**06-25-2015**

Contents


1. More on the Fundamental Group 1
1.1. Basepoint Dependence 1
1.2. Induced Maps Between Fundamental Groups 2
2. The Fundamental Group of a Circle 2
2.1. Path and Homotopy Lifting 3
2.2. The Endpoint Map 3
2.3. The Degree of a Map $S^1 \to S^1$ 4
2.4. The Brouwer Fixed Point Theorem 4
3. Connectedness and $\pi_0(X)$ 4
4. A "Geodesic" Definition of $\pi_n(X)$ 5
4.1. Mapping Spaces and the Compact-Open Topology 5
4.2. Higher Homotopy Groups 6


## 1. More on the Fundamental Group

Recall that for a topological space $X$ and a point $x \in X$, the **fundamental group of $X$ based at $x$**, denoted $\pi_1(X, x)$, is the group of homotopy classes of paths $f : I \to X$ such that $f(0) = f(1) = x$ (loops at $x$). Given a path $f : I \to X$, denote the class of $f$ by $[f]$. Multiplication is given by concatenation, i.e. for loops $f, g : I \rightrightarrows X$,

$$(g \cdot f)(t) := \begin{cases} f(2t) & 0 \leqq t \leqq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leqq t \leqq 1 \end{cases}$$

and $[g] \cdot [f] := [g \cdot f]$.

"I shall feel free to skip all proofs that I find boring."

### 1.1. Basepoint Dependence.

The basepoint $x \in X$ in the definition of $\pi_1(X, x)$ is often immaterial. Suppose $x, y \in X$ and $j : x \to y$ is a path. Then we can define an isomorphism $j_* : \pi_1(X, x) \to \pi_1(X, y)$ between the fundamental groups based at $x$ and $y$ by $j_*[f] := [j][f][j^{-1}]$.

**Exercise:** Check that this is a group homomorphism with inverse $(j^{-1})_*$.

Thus, if $X$ is a topological space and $A \subseteqq X$ is a subspace such that any two $x, y \in A$ can be connected with a path, then the fundamental groups $\pi_1(X, a)$ for $a \in A$ are all isomorphic. Note, however, that the isomorphism $j_*$ induced by a path $j : x \to y$ may depend on the choice of (homotopy class) of the path $j$.

---

*Date*: June 29, 2015.

**Exercise:** Check that if $\pi_1(X,x)$ is abelian and $j,k : x \rightrightarrows y$ are any two paths in $X$ from $x$ to $y$, then $j_* = k_*$.

1.2. **Induced Maps Between Fundamental Groups.** First, some notation. $(X,x)$ shall mean that $X$ is a topological space and that $x \in X$. $p : (X,x) \to (Y,y)$ shall mean $p$ is a continuous map $X \to Y$ that sends $x$ to $y$ (i.e. $p(x) = y$).

Suppose that $p : (X,x) \to (Y,y)$. There is an induced group homomorphism $p_* : \pi_1(X,x) \to \pi_1(Y,y)$ defined by $p_*[f] := [p \circ f]$. Observe

$$I \xrightarrow{f} X \xrightarrow{p} Y,$$

so taking $p \circ f$ pushes $f$ forward to a loop in $Y$.

**Exercise:** Check that

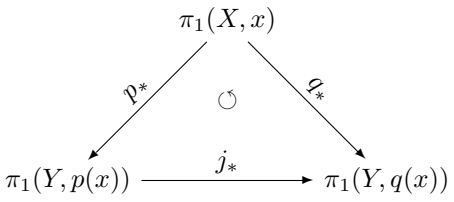
(1) $p_*$ is a group homomorphism.
(2) $(\mathrm{id}_X)_*$ is the identity homomorphism of $\pi_1(X,x)$.
(3) If $(X,x) \xrightarrow{f} (Y,y) \xrightarrow{g} (Z,z)$, then $g_* \circ f_* = (g \circ f)_*$.

Suppose that $p,q : X \to Y$ are continuous maps and that $h : p \Rightarrow q$ is a homotopy from $p$ to $q$. We would like to say that $p$ and $q$ induce the same group homomorphisms $p_*$ and $q_*$ out of $\pi_1(X,x)$, but this doesn't quite make sense since the codomain of $p_*$ is $\pi_1(Y,p(x))$, while the codomain of $q_*$ is $\pi_1(Y,q(x))$, which are not necessarily equal. Fortunately, the next best thing happens.

