

REPRESENTATION OF THE BRAID GROUP

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ABSTRACT. In this expository paper, we study the braid group B_n on n strands. First, we define B_n and show that there exists a group homomorphism between B_n and the mapping class group of the n -punctured disk D_n^2 . The mapping class group then acts on the fundamental group $\pi_1(D_n^2)$, which is naturally isomorphic to the free group F_n on n generators. Thus, B_n can be represented as a subgroup of $\text{Aut}(F_n)$, the automorphism group of the free group, aligning with the Artin representation of B_n that is defined in purely algebraic terms.

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1. INTRODUCTION TO THE BRAID GROUPS

There are several geometric interpretations of the braid group, some of which we will explore in this paper. Before defining the braid group, we first introduce the notion of a braid. We can think of a braid as braided strings.

Definition 1.1. A *braid* on n strands consists of n non-intersecting strings suspended between two parallel horizontal bars, such that each string runs monotonically from the top bar to the bottom bar. Two braids are considered equivalent if one can be continuously deformed into the other while keeping the endpoints of the strings fixed.

We can represent a braid geometrically through a braid diagram.

Definition 1.2. A *braid diagram* is a projection of the braid onto the plane such that no two crossings in the braid occur at the same height.

The set of braids on n strands forms a group under concatenation of braid diagrams, called the *braid group* B_n . The group operation is defined by stacking one braid beneath another, producing a new braid diagram.

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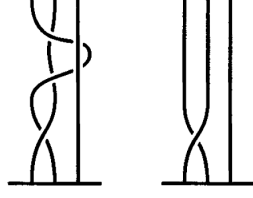


FIGURE 1. These two braids are equivalent through continuous deformation. Image credit of [1] Figure 5.33

Definition 1.3. The *braid group* B_n on n strands is the group whose elements are equivalence classes of braid diagrams on n strands, with the group operation given by concatenation, i.e. stacking one braid diagram beneath the other.

Note that the group operation is associative; the *empty braid* (all strands vertical, with no crossings) serves as the identity element; and each braid has an inverse obtained by reflecting the braid vertically.

We now describe the braid group algebraically in terms of generators and relations, as originally defined by Emil Artin in [2].

Proposition 1.4. The braid group B_n on n strands is the group generated by $\sigma_1, \dots, \sigma_{n-1}$ with relations:

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i & \text{for } |i - j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } 1 \leq i \leq n - 2. \end{aligned}$$

We give a sketch on how the braid group has such a representation. Intuitively, we can interpret σ_i to denote a crossing in which the i th strand passes *over* the $(i + 1)$ st strand. Similarly, its inverse σ_i^{-1} denotes the i th strand crossing *under* the $(i + 1)$ st strand. One can check that concatenating these two braid diagrams gives rise to the empty braid. The identity element is represented by the empty word.

Example 1.5. Consider the braid group B_4 with generators $\sigma_1, \sigma_2, \sigma_3$. Let

$$\beta = \sigma_1 \sigma_3 \sigma_2 \quad \text{and} \quad \alpha = \sigma_2^{-1} \sigma_3^{-1} \sigma_1^{-1}.$$

Note that β and α are inverses of each other in B_4 . Thus, concatenating them gives the identity braid:

$$\beta \cdot \alpha = \beta \cdot \beta^{-1} = (\sigma_1 \sigma_3 \sigma_2) (\sigma_2^{-1} \sigma_3^{-1} \sigma_1^{-1}) = id.$$

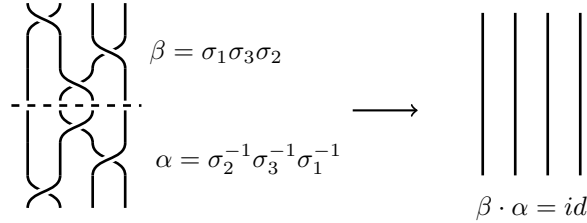


FIGURE 2. Illustration of inverses and the identity in Example 1.5

Next, we illustrate the two relations. The first relation describes the fact that distant crossings commute, while the second describes how adjacent crossings interact. These two relations formally define when two braids are equivalent.

Example 1.6. Consider the braid group B_4 . By the relations in its representation $\sigma_1\sigma_3 = \sigma_3\sigma_1$ and $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. Geometrically, as seen below, the braid represented by $\sigma_1\sigma_3$ can be continuously deformed into the braid represented by $\sigma_3\sigma_1$. Similarly, the braid represented by $\sigma_1\sigma_2\sigma_1$ can be continuously deformed into the braid represented by $\sigma_2\sigma_1\sigma_2$.

