# THE EULER CHARACTERISTIC

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Abstract. This paper introduces the fundamental concepts needed to understand Euler's characteristic and its properties. It covers essential ideas from topology, including the triangulation of surfaces, and graph theory, leading to the definition and key applications of Euler's characteristic.

Contents



## 1. Introduction

In mathematics, numerical invariants serve as powerful tools for classifying different types of mathematical objects. An important numerical invariant is the Euler characteristic, which helps compare and classify spaces based on their structure. For example, orientable surfaces are completely identified by their Euler characteristic. This paper will introduce fundamental concepts to understand the Euler characteristic, its key properties, and its applications.

We begin with a brief introduction to topology, which includes the comparison of topological spaces through homeomorphisms. Next, we introduce Euclidean space and simplices, which provide the framework for triangulating topological spaces and thus deriving the Euler characteristic. We then incorporate ideas from graph theory, particularly concepts involving vertices, edges, faces, and connectedness, to deepen our understanding of these structures. Finally, we formally define the Euler characteristic, introduce its key properties, and demonstrate its wide-ranging applications, including the fundamental result that triangulable homeomorphic topological spaces share the same Euler characteristic.

## 2. Background to Topology

To understand the Euler characteristic in its full generality, we first establish the fundamental language of topology. In this section, we will be defining topological spaces, the open sets that form their structure, and the notions of continuity and homeomorphism, which allow us to compare different spaces.

**Definition 2.1.** A **topological space** $(X, O)$, consists of a set $X$ and a **topology** $O$, a collection of subsets of $X$, with the following properties:

- The countable union of any sets in $O$ is in $O$
- A finite intersection of sets in $O$ is in $O$
- Both $\emptyset, X \in O$.

**Example 2.2.** A topological space $X$, with an unspecified topology, or when said to have the "normal/usual topology" (especially when $X \subset \mathbb{R}^n$), often refers to the *standard topology* induced from $\mathbb{R}^n$. This topology is generated by arbitrary countable unions and finite intersections of open balls in $X$. An open ball in $X$ is a subset of $X$ defined for a point $x \in X$ and radius $r > 0$ by

$$B_r(x) = \{y \in X \mid d(x, y) < r\},$$

where $d(x, y)$ is the standard Euclidean metric given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2},$$

with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

**Definition 2.3.** If $A \subset X$ is in $O$, $A$ is **open**. Furthermore, it's complement, $X - A$, is **closed**.

**Theorem 2.4.** *Every open set in $\mathbb{R}$ is a countable union of a collection of disjoint open intervals.*

*Proof.* Let $A \subset \mathbb{R}$ be an open set. If $A$ is an open interval, we are done. Otherwise, there exist $x, y \in A$ such that $(x, y) \not\subset A$.

For any point $p \in A$, define the maximal open interval $I_p = (a_p, b_p)$ containing $p$ that is still contained in $A$:

$$a_p = \inf\{a' \mid (a', p) \subset A\}, \quad b_p = \sup\{b' \mid (p, b') \subset A\}.$$

If $a \in A$ for all $a \in \mathbb{R}$ and $a < p$, let $a_p = -\infty$. Similarly, if $b \in A$ for all $b \in \mathbb{R}$ and $b < p$, let $b_p = \infty$.

Because $A$ is open under the normal topology in $\mathbb{R}$, there exist $a, b \in \mathbb{R}$ such that $(a, p) \subset A$ and $(p, b) \subset A$. Suppose for the sake of contradiction that $a_p$ does not exist. Therefore for any $a \in \mathbb{R}$ such that $(a, p) \subset A$, there exists $a' < a$ such that $(a', p) \subset A$. Therefore, for any $a \leq p$, $a \in A$. Thus, $a_p = -\infty$ by construction. Therefore, $a_p$ exists for any $p \in A$. A symmetrical proof follows for the existence of $b_p$. Thus, for any $p \in A$, there exists $a_p, b_p$ with $a_p < p < b_p$ and $I_p = (a_p, b_p) \subset A$.

Consider $x, y \in A$ with $(x, y) \not\subset A$, and let $I_x = (a_x, b_x)$ and $I_y = (a_y, b_y)$ be the corresponding maximal open intervals. Since there exists $b' \in (x, y)$ such that $b' \not\subset A$, $b_x \leq b' \leq a_y$.

