SOME SPECTRAL BOUNDS ON GRAPH INVARIANTS

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ABSTRACT. We investigate relationships between graph structure and the spectra of standard graph matrices (the adjacency matrix A and the Laplacian $\mathcal{L}$). Building on classical matrix-analytic techniques, we derive a collection of spectral inequalities that connect eigenvalue extrema and sums to combinatorial quantities such as clique number, co-clique number, and bounds on the Cheeger constant. Our arguments are presented with an emphasis on explicit, approachable proofs, making the techniques accessible to readers with varied backgrounds.

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1. INTRODUCTION

Graphs are combinatorial objects, but translating them to matrices exposes a linear–algebraic toolkit that is easier to work with and scales efficiently. The adjacency matrix A, degree matrix D, and Laplacian, $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$, preserve the underlying structure while enabling standard calculations: powers of A count walks, quadratic forms ($x^\top Ax$, $x^\top Lx$) aggregate adjacency and variation, and eigenvalues summarize global behavior through Rayleigh quotients and orthogonality. We begin by fixing notation and a few elementary facts, then build intuition on explicit spectra for basic families. Building on this, we connect the second eigenvalue to the cheeger constant, and the adjacency spectra yields bounds on the clique and co-clique numbers via Motzkin–Straus, an argument that builds on a previous

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bound given by Herbert Wilf. Throughout, the simplifying identity on d–regular graphs, $\mathcal{L} = I - \frac{1}{d}A$, allows adjacency and normalized–Laplacian statements to be translated without loss.

2. Preliminaries

This section fixes notation on graphs and the matrix objects we use and collects a few elementary facts that will be used repeatedly. Namely, we will define the concept of a graph, as well as construct relevant matrices.

2.1. Fundamental Objects.

Definition 2.1. A *graph* is a tuple, $G = (V, E)$. We say that the *size* of a graph is, $n = |V|$, and refer to elements of V as *vertices*. Elements of E are un-orded pairs of distinct vertices called *edges*. For ease of reference, the vertex set is often thought of as the collection of the first n natural numbers. Two basic associated objects are the *degree*, defined on individual vertices as $d(v) = |\{u \in V \mid \{u, v\} \in E\}|$, and the *volume*, defined on subsets $S \subseteq V$ as $\text{vol}(S) = \sum_{v \in S} d(v)$. We say that a graph is k-regular, if each vertex has degree k.

Some special graphs are given shorthand representations due to their frequent use. The *complete graph*, K_n, has an edge set containing each possible pair of distinct vertices, the *path graph*, P_n, is the graph where $\{i, i+1\} \in E$ for $i < n$, and the *cycle graph*, C_n, which is constructed from P_n by adding an additional edge connecting vertex 1 to vertex n.

Though graphs are useful mathematical objects, when their size increases, their natural structure is somewhat cumbersome. This fact motivates the construction of a nicer object which lends itself more naturally to analysis – matrix representations. These translate the combinatorial structure of vertices and edges into linear algebraic form. As will be discussed in further detail later on, powerful tools from spectral theory become available: eigenvalues provide compact numerical summaries of connectivity and expansion, while eigenvectors reveal hidden geometry and clustering.

Definition 2.2. Given a graph G, we define the *adjacency matrix* entry-wise as:

$$A_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

Even before considering the spectra, the adjacency matrix itself encodes connectivity. In particular, its powers count walks (sequences of edges) between vertices:

Proposition 2.3. *The (i, j) entry of A^k is exactly the number of walks of length k from vertex i to vertex j.*

Proof. By induction on k. For $k = 0$ we have $(A^0)_{ij} = 1$ for $i = j$ and $(A^0)_{ij} = 0$ otherwise, which counts length-0 walks. For $k = 1$ we have $(A)_{ij}$, which trivially counts walks of length 1.

Assume the claim holds for k. Then by the definition of matrix multiplication,

$$(A^{k+1})_{ij} = \sum_r (A^k)_{ir} \, A_{rj}.$$

By the induction hypothesis $(A^k)_{ir}$ is the number of walks of length k from i to r, and each such walk extends to j in exactly A_{rj} ways. Summing over r counts all walks of length $k+1$ from i to j, so the identity holds for $k+1$. □

Translating a graph into its adjacency matrix does not throw information away—it reorganizes it. The matrix captures the same object as the graph: who is connected to whom and how local neighborhoods fit together. Basic graph actions have clean matrix counterparts too, so constructions in the graph world become modular updates in the matrix world. Even repeated matrix multiplication mirrors combinatorial growth by counting possible step-by-step moves through the network.