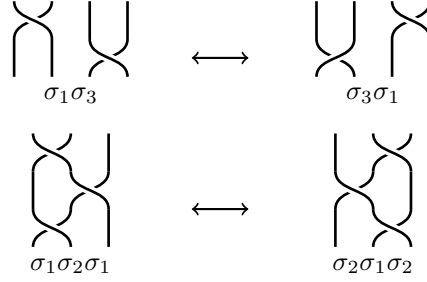


FIGURE 3. The two braid relations

2. MAPPING CLASS GROUP OF AN n -PUNCTURED DISK

So far we have viewed braid groups both geometrically and algebraically. We now connect them with topology by introducing mapping class groups, which provide a powerful framework for studying braids. In this section we show that there exists a group homomorphism between B_n and the mapping class group of a n -punctured disk D_n^2 . We begin by defining the mapping class group.

Definition 2.1. Let X and Y be topological spaces. A map $\varphi : X \rightarrow Y$ is called a *homeomorphism* if φ is a continuous bijection, and the inverse map $\varphi^{-1} : Y \rightarrow X$ is also continuous. If such a map exists, we say that X and Y are *homeomorphic*.

Definition 2.2. Let X and Y be topological spaces, and let $f, g : X \rightarrow Y$ be two homeomorphisms. We say that f and g are *isotopic* if there exists a continuous map

$$F : X \times [0, 1] \rightarrow Y$$

such that

- (1) $F(x, 0) = f(x)$ for all $x \in X$,
- (2) $F(x, 1) = g(x)$ for all $x \in X$,
- (3) For each $t \in [0, 1]$, the map $F_t : X \rightarrow Y$ defined by $F_t(x) := F(x, t)$ is a homeomorphism.

Such a map F is called an *isotopy* from f to g . Intuitively, f can be continuously deformed into g through homeomorphisms.

Definition 2.3. The *mapping class group* of a topological space X , denoted $\text{Mod}(X)$, is the group

$$\text{Mod}(X) = \text{Homeo}^+(X) / \sim,$$

where $\text{Homeo}^+(X)$ is the group of orientation-preserving homeomorphisms of X , where two homeomorphisms are equivalent if they are isotopic. If X is a manifold

with boundary, then homeomorphisms are required to fix the boundary pointwise throughout the isotopy. The group operation is composition.

Next, we will establish that there exists a group homomorphism from B_n to the mapping class group of the n -punctured disk D_n^2 . To establish the connection, we begin with a classical result known as Alexander's trick, which shows that the mapping class group of the closed disk is trivial.

Lemma 2.4. *The mapping class group of the closed disk is trivial:*

$$\text{Mod}(D^2) = \{\text{id}\}.$$

Proof. Let $f : D^2 \rightarrow D^2$ be a homeomorphism that fixes the boundary pointwise. We want to show that f is isotopic to the identity. For $t \in [0, 1]$ define a radius $r_t := 1 - t \in (0, 1]$. For $x \in D^2$ set

$$H(x, t) = \begin{cases} r_t f\left(\frac{x}{r_t}\right), & \|x\| \leq r_t, \\ x, & \|x\| \geq r_t. \end{cases}$$

Define $H(x, 1) := x$ for all $x \in D^2$. Intuitively, for each t we apply f to the smaller concentric disk of radius r_t and then rescale back to radius r_t , leaving the outer annulus $\{r_t \leq \|x\| \leq 1\}$ fixed pointwise.

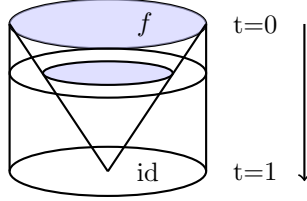


FIGURE 4. Cylinder $D^2 \times [0, 1]$ with cone inside. The unshaded annulus corresponds to the identity, while the shaded disk shows the shrinking action of f

We now verify the required properties for an isotopy. First we check the end points. For $t = 0$ we have $r_0 = 1$, so

$$H(x, 0) = 1 \cdot f\left(\frac{x}{1}\right) = f(x),$$

for all $x \in D^2$. Thus $H(x, 0) = f(x)$. For $t = 1$, we have $r_1 = 0$, so $H(x, 1) = x$ for all $x \in D^2$. Next note that if $\|x\| = 1$ then for every t we have $\|x\| \geq r_t$, hence $H(x, t) = x$. Thus the isotopy fixes ∂D^2 pointwise.

Next we check the continuity of H . On the regions $\{\|x\| < r_t\}$ and $\{\|x\| > r_t\}$ the map $(x, t) \mapsto H(x, t)$ is a composition of continuous maps, hence continuous. On the boundary $\|x\| = r_t$, the inner formula gives

$$H(x, t) = r_t f\left(\frac{x}{r_t}\right) = r_t \cdot \frac{x}{r_t} = x,$$

since $\frac{x}{r_t} \in \partial D^2$ and f fixes the boundary. Thus the two cases agree, so H is continuous everywhere.

