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## 1. References


(1) *A Concise Course in Algebraic Topology*, J. P. May, available online
(2) *Finite Topological Spaces*, J. P. May, available online
(3) *Topology*, J. Munkres,
(4) *Algebraic Topology*, A. Hatcher, available online


## 2. Homotopy Equivalence

What is topology? Intuitively we know what it means for points to be close together and we want to quantify this intuition. Let $f : X \to Y$ be a function. We ask what it means for $f$ to preserve nearness of points. Most people have seen the following definition from analysis:

**Definition 2.1.** *$f$ is continuous if for all $\epsilon > 0$ and all $x \in X$ there exists $\delta_x > 0$ such that if $|x - x'| < \delta_x$, then $|f(x) - f(x')| < \epsilon$.*

Note that $\epsilon$ is not quantified and that this is not a very practical definition because of all these silly $\delta_x$. We want to redefine this in a more elegant fashion. To do this, we define the distance between two points to be

$$d(x, x') = |x - x'|.$$

The"open balls" are then

$$B(x, \delta) = \{x' \mid d(x, x') < \delta_x\}.$$

Now, we note that Definition 2.1 is equivalent to asking that

$$f(B(x, \delta_x)) \subset B(f(x), \epsilon).$$

For $U \subset Y$, define the preimage of $U$ under $f$ as

$$f^{-1}(U) = \{x \mid f(x) \in U\}.$$

The criterion from Definition 2.1 is equivalent to

$$B(x, \delta_x) \subset f^{-1}(B(f(x), \epsilon)).$$

or even better:

$$f^{-1}(B(f(x), \epsilon)) = \bigcup_{x \in X} B(x, \delta_x).$$

**Definition 2.2.** *An open set $U$ in $Y$ is a union of sets of the form $B(y, \epsilon)$.*

**Theorem 2.3** (Or alternative definition)**.** *$f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is open in $X$ when $U$ is open in $Y$.*

This leads us to the definition of a *topological space*.

**Definition 2.4.** *A topological space $X$ is a set $X$ and a collection of open sets $\mathcal{T} = \{U \mid U \subset X\}$ which we call the* open sets*, such that*

*(i) $\emptyset$ and $X$ are open.*
*(ii) Any union of open sets is open.*
*(iii) Any finite intersection of open sets is open.*

**Definition 2.5.** *A metric space $(X, d)$ is a set $X$ with a distance function*

$$d : X \times X \to [0, \infty)$$

*such that*

*(i) $d(x, y) = 0$ if and only if $x = y$*
*(ii) $d(x, y) = d(y, x)$*
*(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$*

An example is $(\mathbb{R}, d)$ where $d(x, y) = |x - y|$. Any Euclidean space $(\mathbb{R}^n, d)$ with the usual distance function is also a metric space. A metric space is a topological space if we take the set of open sets $\mathcal{T}$ to be the collection of all unions of open balls, including the empty set.

**Definition 2.6.** *For a metric space $X$, the metric topology on $X$ is the topology whose open sets $\mathcal{T}$ are any unions (possibly empty) of open balls $B(x, \delta)$, $x \in X$ and $\delta > 0$.*

Geometry cares about the distance function. Topology doesn't know about it, it only knows about the open sets that arise from the distance function.

Another example to which we will return is that of an Alexandroff space:

**Definition 2.7.** *An Alexandroff space is a topological space $X$ with the additional property that* **any** *intersection of open sets is open.*

Any finite topological space (that is, when the set $X$ is finite) is an Alexandroff space.

There are many constructions on sets which give constructions on spaces. One example is the product of topological spaces. Let $X$ and $Y$ be topological spaces. A set is open in the product topology on $X \times Y$ if it is a union or a finite intersection of sets of the form $U \times V$ where $U$ is open in $X$ and $V$ is open in $Y$. An example which will play a central role is $X \times I$ where $I = [0, 1]$ is the unit interval in $\mathbb{R}$. This is the cylinder on $X$.

**Exercise 2.8.** *Define a metric on the product of two metric spaces and prove that the metric topology defined by your metric is the same as the product topology.*

Given a subset $A$ of a topological space $A \subset X$, we have a topology on $A$ given by that of $X$. Open sets in $A$ are defined as those sets which can be expressed as $A \cap U$ for $U$ open in $X$. For a metric space, this is intuitive. The restriction of the distance function on $X$ to the subset $A$ is a distance function on $A$.

**Definition 2.9.** *A homotopy between two continuous functions $f, g : X \to Y$ is a continuous function $h : X \times I \to Y$ such that $f(x) = h(x, 0)$ and $g(x) = h(x, 1)$. We write $f \simeq g$*

From now on, all maps and function are assumed to be continuous.

**Definition 2.10.** *A map $f : X \to Y$ is a homeomorphism if there exists $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In this case, we write $X \cong Y$.*

**Definition 2.11.** *A map $f : X \to Y$ is a homotopy equivalence if there exists $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. In this case, we write $X \simeq Y$.*

**Definition 2.12.** *If $X \simeq *$, then $X$ is contractible.*

**Example 2.13.** *Let $\mathbf{0} \in \mathbb{R}^n$ be the vector $(0, \ldots, 0)$. Let*

$$f : \mathbb{R}^n \to \{\mathbf{0}\}$$

*be the constant map. Let*

$$g : \{\mathbf{0}\} \to \mathbb{R}^n$$

*be the inclusion. These are both continuous maps. In fact, they give a homotopy equivalence between $\mathbb{R}^n$ and the one point set $\{\mathbf{0}\}$ so that $\mathbb{R}^n$ is contractible. Indeed, $h(\mathbf{x}, t) = \mathbf{x} \cdot t$ is a homotopy from $\mathrm{id}_{\mathbb{R}}$ to $g \circ f$, and $f \circ g = \mathrm{id}$ on $\{\mathbf{0}\}$.*

**Exercise 2.14.** *Homotopy is an equivalence relation on continuous maps. The set of equivalence classes of maps $f : X \to Y$ is denoted $[X, Y]$.*

## 3. Fundamental Theorem of Calculus

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial $f(x) = x^n + c_1 x^{n-1} + \ldots + c_1 x + c_0$, where $c_i \in \mathbb{C}$. A complex number $c$ is a root of $f$ if $f(c) = 0$.