This faithfulness matters. It means we can study a graph without leaving its structure behind, while gaining access to the organizing power of linear algebra: clear data layouts, scalable computations, and principled ways to summarize complex connectivity. The spectral viewpoint developed next builds on this foundation, using the matrix not just as a record of edges, but as a compact lens for comparing, classifying, and reasoning about graphs.

In many graphs, especially those with uneven degrees, it is helpful to study a matrix that reflects not just adjacency but also how connections scale with vertex degree. The Laplacian serves this role.

Definition 2.4. Given a graph G, we first define the matrix,

$$L_{ij} = \begin{cases} d(i) & \text{if } i = j \\ -1 & \text{if } \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

Additionally, let D be the matrix with value $D_{ij} = 0$ for $i \neq j$, and $D_{ij} = d(i)$ for $i = j$.

The *Laplacian matrix* is defined as $D^{-\frac{1}{2}} L D^{-\frac{1}{2}}$, $I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$ or explicitly as

$$\mathcal{L}_{ij} = \begin{cases} 1 & \text{if } i = j \\ -\frac{1}{\sqrt{d(i)d(j)}} & \text{if } \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

Note that in some literature, the matrix L is referred to as the *Combinatorial* Laplacian, and $\mathcal{L}$ is referred to as the *Normalized* Laplacian. For the purposes of this paper, we will strictly refer to $\mathcal{L}$ as the Laplacian. We conclude this subsection having fixed the matrix objects that encode a graph's structure: the adjacency matrix A and the Laplacian $\mathcal{L}$. With notation set, we next develop the linear-algebraic tools needed to extract spectral information from these matrices.

2.2. **Fundamental Properties.** We now record some of the fundamental algebraic properties which will aid our spectral analysis. Firstly, the spectral theorem for real symmetric matrices, which supplies the linear-algebraic structure used repeatedly below. In concrete terms, the theorem guarantees that any real symmetric matrix admits an orthonormal basis of eigenvectors and hence can be diagonalized by an orthogonal change of coordinates.

Theorem 2.5. *Let A be a real symmetric matrix. Then A has n real eigenvalues, and there exists a real orthonormal eigenbasis of A. Furthermore, A can be represented as*

$$A = UEU^T,$$

where U is an orthogonal matrix, and E is a real valued diagonal matrix. The multiset of non-zero values contained in E is called the spectra *of A.*

As each of the graph matrices that we introduced are clearly real-symmetric, this guarantees that each graph admits a multi-set of n real eigenvalues. As a standard convention, we will refer to the eigenvalues of $\mathcal{L}$ using the Greek letter λ, indexed in increasing order from 1 to n, and the eigenvalues of A using the Greek letter μ indexed in the same way. A proof of Theorem 2.5 can be found in section 7B of [2].

We now introduce the Rayleigh Quotient, which, in general, measures the degree to which a matrix augments a vector. For the Laplacian, the Rayleigh quotient measures the extent to which the values of a vertex weight vector, f remain consistent across well-connected regions of the graph, with smaller values indicating that f places most of its weight on well connected subsets.

Definition 2.6. Given a real symmetric matrix, M, the *Rayleigh Quotient*, of the matrix is a function $R_M : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$, such that

$$R_M(x) = \frac{x^T M x}{x^T x}.$$

Because the Rayleigh quotient evaluates to an eigenvalue whenever x is an eigenvector, it follows that minimizing or maximizing the quotient over all directions must produce the extreme eigenvalues of a symmetric matrix. Moreover, if the search is restricted to vectors orthogonal to the eigenspaces already identified, the next eigenvalues emerge in order. This principle is made precise in the Courant–Fischer theorem, which characterizes the smallest, largest, and successive eigenvalues as extremal values of the Rayleigh quotient subject to such orthogonality constraints.

