1. More on the Fundamental Group

Recall that for a topological space \( X \) and a point \( x \in X \), the **fundamental group of \( X \) based at \( x \)**, denoted \( \pi_1(X,x) \), is the group of homotopy classes of paths \( f : I \to X \) such that \( f(0) = f(1) = x \) (loops at \( x \)). Given a path \( f : I \to X \), denote the class of \( f \) by \([f]\). Multiplication is given by concatenation, i.e. for loops \( f,g : I \to X \),

\[
(g \cdot f)(t) := \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}
\]

and \([g] \cdot [f] := [g \cdot f]\).

"I shall feel free to skip all proofs that I find boring."

1.1. **Basepoint Dependence.** The basepoint \( x \in X \) in the definition of \( \pi_1(X,x) \) is often immaterial. Suppose \( x,y \in X \) and \( j : x \to y \) is a path. Then we can define an isomorphism \( j_* : \pi_1(X,x) \to \pi_1(X,y) \) between the fundamental groups based at \( x \) and \( y \) by \( j_*[f] := [j][f][j^{-1}] \).

**Exercise:** Check that this is a group homomorphism with inverse \((j^{-1})_*\).

Thus, if \( X \) is a topological space and \( A \subseteq X \) is a subspace such that any two \( x,y \in A \) can be connected with a path, then the fundamental groups \( \pi_1(X,a) \) for \( a \in A \) are all isomorphic. Note, however, that the isomorphism \( j_* \) induced by a path \( j : x \to y \) may depend on the choice of (homotopy class) of the path \( j \).
Exercise: Check that if \( \pi_1(X, x) \) is abelian and \( j, k : x \to y \) are any two paths in \( X \) from \( x \) to \( y \), then \( j_*= k_* \).

1.2. Induced Maps Between Fundamental Groups. First, some notation. \((X, x)\) shall mean that \( X \) is a topological space and that \( x \in X \). \( p : (X, x) \to (Y, y) \) shall mean \( p \) is a continuous map \( X \to Y \) that sends \( x \) to \( y \) (i.e. \( p(x) = y \)).

Suppose that \( p : (X, x) \to (Y, y) \). There is an induced group homomorphism \( p_* : \pi_1(X, x) \to \pi_1(Y, y) \) defined by \( p_*[f] := [p \circ f] \). Observe \( I \xleftarrow{f} X \xrightarrow{p} Y \), so taking \( p \circ f \) pushes \( f \) forward to a loop in \( Y \).

Exercise: Check that
1. \( p_* \) is a group homomorphism.
2. \( (\text{id}_X)_* \) is the identity homomorphism of \( \pi_1(X, x) \).
3. If \( (X, x) \xleftarrow{f} (Y, y) \xrightarrow{g} (Z, z) \), then \( g_* \circ f_* = (g \circ f)_* \).

Suppose that \( p, q : X \to Y \) are continuous maps and that \( h : p \Rightarrow q \) is a homotopy from \( p \) to \( q \). We would like to say that \( p \) and \( q \) induce the same group homomorphisms \( p_* \) and \( q_* \) out of \( \pi_1(X, x) \), but this doesn’t quite make sense since the codomain of \( p_* \) is \( \pi_1(Y, p(x)) \), while the codomain of \( q_* \) is \( \pi_1(Y, q(x)) \), which are not necessarily equal. Fortunately, the next best thing happens.

Let \( x \in X \) be a chosen basepoint. Fixing \( x \), the homotopy \( h \) defines a path \( j(t) := h(x, t) : I \to Y \) from \( p(x) \) to \( q(x) \). This yields an isomorphism \( j_* : \pi_1(Y, p(x)) \cong \pi_1(Y, q(x)) \), and we have \( j_* \circ p_* = q_* \), i.e. the triangle below commutes:

\[
\begin{array}{ccc}
\pi_1(X, x) & \xrightarrow{\pi} & \mathbb{Z} \\
\downarrow{\gamma_*} & & \downarrow{\cong} \\
\pi_1(Y, p(x)) & \xrightarrow{j_*} & \pi_1(Y, q(x))
\end{array}
\]

Exercise: Prove this.