Next we check that for all $t \in [0, 1)$, the map $H(\cdot, t)$ is a homeomorphism. On the

annulus $\{r_t \leq \|x\| \leq 1\}$ we have $H(x, t) = x$, so it restricts to the identity. On the inner disk $\{\|x\| \leq r_t\}$ we can write

$$H(x, t) = r_t f\left(\frac{x}{r_t}\right),$$

which is the conjugate of f by the radial rescaling $x \mapsto \frac{x}{r_t}$. Thus $H(\cdot, t)$ is a homeomorphism on the inner disk as well. Since the two definitions agree on the boundary $\|x\| = r_t$, it follows that $H(\cdot, t)$ is a homeomorphism of all of D^2 .

Moreover, its inverse is given explicitly by

$$H(\cdot, t)^{-1}(y) = \begin{cases} r_t f^{-1}\left(\frac{y}{r_t}\right), & \|y\| \leq r_t, \\ y, & \|y\| \geq r_t, \end{cases}$$

which is continuous for the same reasons that H is continuous. Hence $H(\cdot, t)$ is a homeomorphism for each $t \in [0, 1]$.

Thus we have proved that every boundary-preserving homeomorphism of the disk is isotopic to the identity, so $\text{Mod}(D^2) = \{\text{id}\}$. $\square$

Definition 2.5. Let $(G, \cdot)$ and $(H, \cdot)$ be two groups. A map $\varphi : G \rightarrow H$ is called a *group homomorphism* if for all $g_1, g_2 \in G$,

$$\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2).$$

In other words, φ preserves the group operation. If φ is also bijective, it is called a *group isomorphism*.

Theorem 2.6. *There exists a group homomorphism*

$$\rho : \text{Mod}(D_n^2) \longrightarrow B_n,$$

sending each mapping class to a braid.

Proof. Let $[f] \in \text{Mod}(D_n^2)$ be a mapping class, represented by a homeomorphism

$$f : D_n^2 \rightarrow D_n^2$$

that fixes the boundary ∂D^2 pointwise. Fill in the punctures by marked points $p_1, p_2, \dots, p_n$, and extend f to a homeomorphism $f' : D^2 \rightarrow D^2$ of the closed disk. f' necessarily permutes the n marked points $\{p_1, \dots, p_n\}$.

By Lemma 2.4, f' is isotopic to the identity id_{D^2} . Namely, there exists an isotopy

$$F_t : D^2 \rightarrow D^2, \quad t \in [0, 1],$$

such that $F_0 = \text{id}_{D^2}$ and $F_1 = f'$ (note that we reversed the time parameter in Lemma 2.4). Now, consider the paths traced by the marked points under this isotopy:

$$\gamma_i(t) = F_t(p_i), \quad i = 1, \dots, n.$$

The n paths $(\gamma_1, \dots, \gamma_n)$ form a braid in B_n . Define

$$\rho([f]) := (\gamma_1, \dots, \gamma_n) \in B_n.$$

Next we show that this map is well-defined. Suppose $f_0, f_1 : D_n^2 \rightarrow D_n^2$ are isotopic homeomorphisms relative to the boundary. Then any extensions $f'_0, f'_1 : D^2 \rightarrow D^2$ are isotopic in D^2 . Thus the induced braids of f'_0 and f'_1 are isotopic and represent the same element in B_n :

$$f_0 \simeq f_1 \text{ in } D_n^2 \implies \rho([f_0]) = \rho([f_1]).$$

Finally, composition of mapping classes corresponds to concatenation of isotopies, which corresponds to multiplication in B_n . Therefore, ρ is a group homomorphism. $\square$

Example 2.7. Let D^2 be a closed disk with three mark points P_1, P_2, P_3 . Suppose f is a half twist between P_1 and P_2 defined as follow:

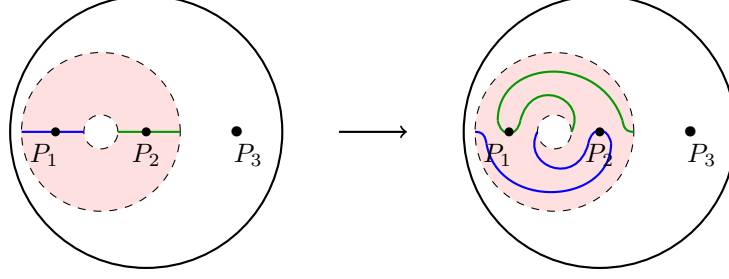


FIGURE 5. Illustration of a half twist. The green and blue lines are drawn to illustrate the effect of the half twist

Note that $f(P_1) = P_2$, $f(P_2) = P_1$, and $f(P_3) = P_3$. Outside the pink annulus, f restricts to the identity. On the annulus, f restricts to a homeomorphism. Since $f|_{\partial(\text{Space})} = id$, f is a homeomorphism on D^2 . Now apply the Alexander's trick (Lemma 2.4) to f , and we obtain the braid induced by the half twist.