Theorem 2.7 (Courant–Fischer)**.** *Let M be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ and corresponding eigenvectors $v_1, \ldots, v_n$. Then,*

$$\lambda_1 = \min_{\|x\|=1} x^T M x = \min_{x \neq 0} \frac{x^T M x}{x^T x}, \quad \lambda_2 = \min_{\substack{\|x\|=1 \\ x \perp v_1}} x^T M x = \min_{\substack{x \neq 0 \\ x \perp v_1}} \frac{x^T M x}{x^T x}$$

$$\lambda_n = \max_{\|x\|=1} x^T M x = \max_{x \neq 0} \frac{x^T M x}{x^T x},$$

$$\lambda_i = \min_{\substack{x \neq 0 \\ x \perp v_1, \ldots, v_{i-1}}} \frac{x^\top M x}{x^\top x}, \quad \text{for } 1 \leq i \leq n$$

A proof of the Courant Fischer eigenvalue characterization can be found in lecture 2 of [4]. Specializing Theorem 2.7 to the Laplacian shows that its eigenvalues are obtained by minimizing the associated Rayleigh quotient over orthogonality-constrained vectors. Writing this quotient in terms of the original Laplacian and the degree matrix makes explicit that the relevant quantity is the total squared difference of a function across edges, normalized by its degree-weighted mass. The following holds taking $f = D^{\frac{1}{2}} x$:

$$R_{\mathcal{L}}(x) = \frac{x^T \mathcal{L} x}{x^T x} = \frac{x^T D^{-\frac{1}{2}} L D^{-\frac{1}{2}} x}{x^T x} = \frac{f^T L f}{(D^{\frac{1}{2}} f)^T D^{\frac{1}{2}} f} = \frac{\sum_{\{u,v\} \in E} \left(f(u) - f(v)\right)^2}{\sum_{v \in V} d(v) f(v)^2}.$$

Thus, λ_n is the direction in which the total squared difference across edges is largest. Viewed as an embedding of vertices on the real line, the eigenvector,

v_n, therefore maximizes the aggregate squared edge-length under the unit-norm constraint. This gives a useful extremal picture complementary to the usual "minimization" viewpoint: while the eigenvector corresponding to λ_2 minimizes edge-variation subject to orthogonality and so is suited to partitioning, the top eigenvector stretches adjacent vertices apart as much as possible and hence highlights extremes and antipodal structure in the graph. This fact lends itself naturally to the optimal visualization of certain graphs. For explicit computation and exposition please see lecture 1 of [8]. In proofs one can exploit this maximization to construct cuts or test functions that witness large combinatorial separations or to reason about graph diameter-like behavior in algebraic terms.

3. Computation and Intuition

In the previous section we established the variational framework and min–max principles that link eigenvalues to combinatorial features of graphs. We now illustrate these ideas by computing the spectra of several classical graph families. Working through these examples not only demonstrates the mechanics of spectral calculations but also develops intuition for how graph structure is reflected in eigenvalues and eigenvectors. This intuition will serve as a guide in the remainder of the paper, where more general bounds and applications rely on recognizing the same patterns that appear transparently in these well-understood cases.

Remark 3.1. Spectral derivations are simplest for highly symmetric or d-regular graphs, since $D = dI$ reduces $\mathcal{L}$ to $I - \frac{1}{d}A$, so diagonalizing A immediately yields the spectra of $\mathcal{L}$. As graphs depart from regularity, D is no longer a scalar and the edge weights $\frac{1}{\sqrt{d(i)d(j)}}$ vary, breaking uniform recurrences and many symmetry-based arguments. Closed-form spectra then become rare.

First, let's consider the $(n-1)$-regular complete graph, K_n.

Proposition 3.2 (K_n Spectra)**.** *The spectrum of K_n is 0 with multiplicity 1, and $\frac{n}{n-1}$ with multiplicity $n-1$.*

Proof. Since K_n is $(n-1)$-regular we have $D = (n-1)I$ and hence

$$L = I - D^{-1/2}AD^{-1/2} = I - \frac{1}{n-1}A.$$

Note that A satisfies $A\mathbf{1} = (n-1)\mathbf{1}$ and $Av = -v$ for every v with $v \cdot \mathbf{1} = 0$. Thus A has eigenvalues $n-1$ with multiplicity 1 and -1 with multiplicity $n-1$. Consequently $D^{-1/2}AD^{-1/2} = \frac{1}{n-1}A$ has eigenvalues 1 and $-\frac{1}{n-1}$, and subtracting from I yields the claimed spectrum for L. □

Then we consider the cycle on n vertices, C_n, which is slightly more complicated.

Proposition 3.3 (C_n Spectra)**.** *The spectrum of C_n can be expressed as follows:*

$$\lambda_k = 1 - \cos\Big(\tfrac{2\pi(k-1)}{n}\Big), \qquad k \in [n].$$

The multiplicity of such eigenvalues is 1 exactly at $k = 1$ (and at $k = \frac{n}{2} + 1$ when n is even) and is 2 otherwise, with each such eigenvalue occurring at the paired indices k and $n - k + 2$.