2. The Fundamental Group of a Circle

Let \( S^1 \in \mathbb{C} \) be the set of unit length complex numbers. It is a circle. For each \( n \in \mathbb{Z} \), define \( f_n(t) := e^{2\pi int} \).

Exercise: Check that \( [f_m] \cdot [f_n] = [f_{m+n}] \).

Theorem. Equip \( \mathbb{Z} \) with its additive group structure. The map \( i : \mathbb{Z} \cong \pi_1(S^1, 1) \) defined by \( i(n) := [f_n] \) is an isomorphism of groups.

The standard proof of this fact uses covering space theory. A map \( p : E \to B \) is said to be a covering map if each point \( y \in B \) has an open neighborhood \( U \ni y \) such that the inverse image \( p^{-1}U \) can be written as a disjoint union \( \bigsqcup_{i \in I} V_i \subseteq X \).
of open sets $V_i$ such that for each $i$, $p$ restricts to a homeomorphism $p| : V_i \to U$. One sometimes says $U$ is **evenly covered** by $p$, and that the $V_i$ are **slices of** $U$. The intuition is that we regard $E$ as living “above” $B$ via $p$, and that each evenly covered neighborhood $U$ has a stack of (“pancakes”) copies of $U$ living above it.

In particular, we have the covering map

$$p(r) = e^{2\pi i r} : (\mathbb{R}, 0) \to (S^1, 1).$$

**Exercise:** Check this is a covering map.

### 2.1. Path and Homotopy Lifting

The important thing about covering maps for us is that given any $f : (I, 0) \to (S^1, 1)$, there is a unique lift $\tilde{f} : (I, 0) \to (\mathbb{R}, 0)$ of $f$ such that $p \circ \tilde{f} = f$ and $\tilde{f}(0) = 0$:

$$\begin{array}{ccc}
(I, 0) & \xrightarrow{f} & (S^1, 1) \\
\vdots & \Downarrow p & \vdots \\
(\mathbb{R}, 0) & \xrightarrow{} & (\mathbb{R}, 0)
\end{array}$$

For example, $f_n(t) = e^{2\pi i nt}$ lifts (uniquely) to $\tilde{f}_n(t) = nt$. More generally, if $p : (E, e) \to (B, b)$ is a covering map and $f : (I, 0) \to (B, b)$ is a path, then there is a unique lift $\tilde{f} : (I, 0) \to (E, e)$ such that $p \circ \tilde{f} = f$.

Here is a sketch of how to define a lift for a general path $f : (I, 0) \to (B, b)$. To begin, partition $I$ into small subintervals $I_1, \ldots, I_n \subseteq I$ (written in ascending order) such that each set $f(I_j)$ is contained in an open subset $U \subset S^1$ (depending on $j$) that is evenly covered by $p$. Next, define the lift of $f$ in stages. First, define $\tilde{f}(0) := e$. Then assuming that $\tilde{f}$ has been defined on $I_1 \cup \cdots \cup I_k$, extend to $I_{k+1}$ by choosing an evenly covered neighborhood $U$ containing $f(I_{k+1})$, choosing the slice $V$ containing $\tilde{f}$ (right endpoint of $I_k$), and setting $\tilde{f} := (p|_V)^{-1} \circ f$ on $I_{k+1}$.

One can check that this process defines the unique lift of $f$ to $\mathbb{R}$ starting at $0$. NB: the technical fact that allows us to chop up $I$ in this way is the Lebesgue Number Lemma, which depends crucially on the fact that $I$ is **compact**, i.e. every open cover of $I$ has a finite subcover. For more details, see Munkres’ *Topology*.

**Exercise:** Prove that a subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. (This is called the Heine-Borel Theorem).

It turns out we also lift homotopies $h : (I \times I, (0, 0)) \to (S^1, 1)$ up $p$.

“If I like an argument enough to do it once, I like it enough to do it twice.”

As before, partition $I \times I$ into small subsquares $I_j \times I_k$ such that for each pair of indices $(j, k)$, the set $h(I_j \times I_k)$ is contained in an open subset $U \subset S^1$ that is evenly covered by $p$. Now lift the subsquares, row by row and one at a time, using the local homeomorphisms of $p$ onto its evenly covered neighborhoods.