Suppose for contradiction that $I_x \cap I_y \neq \emptyset$. Pick $k \in I_x \cap I_y$. By construction of $I_x$ and $I_y$, we have:

$$(x, k) \subset A, \quad (k, y) \subset A, \quad \text{and} \quad k \in A.$$

This implies

$$(x, k) \cup \{k\} \cup (k, y) = (x, y) \subset A,$$

which is a contradiction with our choice of $x$ and $y$. Therefore, $I_x \cap I_y = \emptyset$ whenever $(x, y) \not\subset A$.

Let $x, y \in A$ with $(x, y) \subset A$, and let $I_x = (a_x, b_x)$ and $I_y = (a_y, b_y)$ be their maximal open intervals correspondingly. Consider the claim that $I_x = I_y$. Suppose for the sake of contradiction that $a_x \neq a_y$. Let $a_x < a_y$. We know that $a_x \leq x$ since $(x, y) \subset A$. Since $(a_x, x) \subset A$, $x \in A$, and $(x, y) \subset A$, $(a_x, y) \subset A$, which contradicts the definition of $a_y$. An analogous proof follows when $a_y < a_x$, as $(a_y, x) \subset A$ would contradict the definition of $a_x$. Therefore $a_x = a_y$ and a symmetrical proof follows for $b_x = b_y$. Thus, if $(x, y) \subset A$, then $I_x = I_y$.

By creating a maximal interval for every rational $p \in A$, we observe that $I_p$ is either equal to another open maximal interval, or disjoint from all other non-equivalent open maximal intervals, i.e disjoint from all $I_q$ where $q \in A$ and $I_q \neq I_p$. If it is equal, we can take one copy of such maximal interval, and thus, $A$ is the countable union of its maximal disjoint open intervals:

$$A = \bigcup_{p \in A} I_p.$$

□

**Definition 2.5.** A function $f : X \to Z$ is **continuous** if for all open sets $Y$ in $Z$, $f^{-1}(Y)$ is open in $X$.

**Definition 2.6.** A function $f : X \to Y$ is a **homeomorphism** if $f$ is continuous and bijective, and its inverse function, $f^{-1} : Y \to X$ is also continuous. We say that $X, Y$ are **homeomorphic** to each other if such a function exists.

## 3. Euclidean Space and Simplicial Complexes

In this section, we specialize to Euclidean spaces and introduce the ideas of simplices and simplicial complexes. Let $v_0, v_1, ..., v_k$ be points of Euclidian $n$-space, $\mathbb{E}^n$. This framework creates a bridge between the topological surfaces and the combinatorial structures needed to compute Euler's characteristic.

**Definition 3.1.** The **hyperplane** spanned by these points consists of all linear combinations, $\lambda_0 v_0 + \lambda_1 v_1 + ... + \lambda_k v_k$ where $\lambda_i \in [0, 1]$ for all $0 \leq i \leq k$ and $\sum_{i=0}^{k} \lambda_i = 1$.

**Definition 3.2.** $\{v_0, v_1, ..., v_k\}$ are in **general position** if any proper subset spans a strictly smaller hyperplane.

**Definition 3.3.** A **convex set** is a set such that the points on the line segment connecting any two points in the set lie in the set.

**Example 3.4.** Figure 1 shows two sets of points in $\mathbb{R}^3$. On the left, the points lie inside a solid torus, and on the right, inside a solid sphere. The sphere is convex because any line segment between two points stays entirely within it. The torus, however, is not convex as line segments connecting points on opposite sides, like those shown in red, pass outside the torus.

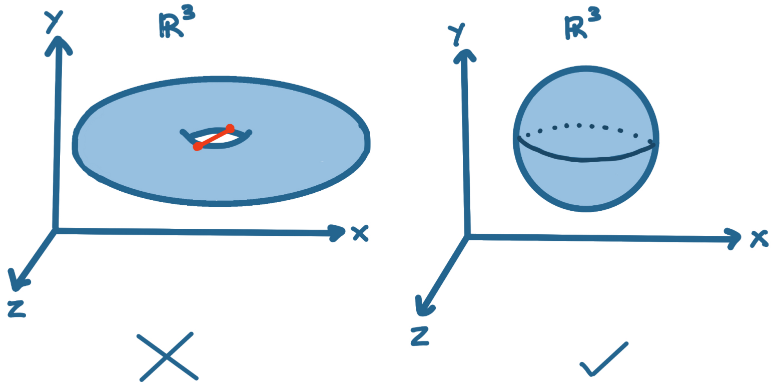


FIGURE 1. Example and Non-Example of Convex Sets

**Definition 3.5.** A **$k$-dimensional simplex** is the smallest convex set containing $v_0, ..., v_k$ in general position where the points make up the vertices of this simplex.