Proof. Here $D = 2I$, so $\mathcal{L} = I - \frac{1}{2}A$. For each $k \in [n]$ define, for $u \in \{0, \ldots, n-1\}$,

$$x^{(k)}(u) = \sin\Big(\tfrac{2\pi(k-1)u}{n}\Big), \qquad y^{(k)}(u) = \cos\Big(\tfrac{2\pi(k-1)u}{n}\Big).$$

The elementary identities

$$x^{(k)}(u-1) + x^{(k)}(u+1) = 2\cos\Big(\tfrac{2\pi(k-1)}{n}\Big)x^{(k)}(u),$$

$$y^{(k)}(u-1) + y^{(k)}(u+1) = 2\cos\Big(\tfrac{2\pi(k-1)}{n}\Big)y^{(k)}(u)$$

imply, for every vertex u,

$$[\mathcal{L}x^{(k)}](u) = x^{(k)}(u) - \tfrac{1}{2}\big(x^{(k)}(u-1) + x^{(k)}(u+1)\big) = \Big(1 - \cos\tfrac{2\pi(k-1)}{n}\Big)x^{(k)}(u),$$

and the same holds for $y^{(k)}$. Thus $\lambda_k = 1 - \cos(2\pi(k-1)/n)$. The multiplicity statement follows from $\cos\big(2\pi(k-1)/n\big) = \cos\big(2\pi(n-k+1)/n\big)$, so that k and $n-k+2$ are paired, with singletons at $k = 1$ and, when n is even, $k = \frac{n}{2} + 1$ $\square$

Finally, following the pattern stated in Remark 3.1, we see that P_n, as the least regular of the three graphs we consider, yields the most difficult explicit computation.

Proposition 3.4 (P_n Spectra)**.** *The spectrum of P_n can be expressed as follows:*

$$\lambda_k = 1 - \cos\Big(\tfrac{(k-1)\pi}{n-1}\Big), \quad k \in [n]$$

Each has multiplicity of 1.

Proof. Let $\mathcal{L}x = \lambda x$ and set $y = D^{-1/2}x$. Then $D^{-1/2}AD^{-1/2}x = (1-\lambda)x$ is equivalent to

$$Ay = \kappa D y, \qquad \kappa = 1 - \lambda.$$

Writing coordinates gives

$$y_2 = \kappa y_1, \qquad y_{i-1} + y_{i+1} = 2\kappa y_i \ (2 \le i \le n-1), \qquad y_{n-1} = \kappa y_n.$$

The interior recurrence $y_{i+1} - 2\kappa y_i + y_{i-1} = 0$ has characteristic equation $r^2 - 2\kappa r + 1 = 0$. With $\kappa = \cos t \in [-1, 1]$ we obtain the real solutions

$$y_i = C\cos((i-1)t) + D\sin((i-1)t).$$

The left boundary $y_2 = \kappa y_1$ forces $D \sin t = 0$. For $\sin t \neq 0$ we take $D = 0$, hence $y_i = C\cos((i-1)t)$. The right boundary $y_{n-1} = \kappa y_n$ becomes

$$\cos((n-2)t) = \cos t\, \cos((n-1)t) \quad \Longleftrightarrow \quad \sin((n-1)t)\sin t = 0.$$

Excluding $\sin t = 0$ (the endpoints $t = 0, \pi$ correspond to the two boundary eigenvalues), we get $(n-1)t = m\pi$ for $m = 1, \ldots, n-1$. Reindex by $k = m+1$ to obtain

$$t_k = \tfrac{(k-1)\pi}{n-1}, \qquad \kappa_k = \cos t_k, \qquad \lambda_k = 1 - \kappa_k = 1 - \cos\Big(\tfrac{(k-1)\pi}{n-1}\Big), \quad k = 1, \ldots, n.$$

These n distinct values yield n independent eigenvectors $x^{(k)} = D^{1/2}y^{(k)}$, so each λ_k has multiplicity 1. $\square$

We computed explicit spectra of the Laplacian for three canonical families, K_n, C_n, and P_n, and saw how symmetry streamlines the algebra: in the d-regular cases (K_n and C_n) the identity $L = I - \frac{1}{d}A$ reduces the task to diagonalizing A, while the less regular path P_n requires a boundary–value recurrence that still yields closed forms and a simple spectrum. The cycle's pairing k with $n-k+2$ explains its multiplicities (with singletons at $k=1$ and, when n is even, $k=\frac{n}{2}+1$), illustrating how graph symmetry shapes eigenvalue structure. We also note that *adjacency* spectra are often simpler and more intuitive to compute.