Let $x \in X$ be a chosen basepoint. Fixing $x$, the homotopy $h$ defines a path

$$j(t) := h(x,t) : I \to Y$$

from $p(x)$ to $q(x)$. This yields an isomorphism $j_* : \pi_1(Y,p(x)) \xrightarrow{\cong} \pi_1(Y,q(x))$, and we have $j_* \circ p_* = q_*$, i.e. the triangle below commutes:




**Exercise:** Prove this.

## 2. The Fundamental Group of a Circle

Let $S^1 \in \mathbb{C}$ be the set of unit length complex numbers. It is a circle. For each $n \in \mathbb{Z}$, define $f_n(t) := e^{2\pi i n t}$.

**Exercise:** Check that $[f_m] \cdot [f_n] = [f_{m+n}]$.

**Theorem.** *Equip $\mathbb{Z}$ with its additive group structure. The map $i : \mathbb{Z} \xrightarrow{\cong} \pi_1(S^1,1)$ defined by $i(n) := [f_n]$ is an isomorphism of groups.*

The standard proof of this fact uses covering space theory. A map $p : E \to B$ is said to be a **covering map** if each point $y \in B$ has an open neighborhood $U \ni y$ such that the inverse image $p^{-1}U$ can be written as a disjoint union $\coprod_{i \in I} V_i \subseteq X$

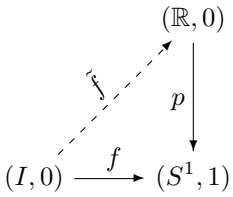
of open sets $V_i$ such that for each $i$, $p$ restricts to a homeomorphism $p| : V_i \xrightarrow{\cong} U$. One sometimes says $U$ is **evenly covered** by $p$, and that the $V_i$ are **slices** of $U$. The intuition is that we regard $E$ as living "above" $B$ via $p$, and that each evenly covered neighborhood $U$ has a stack of ("pancakes") copies of $U$ living above it.

In particular, we have the covering map

$$p(r) = e^{2\pi ir} : (\mathbb{R}, 0) \to (S^1, 1).$$

**Exercise:** Check this is a covering map.

2.1. **Path and Homotopy Lifting.** The important thing about covering maps for us is that given any $f : (I, 0) \to (S^1, 1)$, there is a unique **lift** $\tilde{f} : (I, 0) \to (\mathbb{R}, 0)$ of $f$ such that $p \circ \tilde{f} = f$ and $\tilde{f}(0) = 0$:




For example, $f_n(t) = e^{2\pi int}$ lifts (uniquely) to $\tilde{f_n}(t) = nt$. More generally, if $p : (E, e) \to (B, b)$ is a covering map and $f : (I, 0) \to (B, b)$ is a path, then there is a unique lift $\tilde{f} : (I, 0) \to (E, e)$ such that $p \circ \tilde{f} = f$.

Here is a sketch of how to define a lift for a general path $f : (I, 0) \to (B, b)$. To begin, partition $I$ into small subintervals $I_1, \ldots, I_n \subsetneqq I$ (written in ascending order) such that each set $f(I_j)$ is contained in an open subset $U \subset S^1$ (depending on $j$) that is evenly covered by $p$. Next, define the lift of $f$ in stages. First, define $\tilde{f}(0) := e$. Then assuming that $\tilde{f}$ has been defined on $I_1 \cup \cdots \cup I_k$, extend to $I_{k+1}$ by choosing an evenly covered neighborhood $U$ containing $f(I_{k+1})$, choosing the slice $V$ containing $\tilde{f}$(right endpoint of $I_k$), and setting $\tilde{f} := (p|_V)^{-1} \circ f$ on $I_{k+1}$.

One can check that this process defines the unique lift of $f$ to $\mathbb{R}$ starting at 0. NB: the technical fact that allows us to chop up $I$ in this way is the Lebesgue Number Lemma, which depends crucially on the fact that $I$ is **compact**, i.e. every open cover of $I$ has a finite subcover. For more details, see Munkres' *Topology*.

**Exercise:** Prove that a subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. (This is called the Heine-Borel Theorem).

It turns out we also lift homotopies $h : (I \times I, (0, 0)) \to (S^1, 1)$ up $p$.

"If I like an argument enough to do it once, I like it enough to do it twice."

As before, partition $I \times I$ into small subsquares $I_j \times I_k$ such that for each pair of indicies $(j, k)$, the set $h(I_j \times I_k)$ is contained in an open subset $U \subset S^1$ that is evenly covered by $p$. Now lift the subsquares, row by row and one at a time, using the local homeomorphisms of $p$ onto its evenly covered neighborhoods.