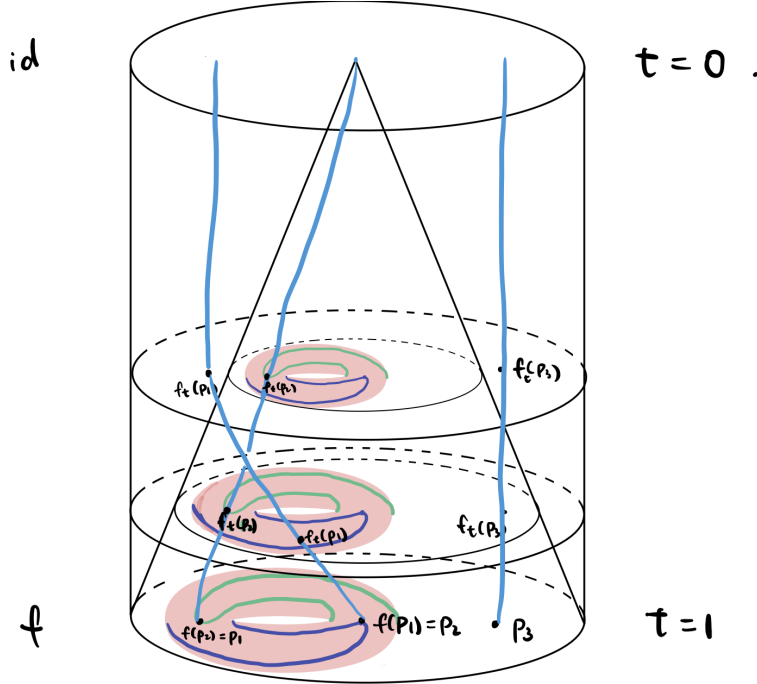


FIGURE 6. Applying the Alexander's trick (Lemma 2.4) to the half twist. Image credit of Zhong Zhang

Theorem 2.8. *The braid group B_n is isomorphic to the mapping class group of the n -punctured disk D_n^2 :*

$$B_n \cong \text{Mod}(D_n^2).$$

Theorem 2.8 establishes that the braid group B_n can be understood as the mapping class group of a disk with n punctures. The proof of this theorem is quite involved, as it requires introducing an additional geometric interpretation of braid groups; therefore, it is omitted here (see [3] for a complete proof of this theorem).

Remark 2.9. Note that the notations for the braid group and the mapping class group are conventionally in opposite order. For the mapping class group, the composition of homeomorphisms is written so that the function written on the left is applied *after* the one on the right, i.e.,

$$(f \circ g)(x) = f(g(x)).$$

In contrast, for the braid group, concatenation is read from left to right: the braid $\sigma_i \sigma_j$ means that σ_i is performed *first*, followed by σ_j .

3. FUNDAMENTAL GROUP OF AN n -PUNCTURED DISK

Now that we have established the connection between braid group and the mapping class group of n -punctured disk, our next step is to explore how these homeomorphisms affect the underlying geometry of the disk. To do this, we introduce the concept of the fundamental group, a powerful algebraic invariant that captures the 'holes' in a topological space.

Definition 3.1. Let X be a topological space and let $\gamma_1, \gamma_2 : [0, 1] \rightarrow X$ be two paths such that

$$\gamma_1(1) = \gamma_2(0).$$

The *concatenation* of γ_1 and γ_2 , denoted $\gamma_1 * \gamma_2$, is the path

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t), & 0 \leq t \leq \frac{1}{2}, \\ \gamma_2(2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Intuitively, the concatenated path moves through the first path and then the second path at twice the original speed.

Definition 3.2. Let X be a topological space and let $x_0 \in X$ be a chosen basepoint. Consider the set of all continuous maps (loops)

$$\gamma : [0, 1] \rightarrow X \quad \text{with} \quad \gamma(0) = \gamma(1) = x_0.$$

Two loops γ_0, γ_1 are said to be *homotopic relative to endpoints* if there exists a homotopy

$$H : [0, 1] \times [0, 1] \rightarrow X$$

such that $H(s, 0) = \gamma_0(s)$, $H(s, 1) = \gamma_1(s)$, and $H(0, t) = H(1, t) = x_0$ for all $t \in [0, 1]$.