### 2.2. The Endpoint Map

Now, given any loop $f : (I, 0) \to (S^1, 1)$, we can define a lift $\tilde{f} : (I, 0) \to (\mathbb{R}, 0)$ for $f$, and since $(p \circ \tilde{f})(1) = f(1) = 1$, we must have $\tilde{f}(1) \in p^{-1}\{1\} = \mathbb{Z}$. It turns out this integer is homotopy invariant, i.e. it is the same for all $g$ homotopic to $f$ via a homotopy that fixes endpoints. For, if $h : (I \times I, (0, 0)) \to (S^1, 1)$ is a homotopy $f \Rightarrow g$ that fixes endpoints, then its lift
has a fixed point, i.e. there is \( x \) preserving) map \( f \) inverse \( i \). 2.4. Exercise: Prove this satisfies the axioms we gave last time for the degree function. This is what we needed to prove the Fundamental Theorem of Algebra.

2.3. The Degree of a Map \( S^1 \to S^1 \). To define the degree of a (non-basepoint preserving) map \( f : S^1 \to S^1 \), choose a path \( \gamma : f(1) \to 1 \), and then set
\[
deg(f) := i^{-1} \gamma_*[f] \in \pi_1(S^1, 1).
\]
This is what we needed to prove the Fundamental Theorem of Algebra.

Exercise: Prove this satisfies the axioms we gave last time for the degree function.

2.4. The Brouwer Fixed Point Theorem. Define
\[ D^2 := \{ a + bi \in \mathbb{C} | a^2 + b^2 \leq 1 \}, \]
the disc of all complex numbers of norm not exceeding 1.

Theorem (Brouwer). Suppose that \( f : D^2 \to D^2 \) is a continuous map. Then \( f \) has a fixed point, i.e. there is \( x \in D^2 \) such that \( f(x) = x \).

Proof. Suppose for contradiction that \( f : D^2 \to D^2 \) does not have a fixed point. Then we can define a continuous map \( r : D^2 \to S^1 = \partial D^2 \) as follows: for any \( x \in D^2 \), our assumption that \( f(x) \neq x \) implies there is a well-defined ray starting at \( f(x) \) passing through \( x \). Let \( r(x) \) be where this ray intersects \( S^1 \). Let \( s : S^1 \to D^2 \) be the inclusion map, i.e. \( s(x) = x \). Then \( r \circ s = \text{id} \) because \( s(x) \) is already on the boundary \( S^1 \). The fundamental group construction preserves this relationship, i.e. the composite of
\[
(S^1, 1) \xrightarrow{r_*} \pi_1(D^2, 1) \xrightarrow{s_*} \pi_1(S^1, 1)
\]
must also the identity homomorphism. On the other hand, we know that \( \pi_1(D^2, 1) \cong 0 \) (it’s contractible), and hence \( r_* \circ s_* \) must be the zero homomorphism since it factors through 0. Contradiction. \[ \square \]

Exercise: Write down an explicit formula for \( r(x) \), and check it is continuous.

3. Connectedness and \( \pi_0(X) \)

A space \( X \) is connected if the only open and closed subsets of \( X \) are \( \emptyset \) and \( X \) itself. Equivalently, define a separation of \( X \) to be a pair \( (U, V) \) of open subsets of \( X \) such that \( U \cup V = X \), \( U \cap V = \emptyset \), and \( U, V \neq \emptyset \). Then \( X \) is connected if and only if it has no separation.

Exercise: Prove this.
Write $x \sim y$ if $x$ and $y$ are elements of some connected subspace of $X$. Define $\pi'_0(X) := X/\sim = \{\sim\text{ equivalence classes of } X\}$. The equivalence classes of $\sim$ are called the connected components of $X$.

A space $X$ is path connected if for any points $x, y \in X$, there is some path $f : I \to X$ such that $f(0) = x$ and $f(1) = y$. Write $x \approx y$ if there is a path $f : I \to X$ connecting $x$ to $y$. Define $\pi_0(X) := X/\approx = \{\approx\text{ equivalence classes of } X\}$. The equivalence classes of $\approx$ are called the path components of $X$.

Warning: in general, $\sim$ and $\approx$ are not the same relation on $X$. $x \approx y$ implies $x \sim y$, but not conversely – see the “topologist’s sine curve”.