2.2. **The Endpoint Map.** Now, given any loop $f : (I, 0) \to (S^1, 1)$, we can define a lift $\tilde{f} : (I, 0) \to (\mathbb{R}, 0)$ for $f$, and since $(p \circ \tilde{f})(1) = f(1) = 1$, we must have $\tilde{f}(1) \in p^{-1}\{1\} = \mathbb{Z}$. It turns out this integer is homotopy invariant, i.e. it is the same for all $g$ homotopic to $f$ via a homotopy that fixes endpoints. For, if $h : (I \times I, (0, 0)) \to (S^1, 1)$ is a homotopy $f \Rightarrow g$ that fixes endpoints, then its lift

$\tilde{h}$ must also be because $\tilde{h}(0,t)$ and $\tilde{h}(1,t)$ are both lifts of the constant paths to the endpoints of $f$, and the lift of a constant path is constant (by the uniqueness of path lifts). Thus $\tilde{h}$ is a homotopy between $\tilde{f}$ and $\tilde{g}$ that fixes endpoints, and in particular, $\tilde{f}(1) = \tilde{g}(1)$.

It follows that there is a well-defined endpoint map $j : \pi_1(S_1, 1) \to (\mathbb{R}, 0)$ satisfying $j([f]) := \tilde{f}(1)$. In particular, $j[f_n] = \tilde{f}_n(1) = n$ since $\tilde{f}_n = nt$.

We now check that $j$ is the inverse to $i : \mathbb{Z} \to \pi_1(S^1, 1)$ in the theorem. First of all, $(j \circ i)(n) = j([f_n]) = n$, so that $i$ is a section (right inverse) of $j$. This proves $j$ is surjective. For injectivity, suppose $[f], [g] \in \pi_1(S^1, 1)$ and $\tilde{f}(1) = j([f]) = j([g]) = \tilde{g}(1)$. Then $\tilde{f}$ and $\tilde{g}$ have the same endpoints, and since $\mathbb{R}$ is convex, we can define a homotopy $h$ between $\tilde{f}$ and $\tilde{g}$ by taking straight lines. Then $p \circ h$ is a homotopy between $f$ and $g$, so $[f] = [g]$. Thus $j$ is a bijective function, and has set-theoretic inverse $i$. Therefore $i$ is a bijective group homomorphism, hence an isomorphism.

2.3. **The Degree of a Map $S^1 \to S^1$.** To define the degree of a (non-basepoint preserving) map $f : S^1 \to S^1$, choose a path $\gamma : f(1) \to 1$, and then set

$$\deg(f) := i^{-1}\gamma_*[f] \in \pi_1(S^1, 1).$$

This is what we needed to prove the Fundamental Theorem of Algebra.

**Exercise:** Prove this satisfies the axioms we gave last time for the degree function.

2.4. **The Brouwer Fixed Point Theorem.** Define

$$D^2 := \{a + bi \in \mathbb{C} \mid a^2 + b^2 \leqq 1\},$$

the disc of all complex numbers of norm not exceeding 1.

**Theorem** (Brouwer)**.** *Suppose that $f : D^2 \to D^2$ is a continuous map. Then $f$ has a fixed point, i.e. there is $x \in D^2$ such that $f(x) = x$.*

*Proof.* Suppose for contradiction that $f : D^2 \to D^2$ does not have a fixed point. Then we can define a continuous map $r : D^2 \to S^1 = \partial D^2$ as follows: for any $x \in D^2$, our assumption that $f(x) \neq x$ implies there is a well-defined ray starting at $f(x)$ passing through $x$. Let $r(x)$ be where this ray intersects $S^1$. Let $s : S^1 \to D^2$ be the inclusion map, i.e. $s(x) = x$. Then $r \circ s = \mathrm{id}$ because $s(x)$ is already on the boundary $S^1$. The fundamental group construction preserves this relationship, i.e. the composite of

$$(S^1, 1) \xrightarrow{s_*} \pi_1(D^2, 1) \xrightarrow{r_*} \pi_1(S^1, 1)$$

must also the identity homorphism. On the other hand, we know that $\pi_1(D^2, 1) \cong 0$ (it's contractible), and hence $r_* \circ s_*$ must be the zero homomorphism since it factors through 0. Contradiction. □

**Exercise:** Write down an explicit formula for $r(x)$, and check it is continuous.

## 3. Connectedness and $\pi_0(X)$

A space $X$ is **connected** if the only open and closed subsets of $X$ are $\varnothing$ and $X$ itself. Equivalently, define a **separation** of $X$ to be a pair $(U, V)$ of open subsets of $X$ such that $U \cup V = X$, $U \cap V = \varnothing$, and $U, V \neq \varnothing$. Then $X$ is connected if and only if it has no separation.

