about the new concepts that we see. One* should *see examples before one learns the language, and this is what we are doing. However, the language that was developed to do algebraic topology is now used in all of the other fields, except perhaps analysis. This is the language of category theory, which was developed by Eilenberg and MacLane (in* General theory of natural equivalences*). We will see illustrations of that language throughout the course.*

4. Covering Spaces

Given a base point preserving map $p : X \to Y$, we get a map

$$p_* : [S^n, X] \to [S^n, Y]$$

defined by

$$p_*[f] = [p \circ f]$$

Further, $q_* \circ p_* = (q \circ p)_*$ and $id_* = id$ Consider the natural inclusion

$$i : * \to X$$

and suppose that the unique map

$$r : X \to *$$

is a homotopy equivalence. That is, $i \circ r \simeq \mathrm{id}$. Then

$$\pi_n(X) \xrightarrow{r_*} \pi_n(*) \xrightarrow{i_*} \pi_n(X)$$

is the identity since

$$i_* \circ r_* = (i \circ r)_* = (id)_* = id.$$

This gives the following result.

Lemma 4.1. *If X is contractible, $\pi_n(X) = 0$ for all $n \geq 0$.*

Lemma 4.2. *$\pi_1(S^1, 1) \cong \mathbb{Z}$ and $\pi_q(S^1) = 0$ for $q \geq 2$.*

Proof. This is a covering space argument, $p : \mathbb{R} \to S^1$. Just like we lifted paths, we can lift maps from spheres S^q so that $\pi_q(S^1) \cong \pi_q(\mathbb{R})$ for $q \geq 2$. Since $\mathbb{R}$ is contractible, $\pi_q(\mathbb{R}) = 0$ for all q. □

Definition 4.3. We say that a surjective map $p : E \to B$ is a *cover* if for small enough $U \subset B$, then $p^{-1}(U) \cong U \times F$ where F is discrete space. (That is, for every $b \in B$, there exists an open set U such that $b \in U$ and $p^{-1}(U) \cong U \times F$.)

Lemma 4.4. *For any cover $p : E \to B$, $\pi_q(E, e) \cong \pi_q(B, p(e))$ for $q \geq 2$.*

Example 4.5. Real projective space or $\mathbb{R}P^n$ is

$$\mathbb{R}P^n = S^n/(\sim)$$

where $x \sim y$ if x and y are antipodal points on S^n. The map $p : S^n \to \mathbb{R}P^n$ is a cover. A trivial example is

$$\mathbb{R}P^1 \simeq S^1.$$

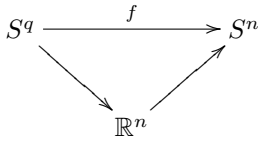
A more interesting example is $p : S^2 \to \mathbb{R}P^2$.

Exercise 4.6. *Prove that $p : S^2 \to \mathbb{R}P^2$ is a cover.*

Example 4.7. $\pi_q(\mathbb{R}P^n) = \pi_q(S^n)$ if $q \geq 2$. In fact, $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$.

Lemma 4.8. $\pi_q(S^n) = 0$ *for* $q < n$

Proof. If $q < n$, a map $f : S^q \to S^n$ is homotopic to a map which misses a point p. We thus can get a factorization



(here, we take the model of $\mathbb{R}^n$ which is $S^n \backslash \{p\}$). Since any map in $S^q \to \mathbb{R}^n$ is null-homotopic, so is $f : S^q \to S^n$. □

Lemma 4.9.

$$\pi_n S^n = \mathbb{Z}$$

Proof. There is no really simple argument. One quick sketch is to construct a map $\Sigma^n : \pi_1 S^1 \to \pi_n S^n$. Given $f : S^1 \to S^1$, we can "supsend" it to a map $\Sigma f : \Sigma S^1 \to \Sigma S^1$, that is, $\Sigma f : S^2 \to S^2$. Iterating this process, we get

$$\Sigma^n f : S^n \to S^n.$$

The claim is that $\Sigma^n : \pi_1 S^1 \to \pi_n S^n$ is an isomorphism. □

Exercise 4.10. *What do we mean above by "suspending maps"? Define the homomorphism $\pi_1 S^1 \to \pi_n S^n$.*

5. The Hopf Fibration

The problem of computing $\pi_q S^n$ is very hard. Here is an example.

Lemma 5.1. $\pi_3 S^2 \cong \mathbb{Z}$. *A generator $h : S^3 \to S^2$ is called the* Hopf fibration.

Definition 5.2. We describe the Hopf fibration. Consider $\mathbb{C}^2 \simeq \mathbb{R}^4$. The unit sphere in $\mathbb{C}^2$ is thus S^3. Let x and y be points on S^3. If $x = \lambda y$ where λ is in the unit circle $S^1 \subset \mathbb{C}$, then we let $x \sim y$. We quotient S^3 by this relation,

$$S^3/(\sim)$$

You can check that $S^3/(\sim) \simeq S^2$. This gives a map

$$\eta : S^3 \to S^3/(\sim) \simeq S^2$$

Check out: http://nilesjohnson.net/hopf.html for a cool video of this map.

Definition 5.3. Complex projective space is defined as follows. Given coordinates $(z_0, \ldots, z_n)$ in $\mathbb{C}^{n+1}$. We let $(z_0, \ldots, z_n) \sim \lambda(z_0, \ldots, z_n)$ for $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then

$$\mathbb{C}P^n = \mathbb{C}^{n+1}/(\sim)$$

Exercise 5.4. *Make sense of*

$$\mathbb{C}P^n \simeq S^{2n+1}/S^1$$