For the first few dimensions:

(1) 0-simplex=points
(2) 1-simplex=closed line segment
(3) 2-simplex=triangle
(4) 3-simplex=tetrahedron

**Definition 3.6.** Suppose $A, B$ are $n$-simplices with $n \geq 2$. If vertices of $B$ form a subset of the vertices of $A$, $B$ is a **face** of $A$.

**Definition 3.7.** A finite collection of simplices in some euclidean space, $\mathbb{E}^n$, is called a **simplicial complex** if whenever a simplex lies in the collection then so does each face, and whenever two simplexes of the collection intersect they do so in a common face.

**Definition 3.8.** A space is **triangulable** if it is homeomorphic to the union of a finite collection of simplices in some Euclidean space where whenever two simplexes of the collection intersect they do so in a common face.

**Definition 3.9.** A **triangulation** of a topological space X consists of a simplicial complex $K$ and a homeomorphism $h : |K| \to X$

**Example 3.10.** Figure 2 below illustrates a triangulation of a cylinder. The union of two triangles with sides identified as in the left most figure forms a shape homeomorphic to a rectangle with 2 opposite sides identified, which in turn is homeomorphic to a cylinder.

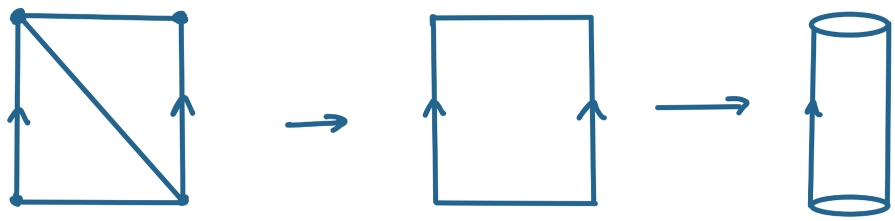

FIGURE 2. Example of Cylinder Triangulation

**Example 3.11.** Figure 3 below illustrates a triangulation of a torus. A torus is homeomorphic to a union of two traingles with sides identified as in the left most figure. This is homeomorphic to a rectangle with opposite sides identified by gluing the opposite ends together, which is homeomorphic to a torus.

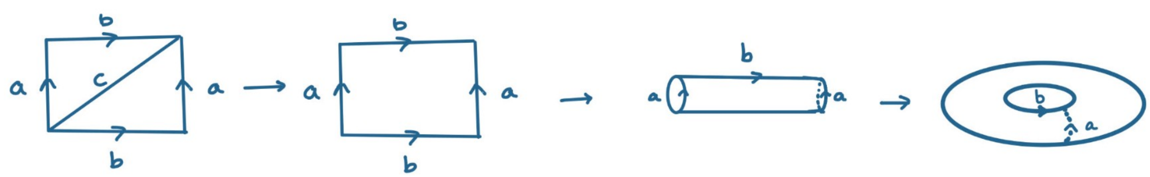


FIGURE 3. Example of Torus Triangulation

## 4. Some Graph Theory

Since simplicial complexes often involve vertices and edges, many ideas from graph theory naturally overlap with our study of Euler's characteristic. In this section, we review key concepts with finite graphs such as paths, cycles, trees, and maximal trees, which will later help us connect combinatorial structure to topological invariants.

**Definition 4.1. A graph** is a set consisting of points which are referred to as **vertices**, and edges, which are lines specific to two vertices in the set.

**Definition 4.2.** A **walk** is between a sequence of vertices where each consecutive pair of vertices in the sequence is connected by an edge in the graph. A walk with no repeating vertices is called a **path**. A walk that starts and ends with the same vertex is called a **cycle.**

**Definition 4.3.** A graph is said to be **connected** if there exists a path between any two vertices.

**Definition 4.4.** A **connected component** of a graph $G$ is a set of vertices such that a walk exists in $G$ between any two vertices in the set, and it would not be possible to include another vertex of $G$ in the set without breaking this property.