**Exercise:** Prove this.

Write $x \sim y$ if $x$ and $y$ are elements of some connected subspace of $X$. Define $\pi_0'(X) := X/\sim = \{\sim \text{ equivalence classes of } X\}$. The equivalence classes of $\sim$ are called the **connected components of** $X$.

A space $X$ is **path connected** if for any points $x, y \in X$, there is some path $f : I \to X$ such that $f(0) = x$ and $f(1) = y$. Write $x \approx y$ if there is a path $f : I \to X$ connecting $x$ to $y$. Define $\pi_0(X) := X/\approx = \{\approx \text{ equivalence classes of X}\}$. The equivalence classes of $\approx$ are called the **path components of** $X$.

Warning: in general, $\sim$ and $\approx$ are **not** the same relation on $X$. $x \approx y$ implies $x \sim y$, but not conversely – see the "topologist's sine curve".

**Exercise:** Suppose $X$ is a space, $(A_i)_{i\in I}$ is a family of connected subspaces of $X$, and there is a point $x \in \bigcap_{i\in I} A_i$. Prove that $\bigcup_{i\in U} A_i$ is a connected subspace of $X$. Deduce that connected components are connected.

**Exercise:** Suppose $X$ and $Y$ are spaces. Prove $\pi_0'(X \coprod Y) \cong \pi_0'(X) \coprod \pi_0'(Y)$.

**Exercise:** Suppose $X$ is a space, and $A$ is a connected subspace of $X$. Prove that the closure of $A$ is also connected. Deduce that the connected components of a space are closed. Show by example that they need not be open (consider $\mathbb{Q}$, for example).

**Exercise:** Show that $I = [0, 1]$ is connected. Show that the continuous image of a connected set is a connected subspace. Conclude that $x \approx y$ implies $x \sim y$.

That said, there is a convenient **local condition** that guarantees $x \approx y$ implies $x \sim y$. Here is the general setup. Let $X$ be a topological space and $x \in X$. A **neighborhood of** $x$ is an open set $U$ such that $x \in U$.[1] Now let $P$ be some property. In general, $X$ is **locally** $P$ if and only if for every $x \in X$ and neighborhood $U$ of $x$, there is a neighborhood $V$ of $x$ such that $x \in V \subseteq U$ and $P$ holds in $V$.

**Exercise:** Prove that if $X$ is locally connected (resp. path connected), then the connected components (resp. path components) of $X$ are open.

**Exercise:** Observe that the connected components of $X$ are partitioned into path components. Prove that if $X$ is locally path connected, then for any $x, y \in X$, $x \sim y$ implies $x \approx y$. Deduce that if $X$ is connected and locally path connected, then it is path connected.

From the exercises, it follows that if $X$ is locally path connected, then the sets $\pi_0'(X)$ and $\pi_0(X)$ are equal.

## 4. A "Geodesic" Definition of $\pi_n(X)$

4.1. **Mapping Spaces and the Compact-Open Topology.** For any spaces $X$ and $Y$, define a space $\mathrm{Map}(X, Y)$ as follows. As a set, it will just be the set of all

[1] Different authors use "neighborhood" differently. For example, Bourbaki defines a neighborhood of a point $x$ to be a set $N$ containing an open set $U$ containing $x$.

continuous maps $X \to Y$. Its topology is a bit more subtle and complicated than the point-set constructions we've encounted so far.

For intuition, think back to metric spaces. If $X$ is compact and $Y$ is a metric space, we can give $\mathrm{Map}(X, Y)$ the uniform metric:

$$d(f, g) := \sup\{d(f(x), g(x)) \,|\, x \in X\}.$$

As always, this will topologize $\mathrm{Map}(X, Y)$.

**Exercise:** Show that $d : Y \times Y \to \mathbb{R}$ is a continuous function, and hence the composite function $d(f(x), g(x))$,

$$X \overset{\langle f,g\rangle}{\to} Y \times Y \overset{d}{\to} \mathbb{R}$$

is, too. Deduce that $\sup_{x\in X} d(f(x), g(x))$ is well-defined. Prove that it defines a metric on $\mathrm{Map}(X, Y)$.

In general, we cannot use this construction. Instead there is the **compact-open topology**. For a compact subset $K \subseteqq X$ and an open subset $U \subseteqq Y$, let

$$U(K, U) := \{f : X \to Y \,|\, f(K) \subseteqq U\}.$$

We can generate a topology on $\mathrm{Map}(X, Y)$ with these sets. The open sets are arbitrary unions of finite intersections of the $U(K, U)$.