The *fundamental group* of X at the basepoint x_0 , denoted $\pi_1(X, x_0)$, is the set of equivalence classes of loops with concatenation as the group operation.

The group operation is well-defined. Suppose $[\gamma_0] = [\gamma_1]$ in $\pi_1(X, x_0)$. Then for any loop γ_2 , we have

$$[\gamma_1] * [\gamma_2] = [\gamma_1 * \gamma_2] = [\gamma_0 * \gamma_2] = [\gamma_0] * [\gamma_2],$$

so concatenation is well-defined on homotopy classes.

We can check the three group axioms and describe the group structure in more detail. The set $\pi_1(X, x_0)$ is a group under concatenation of loops. Closure holds because the concatenation of two loops based at x_0 is again a loop at x_0 . Associativity holds up to homotopy, which suffices since elements are homotopy classes. The identity element is represented by the constant loop $e(t) = x_0$, and the inverse of a loop γ is given by $\gamma^{-1}(t) = \gamma(1-t)$, which traverses γ in the reverse direction.

Next we give two examples of fundamental groups

Example 3.3. Consider a plane $\mathbb{R}^2$ and let some point $x_0 \in \mathbb{R}^2$ be the basepoint. Any loop $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ based at x_0 can be continuously contracted to the constant loop at x_0 . Hence,

$$\pi_1(\mathbb{R}^2, x_0) \cong \{\text{id}\}.$$

Example 3.4. Consider a circle S^1 and let some point $x_0 \in S^1$ be the basepoint. A loop $\gamma : [0, 1] \rightarrow S^1$ based at a point x_0 can be classified by its *winding number*, i.e. the number of times γ winds around the circle (positive for counterclockwise, negative for clockwise). Notice that two loops are homotopic if and only if they have the same winding number. Therefore,

$$\pi_1(S^1, x_0) \cong \mathbb{Z},$$

with group operation given by addition of integers, corresponding to concatenation of loops (adding winding numbers).

Notice that the fundamental group $\pi_1(X, x_0)$ measures the “holes” in a topological space X . Any loop that does not go around a hole in the topological space is homotopic (can be continuously contracted) to the constant loop at x_0 . If a loop cannot be contracted, then there must be a hole that the loop winds around.

As seen in our two examples: the fundamental group of a plane $\pi_1(\mathbb{R}^2, x_0)$ is trivial since the plane has no holes and every loop can shrink to a point. The fundamental group of a circle $\pi_1(S^1, x_0) \cong \mathbb{Z}$, because loops can wind around the circular hole any integer number of times.

Thus, the fundamental group gives an algebraic way to “detect and count” holes in a space.

The objective of the rest of the section is to show that the fundamental group of n -punctured disk—our main subject of study—is equivalent to the free group. Homotopy equivalent spaces have the same fundamental groups, so we are going to deformation retract D_n to a wedge of circles and compute its fundamental group.

Definition 3.5. Let X be a topological space and $A \subseteq X$ a subspace. A *deformation retraction* of X onto A is a continuous map

$$H : X \times [0, 1] \rightarrow X$$

such that for all $x \in X$ and $a \in A$:

$$H(x, 0) = x, \quad H(x, 1) \in A, \quad H(a, t) = a \text{ for all } t \in [0, 1].$$

In this case, we say that X *deformation retracts* onto A , denoted $X \simeq A$.

Lemma 3.6. *Let D_n^2 be the disk with n punctures. Then D_n^2 deformation retracts onto a wedge of n circles:*

$$D_n^2 \simeq \bigvee_{i=1}^n S_i^1.$$

Proof. Choose a basepoint $x_0 \in \partial D^2$. For each puncture p_i , draw a simple path from x_0 to a small circle around p_i , such that these paths are disjoint except at x_0 . Let S_i^1 be a loop around the i -th puncture based at the endpoint of the corresponding path.

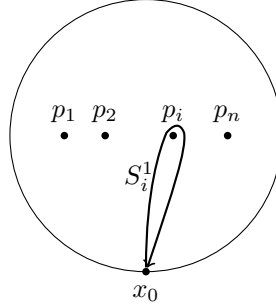


FIGURE 7. An n -punctured disk with the chosen simple path S_i^1

We can then continuously “slide” each point of D_n^2 along its path toward the corresponding loop, collapsing the rest of the disk onto the wedge of circles. This defines a deformation retraction

$$H : D_n^2 \times [0, 1] \rightarrow D_n^2, \quad H(x, 0) = x, \quad H(x, 1) \in \bigvee_{i=1}^n S_i^1,$$

which fixes the wedge of circles pointwise. $\square$

Next, we will introduce what is a free group.