**Definition 4.5.** A **tree** is a connected graph with a single, unique path between each pair of vertices.

**Remark 4.6.** A **tree** is often referred to as a connected graph containing no cycles. Indeed, in a graph with cycles, any vertex in the cycle will have two paths towards it: a path in one direction, and a path in the opposite direction of the loop. See Figure 4 below.

**Example 4.7.** Figure 4 shows an example of a graph that is a tree and an example of a graph that is not a tree.

The graph on the left of figure 4 is an example of a tree as it contains no loops, meaning that there is a unique path between any two vertices.

On the other hand, the right graph contains a cycle, highlighted in yellow. For any vertex in the loop, there are two paths to it. For example from $v_1$ to $v_5$ there are two paths:

- $v_1 \to v_3 \to v_5$
- $v_1 \to v_3 \to v_4 \to v_5$

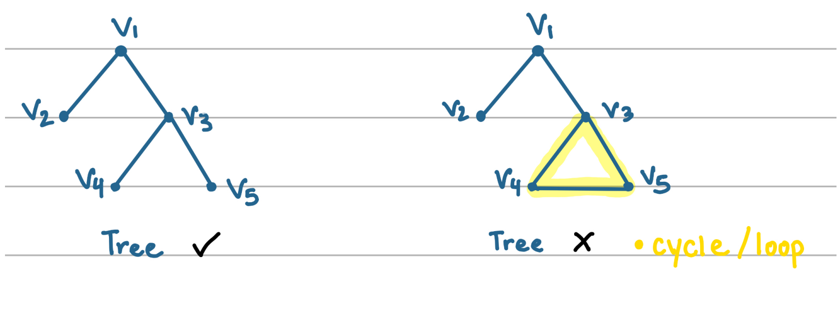


FIGURE 4. Tree vs. Not Tree

**Definition 4.8.** A **maximal tree**, $T$, of a connected graph $G$ is a tree containing all vertices of $G$ such that an edge between two vertices in $T$ also exists as an edge between the two vertices in $G$.

**Example 4.9.** Figure 5 shows an example of 3 graphs and their maximal trees. As previously shown, the graph in the middle of figure 5 is already a tree. Hence its maximal tree is trivially itself.

On the other hand, the graphs on the left and right both contain loops, implying that they are not trees. Notice that the maximal tree is not unique to a graph. For the left graph, another maximal tree could contain the edges: $(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5)$. Similarly, another maximal tree for the right graph could contain the edges: $(v_1, v_2), (v_1, v_3)$, $(v_3, v_4), (v_4, v_5)$.

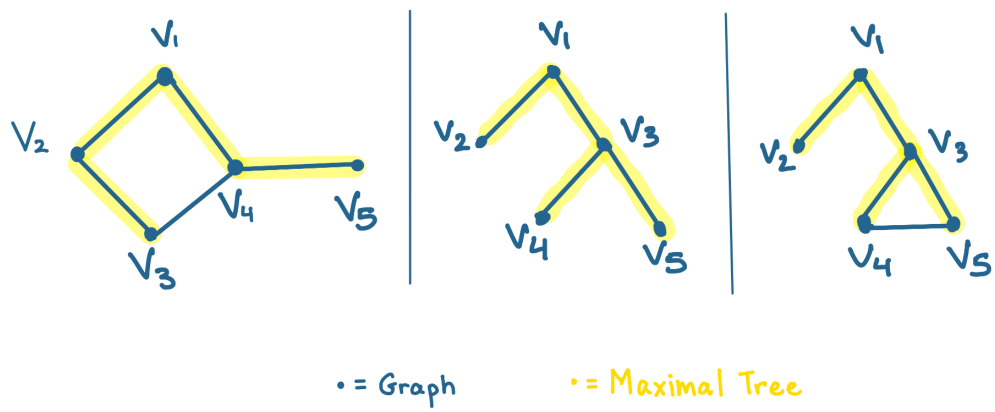


FIGURE 5. Examples of Maximal Tree

**Theorem 4.10.** *For any connected graph $G$, there exists a maximal tree.*

*Proof.* We will prove this statement by **mathematical induction** on the number of vertices $n$ in a connected graph $G$.

**Base Case:** Let $n = 1$. A graph with a single vertex and no edges, is trivially connected and itself forms a tree. Since it includes all the vertices of the graph and contains no cycles, it is a maximal tree. Thus, the base case holds.