**Exercise:** Suppose $X$ is a space, $(A_i)_{i \in I}$ is a family of connected subspaces of $X$, and there is a point $x \in \bigcap_{i \in I} A_i$. Prove that $\bigcup_{i \in I} A_i$ is a connected subspace of $X$. Deduce that connected components are connected.

**Exercise:** Suppose $X$ and $Y$ are spaces. Prove $\pi'_0(X \coprod Y) \cong \pi'_0(X) \coprod \pi'_0(Y)$.

**Exercise:** Suppose $X$ is a space, and $A$ is a connected subspace of $X$. Prove that the closure of $A$ is also connected. Deduce that the connected components of a space are closed. Show by example that they need not be open (consider $\mathbb{Q}$, for example).

**Exercise:** Show that $I = [0,1]$ is connected. Show that the continuous image of a connected set is a connected subspace. Conclude that $x \approx y$ implies $x \sim y$.

That said, there is a convenient local condition that guarantees $x \approx y$ implies $x \sim y$. Here is the general setup. Let $X$ be a topological space and $x \in X$. A neighborhood of $x$ is an open set $U$ such that $x \in U$.\footnote{Different authors use “neighborhood” differently. For example, Bourbaki defines a neighborhood of a point $x$ to be a set $N$ containing an open set $U$ containing $x$.} Now let $P$ be some property. In general, $X$ is locally $P$ if and only if for every $x \in X$ and neighborhood $U$ of $x$, there is a neighborhood $V$ of $x$ such that $x \in V \subseteq U$ and $P$ holds in $V$.

**Exercise:** Prove that if $X$ is locally connected (resp. path connected), then the connected components (resp. path components) of $X$ are open.

**Exercise:** Observe that the connected components of $X$ are partitioned into path components. Prove that if $X$ is locally path connected, then for any $x, y \in X$, $x \sim y$ implies $x \approx y$. Deduce that if $X$ is connected and locally path connected, then it is path connected.

From the exercises, it follows that if $X$ is locally path connected, then the sets $\pi'_0(X)$ and $\pi_0(X)$ are equal.

4. A “Geodesic” Definition of $\pi_n(X)$

4.1. **Mapping Spaces and the Compact-Open Topology.** For any spaces $X$ and $Y$, define a space $\text{Map}(X,Y)$ as follows. As a set, it will just be the set of all
continuous maps $X \to Y$. Its topology is a bit more subtle and complicated than the point-set constructions we’ve encountered so far.

For intuition, think back to metric spaces. If $X$ is compact and $Y$ is a metric space, we can give $\text{Map}(X,Y)$ the uniform metric:

$$d(f, g) := \sup\{d(f(x), g(x)) \mid x \in X\}.$$ 

As always, this will topologize $\text{Map}(X,Y)$.

**Exercise:** Show that $d : Y \times Y \to \mathbb{R}$ is a continuous function, and hence the composite function $d(f(x), g(x))$, $X \xrightarrow{(f, g)} Y \times Y \xrightarrow{d} \mathbb{R}$ is, too. Deduce that $\sup_{x \in X} d(f(x), g(x))$ is well-defined. Prove that it defines a metric on $\text{Map}(X,Y)$.

In general, we cannot use this construction. Instead there is the **compact-open topology**. For a compact subset $K \subseteq X$ and an open subset $U \subseteq Y$, let

$$U(K, U) := \{f : X \to Y \mid f(K) \subseteq U\}.$$

We can generate a topology on $\text{Map}(X,Y)$ with these sets. The open sets are arbitrary unions of finite intersections of the $U(K, U)$.

**Exercise:** Check that this defines a topology. Prove that in the case $X$ is compact and $Y$ is a metric space, the uniform metric topology and the compact-open topology coincide.

The key point about the compact-open topology is that for reasonable enough spaces $X, Y, Z$, there is a bijective correspondence between the continuous maps $X \times Y \to Z$ and the continuous maps $X \to \text{Map}(Y,Z)$. It is given by regarding a function $f(x, y)$ of two variables as the function-valued function $x \mapsto f(x, -)$. In practice, one usually avoids tackling the compact-open topology directly, and instead uses this correspondence to reduce to the better-understood space $X \times Y$.