Definition 3.7. Let S be a set. A *word* over S is a finite sequence of elements from the set

$$S \cup S^{-1} = \{s \mid s \in S\} \cup \{s^{-1} \mid s \in S\},$$

where s^{-1} is a formal inverse of s .

Explicitly, a word is an expression of the form

$$w = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n},$$

where $s_i \in S$ and $\epsilon_i \in \{1, -1\}$ for $i = 1, \dots, n$. A word $w = s_1 s_2 \cdots s_n$ is called *reduced* if it contains no adjacent pairs of the form ss^{-1} or $s^{-1}s$.

Definition 3.8. Let S be a set. The *free group* on S , denoted $F(S)$, is the set of all reduced words over S , with group operation given by *concatenation followed by reduction*. The identity element is the *empty word*, and the inverse of a word $w = s_1 s_2 \cdots s_n$ is $w^{-1} = s_n^{-1} \cdots s_2^{-1} s_1^{-1}$. All elements in S are the *generators* of the free group.

A free group is called ‘free’ because it has no defining relations other than the cancellation of inverses.

Lemma 3.9. *The fundamental group of the n -punctured disk D_n^2 is isomorphic to the free group F_n on n generators $\{a_1, a_2, \dots, a_n\}$.*

Proof. Choose a basepoint $x_0 \in \partial D^2$. By Proposition 3.6, D_n^2 deformation retracts onto a wedge of n circles:

$$D_n^2 \simeq \bigvee_{i=1}^n S_i^1,$$

where each circle S_i^1 corresponds to a loop around the i -th puncture. Let $r : D_n^2 \rightarrow \bigvee_{i=1}^n S_i^1$ be the deformation retraction.

A deformation retraction induces an isomorphism on fundamental groups:

$$r_* : \pi_1(D_n^2, x_0) \longrightarrow \pi_1\left(\bigvee_{i=1}^n S_i^1, x_0\right).$$

The fundamental group of a wedge of n circles is free on n generators. Let a_i be the loop based at x_0 that traverses the i -th circle once. Any loop in the wedge is homotopic (relative to x_0) to a finite concatenation of these loops and their inverses, and no nontrivial reduced word in the a_i and a_i^{-1} is homotopic to the constant loop. Hence,

$$\pi_1\left(\bigvee_{i=1}^n S_i^1, x_0\right) \cong F_n = \langle a_1, a_2, \dots, a_n \rangle.$$

Composing these facts gives the desired isomorphism:

$$\pi_1(D_n^2, x_0) \cong F_n.$$

□

4. MAPPING CLASS GROUP ACTING ON FUNDAMENTAL GROUP

In this section, we will show that a boundary-preserving homeomorphism of D_n^2 induces an automorphism of its fundamental group. In other words, we will explore how the mapping class group acts on the fundamental group of D_n^2 .

Lemma 4.1. *Let X, Y be topological spaces, and let $f : X \rightarrow Y$ be a continuous map that maps point $x_0 \in X$ to point $y_0 \in Y$. Then f induces a group homomorphism:*

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0), \quad [\gamma] \longmapsto [f \circ \gamma].$$

Proof. First, we show that f_* is well-defined. Suppose $\gamma_0, \gamma_1 : [0, 1] \rightarrow X$ are loops based at x_0 that are homotopic relative to endpoints. Then there exists a continuous map $H : [0, 1] \times [0, 1] \rightarrow X$ such that

$$H(s, 0) = \gamma_0(s), \quad H(s, 1) = \gamma_1(s), \quad H(0, t) = H(1, t) = x_0 \quad \text{for all } t.$$

Applying f , we have

$$[f \circ \gamma_0], [f \circ \gamma_1] \in \pi_1(Y, y_0),$$

and $(f \circ H) : [0, 1] \times [0, 1] \rightarrow Y$ is a homotopy between $f \circ \gamma_0$ and $f \circ \gamma_1$. Hence f_* is well-defined.

Next, we check that f_* is a group homomorphism.

- (1) Identity: Since $f(x_0) = y_0$, we have $f_*([c_{x_0}]) = [c_{y_0}]$, where c_{x_0} denotes the constant loop at x_0 .

(2) Compatibility: For $[\alpha], [\beta] \in \pi_1(X, x_0)$,

$$\begin{aligned} f_*([\alpha] \cdot_X [\beta]) &= f_*([\alpha\beta]) = [f \circ (\alpha\beta)] = [(f \circ \alpha)(f \circ \beta)] \\ &= [f \circ \alpha] \cdot_Y [f \circ \beta] = f_*([\alpha]) \cdot_Y f_*([\beta]). \end{aligned}$$

Thus f_* is a group homomorphism. $\square$

Corollary 4.2. *Let X and Y be topological spaces, and let $f : X \rightarrow Y$ be a homeomorphism mapping $x_0 \in X$ to $y_0 \in Y$. Then f induces a group isomorphism*

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0), \quad [\gamma] \mapsto [f \circ \gamma].$$

Proof. Since f is a homeomorphism, it is continuous and has a continuous inverse $f^{-1} : Y \rightarrow X$. By Lemma 4.1, f_* is a homomorphism.