**Inductive Hypothesis:** Assume that every connected graph with at most $m$ vertices has a maximal tree.

**Inductive Step:** Let $G$ be a connected graph with $m + 1$ vertices. We want to show that $G$ contains a maximal tree.

Since $G$ is connected, there exists a path between any pair of vertices. Suppose we remove a vertex $v \in G$, and consider the subgraph $G'$ induced by the remaining $m$ vertices.

Though $G'$ might be a disconnected graph, it must be made of connected components or connected subgraphs. Since every connected component has at most $m$ vertices, using the inductive hypothesis we can create a maximal tree for each of these components. Furthermore, since $G$ was originally connected, there must be a vertex $u_i$ in each connected component $i$ that is connected to $v$. By adding an edge that connects $v$ to $u_i$ for all connected components $i$, the resulting graph, $T$ includes all vertices of $G$, and is connected since all points, if they are from different connected components, can be accessed through a path containing $v$. We prove by contradiction that there are no loops in $T$. Suppose the vertex $v$ was part of the loop. Since $v$ is connected to each connected component by precisely one edge, that means that the connected components were not disconnected, which is a contradiction. Now suppose $v$ is not part of loop, and instead another vertex $u$ is part of a loop. Since $v$ is not in the loop, the loop must be between vertices of the same connected component. Since a maximal tree was constructed for each connected component, there cannot exist a loop between vertices of the same connected component, which is a contradiction. Therefore $T$ does not contain any loops. Since $T$ is connected, does not contain any loops, and contains all vertices of $G$, $T$ is a maximal tree. Therefore any connected graph $G$ with $m+1$ vertices has a maximal tree.

Thus, by the principle of mathematical induction, any connected graph $G$ with $n \in \mathbb{N}$ vertices has a maximal tree.

□

**Lemma 4.11.** *For any tree $\tau$, if $\tau$ has $X$ vertices, then $\tau$ has $X-1$ edges where $X \in \mathbb{N}$.*

*Proof.* We will do this by **mathematical inducton** on the number of vertices. Let $P(X)$ be the statement that for a tree $\tau$ with $X$ vertices, $\tau$ has $X-1$ edges, where $X \in \mathbb{N}$.

**Base Case:** Let $X = 1$. Since an edge only exists between 2 vertices, there are 0 edges. Therefore, the base case, $P(1)$ is true.

**Inductive Hypothesis:** Suppose that $P(k)$ is true. Therefore, for any tree, $\tau$ with $k$ vertices, $\tau$ has $k-1$ edges where $k \in \mathbb{N}$.

**Inductive Step:** Let $\tau$ be a tree with $k+1$ vertices. We want to show that $\tau$ has $k$ edges.

Since $\tau$ is a tree, it is a finite, connected graph with no cycles. If we draw a maximal path in $\tau$, the longest possible path starting from some vertex, it must end at a vertex with no other adjacent vertices. This endpoint cannot be the starting vertex, as that would form a cycle, which contradicts the definition of a tree.

Therefore, there must exist at least one vertex in $\tau$ that is connected to only one edge. Removing this vertex and its corresponding edge, we get a subgraph that remains connected and contains no cycles. Therefore the subgraph is still a tree. This is because removing a vertex cannot introduce a cycle since all other edges and paths remain unchanged.

By the inductive hypothesis, there exist precisely $k-1$ edges in this subgraph. Since we removed one edge and one vertex from the original structure, this implies that the original tree has $k+1$ vertices and $k$ edges. Therefore, $P(k+1)$ is true.

Since $P(k)$ implies $P(k+1)$, and the base case is true, by the Principle of Mathematical Induction, $P(X)$ is true for any $X \in \mathbb{N}$. Therefore, for any tree $\tau$, if $\tau$ has $X$ vertices, $\tau$ has $X-1$ edges. □

## 5. The Euler Characteristic

In this section, we will use ideas from earlier sections to define the Euler characteristic and explore its fundamental properties. This section develops key lemmas and theorems, demonstrating how Euler's characteristic behaves for graphs and surfaces, and establishing its invariance under homeomorphism.

**Definition 5.1.** Let $L$ be a finite, simplical complex of dimension $n$. Then its **Euler characteristic**, $\chi(L)$, is defined by

$$\chi(L) = \sum_{i=0}^{n} (-1)^i \alpha_i$$

where $\alpha_i$ is the number of $i$ simplexes in $L$.