For spaces with basepoints, say $(X, x)$ and $(Y, y)$, we can look at the subspace of basepoint-preserving maps

$$\text{Map}_*(((X, x), (Y, y)) := \{f \in \text{Map}(X,Y) \mid f(x) = y\} \subseteq \text{Map}(X,Y).$$

$\text{Map}_*((X, x), (Y, y))$ has a canonical basepoint, namely the constant function to $y$, so it is usually regarded as a based space. Note that the basepoints $x$ and $y$ are usually suppressed in the notation. Define

$$\Omega(X, x) := \text{Map}_*((S^1, 1), (X, x)).$$

As above, one usually shortens to $\Omega X$. This is called the **loop space** of $X$.

### 4.2. Higher Homotopy Groups

As a first step, let’s reinterpret $\pi_1$ in terms of $\pi_0$ and $\Omega$. Given a based space $(X, x)$, $\Omega(X, x)$ is the space of all loops on $X$ based at $x$. It is therefore too big to be $\pi_1(X, x)$; to recover $\pi_1(X, x)$ from $\Omega(X, x)$, we must identify homotopic loops. Note that paths $f : I \to X$ such that $f(0) = f(1)$ are naturally identified with maps $S^1 \to X$ by gluing the two endpoints of $I$ together. So the previous notion of loop and this new notion are equivalent.
Observe, however, a homotopy between two loops \( f, g : (S^1, 1) \Rightarrow (X,x) \) is equivalent to the following data via gluing and the correspondence for mapping spaces:

\[
\begin{align*}
    h : I \times I &\to X & \text{a homotopy fixing endpoints of} \, I \\
    h : S^1 \times I &\to X & \text{a homotopy fixing the basepoint of} \, S^1 \\
    h : I \times S^1 &\to X & \text{a “twisted homotopy” fixing the basepoint of} \, S^1 \\
    \gamma : I &\to \map_*(S^1, X) = \Omega X & \text{a path in} \, \Omega X
\end{align*}
\]

Moreover, the two homotopic loops \( f, g \) correspond to the endpoints of the path \( \gamma \) under this correspondence. Thus, taking homotopy classes of paths as in our original construction of \( \pi_1(X,x) \) is equivalent to taking the path components of \( \Omega X \). We therefore have an alternative definition:

\[
\pi_1(X,x) := \pi_0(\Omega(X,x)).
\]

“I love iterated loop spaces.”

Now, to define the \textbf{higher homotopy groups},\(^2\) inductively take

\[
\pi_n(X, x) := \pi_{n-1}(\Omega(X,x)) \cong \cdots \cong \pi_0(\Omega^n(X,x)).
\]

Observe that since \( \pi_1 \) is a group, so too is \( \pi_n \) for \( n > 1 \), since we can move \( n-1 \) copies of \( \Omega \) inside: \( \pi_n(X,x) \cong \pi_1(\Omega^{n-1}(X,x)) \). It turns out that for \( n > 1 \), \( \pi_n \) is always an abelian group, more on that later. In general, \( \pi_0 \) is naturally only a set.

Later on, we will show that \( \pi_n(S^n) \cong \mathbb{Z} \) for \( n > 0 \), but this will require some more machinery. The reader should be advised that homotopy groups are \textit{extremely difficult} to compute in general. We don’t even know all the groups \( \pi_q(S^n) \).

Finally, we have the following important definition: suppose \( X \) and \( Y \) are spaces and \( f : X \to Y \) is a map between them. Then \( f : X \to Y \) is a \textbf{weak homotopy equivalence} if and only if for every \( x \in X \) and \( n \geq 0 \), the induced map \( f_* : \pi_n(X,x) \to \pi_n(Y,f(x)) \) is an isomorphism. In general, a homotopy equivalence is a weak homotopy equivalence, by basepoint-change isomorphisms. Conversely, between sufficiently nice spaces,\(^3\) a weak homotopy equivalence is an ordinary homotopy equivalence. Thus, in \textit{classical} algebraic topology, there isn’t much of a difference between the two notions.

For \textbf{finite spaces}, however, these concepts are very different!

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\(^2\)Note there are equivalent definitions in terms of maps of the spheres \( S^n \) into a space \( X \).

\(^3\)e.g. spaces of the homotopy type of a CW complex.