**Exercise:** Check that this defines a topology. Prove that in the case $X$ is compact and $Y$ is a metric space, the uniform metric topology and the compact-open topology coincide.

The key point about the compact-open topology is that for reasonable enough spaces $X, Y, Z$, there is a bijective correspondence between the continuous maps $X \times Y \to Z$ and the continuous maps $X \to \mathrm{Map}(Y, Z)$. It is given by regarding a function $f(x, y)$ of two variables as the function-valued function $x \mapsto f(x, -)$. In practice, one usually avoids tackling the compact-open topology directly, and instead uses this correspondence to reduce to the better-understood space $X \times Y$.

For spaces with basepoints, say $(X, x)$ and $(Y, y)$, we can look at the subspace of basepoint-preserving maps

$$\mathrm{Map}_*((X, x), (Y, y)) := \{f \in \mathrm{Map}(X, Y) \,|\, f(x) = y\} \subseteqq \mathrm{Map}(X, Y).$$

$\mathrm{Map}_*((X, x), (Y, y))$ has a canonical basepoint, namely the constant function to $y$, so it is usually regarded as a based space. Note that the basepoints $x$ and $y$ are usually suppressed in the notation. Define

$$\Omega(X, x) := \mathrm{Map}_*((S^1, 1), (X, x)).$$

As above, one usually shortens to $\Omega X$. This is called the **loop space** of $X$.

4.2. **Higher Homotopy Groups.** As a first step, let's reinterpret $\pi_1$ in terms of $\pi_0$ and $\Omega$. Given a based space $(X, x)$, $\Omega(X, x)$ is the space of *all* loops on $X$ based at $x$. It is therefore too big to be $\pi_1(X, x)$; to recover $\pi_1(X, x)$ from $\Omega(X, x)$, we must identify homotopic loops. Note that paths $f : I \to X$ such that $f(0) = f(1)$ are naturally identified with maps $S^1 \to X$ by gluing the two endpoints of $I$ together. So the previous notion of loop and this new notion are equivalent.

Observe, however, a homotopy between two loops $f, g : (S^1, 1) \rightrightarrows (X, x)$ is equivalent to the following data via gluing and the correspondence for mapping spaces:

| | |
|---|---|
| $h : I \times I \to X$ | a homotopy fixing endpoints of $I$ |
| $h : S^1 \times I \to X$ | a homotopy fixing the basepoint of $S^1$ |
| $h : I \times S^1 \to X$ | a "twisted homotopy" fixing the basepoint of $S^1$ |
| $\gamma : I \to \mathrm{Map}_*(S^1, X) = \Omega X$ | a path in $\Omega X$ |

Moreover, the two homotopic loops $f, g$ correspond to the endpoints of the path $\gamma$ under this correspondence. Thus, taking homotopy classes of paths as in our original construction of $\pi_1(X, x)$ is equivalent to taking the path components of $\Omega X$. We therefore have an alternative definition:

$$\pi_1(X, x) := \pi_0(\Omega(X, x)).$$

"I love iterated loop spaces."

Now, to define the **higher homotopy groups**,[2] inductively take

$$\pi_n(X, x) := \pi_{n-1}(\Omega(X, x)) \cong \cdots \cong \pi_0(\Omega^n(X, x)).$$

Observe that since $\pi_1$ is a group, so too is $\pi_n$ for $n > 1$, since we can move $n - 1$ copies of $\Omega$ inside: $\pi_n(X, x) \cong \pi_1(\Omega^{n-1}(X, x))$. It turns out that for $n > 1$, $\pi_n$ is always an abelian group, more on that later. In general, $\pi_0$ is naturally only a set.

Later on, we will show that $\pi_n(S^n) \cong \mathbb{Z}$ for $n > 0$, but this will require some more machinery. The reader should be advised that homotopy groups are *extremely difficult* to compute in general. We don't even know all the groups $\pi_q(S^n)$.

Finally, we have the following important definition: suppose $X$ and $Y$ are spaces and $f : X \to Y$ is a map between them. Then $f : X \to Y$ is a **weak homotopy equivalence** if and only if for every $x \in X$ and $n \geqq 0$, the induced map $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism. In general, a homotopy equivalence is a weak homotopy equivalence, by basepoint-change isomorphisms. Conversely, between sufficiently nice spaces,[3] a weak homotopy equivalence is an ordinary homotopy equivalence. Thus, in *classical* algebraic topology, there isn't much of a difference between the two notions.

For **finite spaces**, however, these concepts are very different!

[2]Note there are equivalent definitions in terms of maps of the spheres $S^n$ into a space $X$.

[3]e.g. spaces of the homotopy type of a CW complex.