**Remark 5.2.** Notice that graphs are a type of simplical complex as it is a finite collection of points and line segments (edges). Therefore the Euler characteristic of a graph makes sense.

**Lemma 5.3.** *For any connected graph $\tau$, $\chi(\tau) \leq 1$. Moreover, $\chi(\tau) = 1$ if and only if $\tau$ is a tree.*

*Proof.* Let $\tau$ be a graph with $X$ vertices and $Y$ edges. By definition, vertices have dimension 0 while edges have dimension 1. Therefore, $\chi(\tau) = (-1)^0 X + (-1)^1 Y = X - Y$

Suppose $\tau$ is a tree. By Lemma 4.11, $Y = X - 1$ Therefore, the Euler characteristic is: $\chi(\tau) = X - (X - 1) = 1$. Therefore, if $\tau$ is a tree, $\chi(\tau) = 1$.

Suppose $\tau$ is not a tree. Therefore, there exists at least one loop in $\tau$. From Theorem 4.10, a maximal tree can be constructed for $\tau$. Suppose we remove $Z \in \mathbb{N}$ edges to form a maximal tree, $\tau'$. Thus, $\tau'$ has $Y - Z$ edges. Furthermore, a maximal tree has the same number of vertices as $\tau$. Thus, its Euler characteristic can be obtained:

$$\chi(\tau') = X - (Y - Z) = X - Y + Z$$

Since $Z \in \mathbb{N}$,

$$\chi(\tau) = X - Y < \chi(\tau')$$

Since $\tau'$ is a tree, we have shown in the previous case that $\chi(\tau') = 1$, therefore

$$\chi(\tau) < 1$$

Therefore for any connected graph $\tau$, $\chi(\tau) \leq 1$ and $\chi(\tau) = 1$ if and only if $\tau$ is a tree. ☐

**Definition 5.4.** A **combinatorial surface** $K$ is a simplicial complex such that $K$ is

- connected
- each of its edges is part of exactly two faces
- it is 2-dimensional
- each vertex is part of at least three triangles.

**Example 5.5.** Figure 6 shows examples of two structure that are combinatorial surfaces and a structure that is not a combinatorial surface. The left structure is a tetrahedron, and the middle shows two tetrahedrons glued together on a common surface. Since both of these figures are simplicial complexes that fulfill the four conditions, they are combinatorial surfaces.

The right structure shows two tetrahedrons that share only a common edge. This common edge between the two tetrahedrons is shared by three surfaces, violating the second condition and is thus not a combinatorial surface.

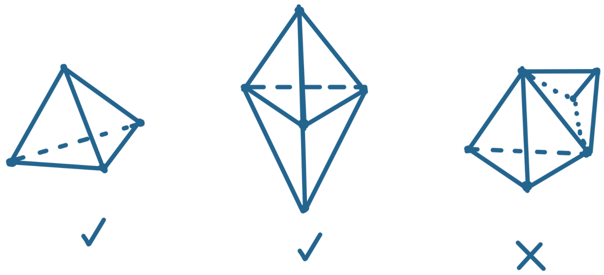

FIGURE 6. Examples and Non-Example of Combinatorial Surfaces

**Definition 5.6.** For any combinatorial surface $K$ we can fix a maximal tree for the graph consisting of its vertices and edges. The **dual graph** is constructed by placing a vertex at the center of each face of $K$. Then, an edge is added between two vertices of adjacent faces if and only if the shared edge is not in the maximal tree.

**Example 5.7.** Figure 7 shows the construction of the dual graph (black), for a tetrahedron (blue) with the vertices $v_1, v_2, v_3, v_4$. The maximal tree (yellow) contains the edges $(v_1, v_4), (v_2, v_4), (v_3, v_4)$. The vertices, $v_5, v_6, v_7, v_8$ are the center vertex for each face of the tetrahedron. More specifically:

- $v_5$ is the center vertex for the face with edges $(v_1, v_2), (v_2, v_4), (v_4, v_1)$.
- $v_6$ is the center vertex for the face with edges $(v_1, v_2), (v_1, v_3), (v_2, v_3)$.
- $v_7$ is the center vertex for the face with edges $(v_4, v_2), (v_4, v_3), (v_2, v_3)$.
- $v_8$ is the center vertex for the face with edges $(v_1, v_3), (v_4, v_3), (v_1, v_4)$.