To show injectivity, suppose

$$f_*([\gamma]) = [f \circ \gamma] = [c_{y_0}].$$

Applying the induced map $(f^{-1})_*$ gives

$$[\gamma] = (f^{-1})_*([f \circ \gamma]) = (f^{-1})_*([c_{y_0}]) = [c_{x_0}].$$

Thus no two distinct elements of $\pi_1(X, x_0)$ map to the same element, so f_* is injective.

To show surjectivity, let $[\delta] \in \pi_1(Y, y_0)$. Consider the loop $f^{-1} \circ \delta$ in X . Then

$$f_*([f^{-1} \circ \delta]) = [f \circ (f^{-1} \circ \delta)] = [\delta].$$

Hence f_* is surjective.

Since f_* is both a homomorphism and a bijection, it is a group isomorphism. $\square$

Remark 4.3. The corollary above shows that a homeomorphism induces an isomorphism of fundamental groups. In fact, a much stronger statement is true: homotopy equivalent spaces have isomorphic fundamental groups. The proof is not included in this paper for the sake of brevity (see section 3 of [11] for a complete proof).

With the group homomorphism between the mapping class group and the braid group defined in Section 2, we can now state the following theorem.

Theorem 4.4. *There is a natural action of the mapping class group of the n -punctured disk on its fundamental group. More precisely, for any basepoint $x_0 \in \partial D^2$, this induces a group homomorphism*

$$B_n \cong \text{Mod}(D_n^2) \longrightarrow \text{Aut}(\pi_1(D_n^2, x_0)),$$

sending the isotopy class of a homeomorphism to the induced automorphism of $\pi_1(D_n^2, x_0)$.

Proof. By Lemma 2.6, there exists a group homomorphism

$$\rho : \text{Mod}(D_n^2) \longrightarrow B_n,$$

where $\text{Mod}(D_n^2)$ denotes isotopy classes of homeomorphisms of D_n^2 fixing the boundary.

Let $[f] \in \text{Mod}(D_n^2)$. By definition, $f : D_n^2 \rightarrow D_n^2$ is a homeomorphism (up to isotopy) that fixes ∂D_n^2 pointwise. Choose a basepoint $x_0 \in \partial D^2$, so that $f(x_0) = x_0$. By Corollary 4.2, f induces a group isomorphism

$$f_* : \pi_1(D_n^2, x_0) \longrightarrow \pi_1(D_n^2, x_0), \quad [\gamma] \mapsto [f \circ \gamma],$$

so that $f_* \in \text{Aut}(\pi_1(D_n^2, x_0))$.

We must check that this assignment is well-defined on mapping classes, i.e., independent of the choice of representative f in its isotopy class. Suppose $f, g \in [f]$, so that there exists a continuous map $H : D_n^2 \times [0, 1] \rightarrow D_n^2$ with

$$H(x, 0) = f(x), \quad H(x, 1) = g(x), \quad H(x, t) = x \text{ for all } x \in \partial D^2.$$

For any loop $\gamma : [0, 1] \rightarrow D_n^2$ based at x_0 , define

$$F(s, t) := H(\gamma(s), t).$$

Then F is a homotopy (relative to the basepoint) between $f \circ \gamma$ and $g \circ \gamma$, since $F(0, t) = F(1, t) = x_0$. Therefore

$$f_*([\gamma]) = [f \circ \gamma] = [g \circ \gamma] = g_*([\gamma]),$$

showing that the induced map on π_1 depends only on the isotopy class $[f]$.

Composing with $\rho : \text{Mod}(D_n^2) \rightarrow B_n$, we obtain the desired, well-defined group homomorphism:

$$B_n \longrightarrow \text{Aut}(\pi_1(D_n^2, x_0)).$$

□

5. THE ARTIN REPRESENTATION

Definition 5.1. The *Artin representation* is the homomorphism

$$\Phi : B_n \rightarrow \text{Aut}(F_n)$$

defined on generators σ_i by:

$$\begin{aligned} x_i &\mapsto x_{i+1}, \\ x_{i+1} &\mapsto x_{i+1}^{-1} x_i x_{i+1}, \\ x_j &\mapsto x_j \quad \text{for } j \neq i, i+1. \end{aligned}$$