**Theorem 3.1.** *$f$ has a root, and hence, $f$ has $n$ roots.*

Given any continuous map $f : S^1 \to S^1$, where

$$S^1 = \{a + bi \in \mathbb{C} \mid a^2 + b^2 = 1\},$$

we will define an invariant for such maps called the *degree*. To each $f$ we will associate an integer $\deg(f) \in \mathbb{Z}$ which will satisfy:

(i) $f \simeq g$ if and only if $\deg(f) = \deg(g)$
(ii) If $f_c$ is $f_c(x) = c$, then $\deg(f_c) = 0$
(iii) If $f_n = x^n$, then $\deg(f_n) = n$

*Proof of Theorem 3.1.* Consider a polynomial $f : \mathbb{C} \to \mathbb{C}$. Assume that $f$ has no roots. Let $\hat{f}(x) = \frac{f(x)}{|f(x)|}$. Then

$$\hat{f} : S^1 \to S^1.$$

Define

$$h(x, t) = f(tx)/|f(tx)|.$$

This makes sense since $f$ has no roots with $|r| \leq 1$. Further, $h(x, 0) = f_c$ for $c = f(0)/|f(0)|$ and $h(x, 1) = \hat{f}(x)$. Then by homotopy invariance $\deg(\hat{f}) = 0$. Now since $f$ has no roots $r$ such that $|r| \geq 1$, we can define $j \colon S^1 \to S^1$ by

$$j(x, t) = k(x, t)/|k(x, t)|$$

where $k\colon S^1 \to \mathbb{C}$ is defined by

$$k(x,t) = t^n f(x/t) = x^n + t(c_1 x^{n-1} + \ldots + t^{n-1} c_0).$$

Then $j(x,0) = x^n$ and $j(x,1) = \hat{f}$. Since, $\deg(f) = 0$, this implies that $n = 0$. Hence, if $f$ has no roots, it is constant. □

## 4. The Fundamental Group

**Definition 4.1.** *A group $G$ is a set together with two functions $m : G \times G \to G$ and $(-)^{-1} : G \to G$ and an element $e \in G$. We write $m(g,h) = gh$ and require*

*(i)* $(gh)k = g(hk)$
*(ii)* $eg = g = ge$
*(iii)* $g^{-1}g = e = gg^{-1}$.

**Definition 4.2.** *A path between $x$ and $y$ is a function $p : I \to X$ where $p(0) = x$ and $p(1) = y$. A homotopy between such paths is a function $h : I \times I \to X$ such that $h(x,0) = p(x)$, $h(x,1) = q(x)$, $h(0,t) = x$ and $h(1,t) = y$.*

**Exercise 4.3.** *This is an equivalence relation on paths from $x$ to $y$.*

**Definition 4.4.** *The set of equivalence classes of paths from $x$ to $y$ is denoted $[x,y]$. We write $[p]$ for the equivalence class of $p$.*

Given a path $p$ from $x$ to $y$ and a path $q$ from $y$ to $z$, the concatenation of $p$ and $q$ is

$$p \cdot q(s) = \begin{cases} p(2s) & 0 \le s \le 1/2 \\ q(2s-1) & 1/2 \le s \le 1. \end{cases}$$

**Exercise 4.5.** *Check that $[p \cdot q] = [p' \cdot q']$ if $[p] = [p']$ and $[q] = [q']$. This gives a multiplication $[x,y] \times [y,z] \to [x,z]$ defined by*

$$[p] \cdot [q] = [p \cdot q]$$

*Check that this is associative. That is, check that $[p \cdot (q \cdot r)] = [(p \cdot q) \cdot r]$. Check that for $c_{p(0)}$ the constant path at $p(0)$,*

$$[p] = [c_{p(0)}] \cdot [p]$$

*and*

$$[p] = [p] \cdot [c_{p(1)}]$$

*For $[p] \in [x,y]$, there is a path $[p]^{-1} \in [y,x]$ given by*

$$p^{-1}(t) = p(1-t).$$

*Further,*

$$[p] \cdot [p]^{-1} = [c_{p(0)}] \in [x,x]$$

*and*

$$[p]^{-1} \cdot [p] = [c_{p(1)}] \in [y,y]$$

**Definition 4.6** (The fundamental group)**.** *Let $x \in X$ be a fixed point. The fundamental group of $X$ is*

$$\pi_1(X,x) = [x,x]$$

We will prove the following fact next time:

**Theorem 4.7.**

$$\pi_1(S^1, 1) = \mathbb{Z}$$

The intuition is that of the *winding number*. That is, the number of times the path $p : S^1 \to S^1$ winds around $S^1$. It may be all twisted, but using homotopies, any path can be "straightened out" to one that winds $n$ times around $S^1$ for $n \in \mathbb{Z}$.