Since the edges $(v_1, v_2)$, $(v_2, v_3)$, $(v_1, v_4)$ are not in the maximal tree, we can connect vertices $v_6$ and $v_5$, $v_6$ and $v_7$, and $v_6$ and $v_8$ respectively to create the dual graph.

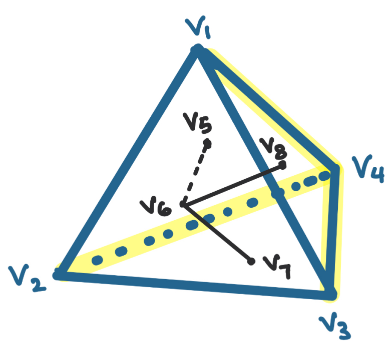


FIGURE 7. Dual Graph of a Tetrahedron

**Lemma 5.8.** *$\chi(K) \leq 2$ for any combinatorial surface $K$.*

*Proof.* Let $K$ be combinatorial surface with $X$ vertices, $Y$ edges, and $Z$ faces.

Consider a maximal tree $\tau$ of $K$. Therefore, $\tau$ has $X$ vertices and $Y'$ edges where $Y' \leq Y$. By Lemma 5.3, $\chi(\tau) = X - Y' = 1$.

Consider the dual graph $\tau'$ of $\tau$.Suppose for contradiction that $\tau'$ is disconnected. Then, there exist at least two adjacent faces in $K$ such that no sequence of edges in $\tau'$ connects their corresponding dual vertices. Since $\tau'$ includes only those edges of $K$ that are not in $\tau$, this implies that any path between these faces in $K$ must pass through an edge of the maximal tree $\tau$.

Consequently, the absence of a connecting path in $\tau'$ implies the existence of a cycle in $\tau$.

However, $\tau$ was defined to be a tree, and trees by definition contain no cycles. This is a contradiction, hence $\tau'$ is connected.

Since there exists a vertex of $\tau'$ for every face of $K$, $\tau'$ has $Z$ vertices. Furthermore, there exists an edge of $\tau'$ for all edges in $K$ that are not in $\tau$. Therefore, $\tau'$ has $Y - Y'$ edges. Therefore, $\chi(\tau') = Z - (Y - Y')$ Observe that

$$\chi(K) = X - Y + Z = (X - Y') + (Z - (Y - Y')) = \chi(\tau) + \chi(\tau')$$

Since $\chi(\tau) = 1$ and $\chi(\tau') \leq 1$,

$$\chi(K) \leq 2$$

Therefore, for any combinatorial surface $K$, $\chi(K) \leq 2$. $\square$

The following lemma gives a way of computing the Euler characteristic of a union.

**Lemma 5.9.** *Let $K, L$ be simplical complexes interesecting in a common subcomplex. Then $\chi(K \cup L) = \chi(K) + \chi(L) - \chi(K \cup L)$.*

*Proof.* Let $K$ have $X_K, Y_K, Z_K \in \mathbb{N}$ vertices, edges, and faces. Similarly, let $L$ have $X_L, Y_L, Z_L \in \mathbb{N}$ vertices, edges and faces. Consider $A = X \cap L$ and let $X_A, Y_A, Z_A \in \mathbb{N}$ denote the number of vertices, edges, and faces in the intersecting subcomplex.

Consider $K \cup L$. Since $K$ and $L$ share $X_A$ vertices, there are $X_K + X_L - X_A$ vertices in $K \cup L$. With similar reasoning, there are $Y_K + Y_L - Y_A$ edges and $Z_K + Z_L - Z_A$ faces in $K \cup L$. Therefore,

$$\chi(K \cup L) = (X_K + X_L - X_A) - (Y_K + Y_L - Y_A) + (Z_K + Z_L - Z_A)$$

By reorganizing the right hand side,

$$\chi(K \cup L) = (X_K - Y_K + Z_K) + (X_L - Y_L - Z_L) - (X_A - Y_A + Z_A)$$

By definition of Euler characteristic,

$$\chi(K \cup L) = \chi(K) + \chi(L) - \chi(A)$$

Since $A = K \cap L$,

$$\chi(K \cup L) = \chi(K) + \chi(L) - \chi(K \cap L)$$

$\square$

**Definition 5.10.** A **barycentric subdivision** of a combinatorial surface is a refinement of its triangulation obtained by the following process:

- Place a new vertex at the center of each face ,
- Place a new vertex at the midpoint of each edge,

- Then, for each triangle, connect:
  - the vertex at the face center to the vertex at the midpoints of the original three edges
  - the vertex at the face center to the vertices of the original three edges
  - each vertex at an edge midpoint to vertices that were part of the original face.