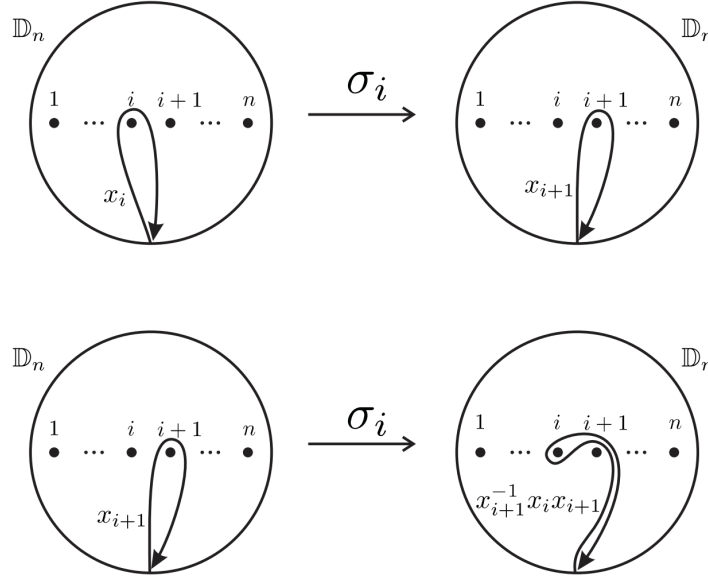


FIGURE 8. Image credit of [12] Figure 1.4

Theorem 5.2. *The induced action of $B_n \cong \text{Mod}(D_n^2)$ on $\pi_1(D_n^2)$ agrees with the Artin representation.*

Proof. Let $x_0 \in \partial D^2$ be a basepoint. By Lemma 3.9, $\pi_1(D_n^2, x_0) \cong F_n$, generated by loops $x_1, \dots, x_n$ around the n punctures. Consider the action of the braid generator $\sigma_i \in B_n$, which corresponds to swapping the i -th and $(i+1)$ -st strands with the i -th crossing over the $(i+1)$ -st. Geometrically, this induces a homeomorphism of D_n^2 up to isotopy which moves the punctures along the braid strands while fixing the boundary.

Tracing the loops around the punctures under this motion:

- The i -th loop x_i is carried along the i -th strand over the $(i+1)$ -st strand, resulting in the loop being conjugated by x_i as $x_i \mapsto x_i x_{i+1} x_i^{-1}$.
- The $(i+1)$ -st loop x_{i+1} is now based around the former i -th puncture, so $x_{i+1} \mapsto x_i$.
- Loops x_j for $j \neq i, i+1$ are unaffected by the braid move, so $x_j \mapsto x_j$.

This exactly matches the definition of $\Phi(\sigma_i)$. Since the geometric action respects the braid group relations, the induced map $B_n \rightarrow \text{Aut}(F_n)$ coincides with the Artin representation. $\square$

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REFERENCES

- [1] Adams, Colin Conrad. "5.4 Braids," in The knot book: an elementary introduction to the mathematical theory of knots. American Mathematical Soc., 2004.
- [2] Artin, Emil. "Theory of braids." *Annals of Mathematics* 48, no. 1 (1947): 101-126.
- [3] Farb, Benson, and Dan Margalit. A primer on mapping class groups. Vol. 49. Princeton university press, 2011.
- [4] Bardakov, Valeriy G., and Paolo Bellingeri. On representations of braids as automorphisms of free groups and corresponding linear representations. Vol. 670. Providence, RI: American Mathematical Society, 2016.
- [5] Birman, Joan S., and Tara E. Brendle. "Braids: a survey." In *Handbook of knot theory*, pp. 19-103. Elsevier Science, 2005.
- [6] Mustata, Maya. Topological tools for distinguishing knot types. <https://math.uchicago.edu/~may/REU2024/REUPapers/Mustata.pdf>
- [7] Myasnikov, Alexei. Notes on free groups. <https://www.math.unl.edu/~mbrittenham2/classwk/990s08/public/myasnikov.1.free.groups.pdf>
- [8] Srivastava, Eha. An introduction to the bureau representation of the artin braid group. <https://math.uchicago.edu/~may/REU2021/REUPapers/Srivastava.pdf>
- [9] Jackson, Craig H. "Braid group representations." Master's thesis, The Ohio State University, 2001.
- [10] Li, Ang. Fundamental groups and the Can Kampen's Theorem. <https://math.uchicago.edu/~may/REU2013/REUPapers/Li.pdf>
- [11] Bloom, Samuel. Fundamental groups and the Can Kampen's Theorem. <https://www.math.uchicago.edu/~may/VIGRE/VIGRE2009/REUPapers/Bloom.pdf>
- [12] González-Meneses, Juan. "Basic results on braid groups." In *Annales Mathématiques Blaise Pascal*, vol. 18, no. 1, pp. 15-59. 2011.