This process ultimately divides each original triangle into six smaller triangles, resulting in a finer triangulation of the original surface.

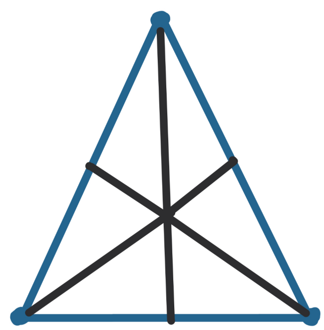

FIGURE 8. Example of Barycentric Subdivision

**Lemma 5.11.** *The Euler characteristic of a combinatorial surface is left unchanged by barycentric subdivision.*

*Proof.* Consider a combinatorial surface with at most $n$ faces. Suppose it has $X$ vertices, $Y$ edges, and $Z$ faces. Let $K'$ be the barycentric subdivision of this surface. By definition of barycentric subdivision, in addition to the vertices in $K$, there also exists a vertex in $K'$ for each edge and face in $K$. Therefore, there are $X + Y + Z$ vertices in $K'$. Since each edge is split into 2 edges and the center vertex in each face is connected to 6 different edges, there are $2Y + 6Z$ edges in $K'$. Lastly, since 6 new faces are created for every face, there are $6Z$ faces in $K'$. Therefore, the Euler characteristic of $K'$ is

$$\chi(K') = (X + Y + Z) - (2Y + 6Z) + 6Z = X - Y + Z$$

Since $\chi(K) = X - Y + Z$,

$$\chi(K') = \chi(K)$$

Therefore the Euler characteristic does not change with barycentric subdivision. □

**Theorem 5.12.** *If two triangulable topological spaces, $X, Y$ are homeomorphic to each other, then they share the same Euler characteristic.*

*Proof.* Let $X$ and $Y$ be triangulable topological spaces that are homeomorphic to each other. Then there exist homeomorphisms

$$f : |K_x| \to X \quad \text{and} \quad g : |K_y| \to Y,$$

where $K_x$ and $K_y$ are combinatorial surfaces. Since $X$ and $Y$ are homeomorphic, their triangulations $K_x$ and $K_y$ are also homeomorphic as topological spaces. Using the following definition and results from piecewise-linear (PL) topology,

- **Definition**: Two simplicial complexes, $K$ and $L$, are **PL homeomorphic** if there exists a subdivion $K'$ of $K$ and a subdivion $L'$ of $L$ such that $K'$ and $L'$ are homeomorphic to each other.
- **Theorem**: Any homeomorphism between finite simplicial complexes can be approximated by a PL homeomorphism. (See [6], and [7] Section 1, Theorem B)
- **Theorem**: Two finite simplicial complexes related by a PL homeomorphism have a common subdivision (See [4], Theorem 1.3).

we can derive that there exist integers $m, n \geq 0$ such that the structure after $m$ barycentric subdivisions of $K_x$, denoted $\mathrm{sd}^m K_x$, and $n$ barycentric subdivisions of $K_y$, denoted $\mathrm{sd}^n K_y$, are simplicially isomorphic:

$$\phi : \mathrm{sd}^m K_x \xrightarrow{\cong} \mathrm{sd}^n K_y.$$

Since simplicially isomorphic structure share the same number of vertices, edges, and faces, they share the same Euler characteristic. Additionally, by Lemma 5.11, the Euler characteristic is invariant under barycentric subdivision. Therefore, we get the following result:

$$\chi(K_x) = \chi\left(\mathrm{sd}^m K_x\right) = \chi\left(\mathrm{sd}^n K_y\right) = \chi(K_y).$$

Thus, the Euler characteristics of $K_x$ and $K_y$, and therefore of $X$ and $Y$, are equal. □

**Remark 5.13.** Additionally, further applications of the Euler characteristic includes the Classification Theorem, which includes the idea that any orientable surface is homeomorphic to a genus-g surface (See [3], Section 7.1).

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