4. A Spectral Bound on the Cheeger Constant

The concept of Cheeger Constant, while fundamental to our discrete analysis, originates from Jeff Cheeger's pioneering work in differential geometry on Riemannian manifolds. Cheeger originally defined what is now known as the Cheeger constant, (in some literature it is referred to as the conductance or isoperimetric constant) of a manifold as the infimum over all smooth hypersurfaces of the ratio of the hypersurface's area to the minimum volume of the regions it separates. This geometric quantity measures how difficult it is to partition a manifold into large pieces with relatively small boundary, capturing the manifold's "bottleneck" properties in a continuous setting. For more precise discussion, see [10]. The discrete graph-theoretic version we study here—where edge cuts replace hypersurfaces, vertex degrees replace volume elements, and set cardinalities replace continuous measures—preserves the essential geometric intuition while making the concept tractable for combinatorial analysis. This discretization has proven remarkably fruitful, as the spectral techniques that work in the smooth setting translate directly to the discrete case with the graph Laplacian.

4.1. Relevant Definitions.

We begin by introducing the necessary machinery to approach and define the Cheeger Constant of a graph.

Definition 4.1. Given a graph $G = (V, E)$, we define the *boundary* of $S \subseteq V$ to be $\partial S = \{\{u, v\} \in E \mid u \in S, v \in V \setminus S\}$, where $V \setminus S$ denotes the complement of S. Note that $\partial S = \partial (V \setminus S)$.

Definition 4.2. Given a graph $G = (V, E)$ and a nonempty proper subset, $S \subset V$, the *Cheeger Constant* of the vertex set S is defined as

$$\varphi(S) = \frac{|\partial S|}{\min\{\mathrm{vol}(S), \mathrm{vol}(V \setminus S)\}}.$$

Similarly, the *Cheeger Constant* of the graph G is defined as

$$\varphi(G) = \min_{S \subset V} \varphi(S).$$

Example 4.3. In the complete graph, K_n, each vertex has degree $n-1$, so $\mathrm{vol}(V) = n(n-1)$. Then, any S of size $K \leq \frac{n}{2}$ admits a boundary of size $k(n-k)$ and a volume of $k(n-1)$. Thus $\varphi(S) = \frac{k(n-k)}{k(n-1)}$, which is minimized by choosing $k = \lfloor \frac{n}{2} \rfloor$, yielding: $\varphi(K_n) = \frac{\lceil \frac{n}{2} \rceil}{n-1}$.

4.2. **The Bound.**

Theorem 4.4. *For any graph G, the graph's Cheeger Constant can be bounded as follows:*

$$\frac{\lambda_2(G)}{2} \leq \varphi(G) \leq \sqrt{2\lambda_2(G)}$$

Proof. First, fix a proper subset, $S \subset V$ satisfying $\varphi(G) = \frac{|\partial S|}{\min\{\mathrm{vol}(S), \mathrm{vol}(V \setminus S)\}}$. Then define a function $f : V \to \mathbb{R}$ such that

$$f(v) = \begin{cases} \frac{1}{\mathrm{vol}(S)} & \text{if } v \in S \\ -\frac{1}{\mathrm{vol}(V \setminus S)} & \text{if } v \notin S \end{cases}$$

Clearly, f satisfies the Courant-Fischer orthogonality condition of being orthogonal to $D\mathbf{1}$, as

$$\sum_{v \in V} d(v)f(v) = \mathrm{vol}(S)\left(\frac{1}{\mathrm{vol}(S)}\right) + \mathrm{vol}(V \setminus S)\left(-\frac{1}{\mathrm{vol}(V \setminus S)}\right) = 0.$$

Note that the expression $f(u) - f(v)$ takes nonzero value, $\frac{1}{\mathrm{vol}(S)} + \frac{1}{\mathrm{vol}(V \setminus S)}$, only when $\{u, v\} \in \partial S$. Thus, we compute the numerator of the Rayleigh Quotient

$$\sum_{\{u,v\} \in E} \big(f(u) - f(v)\big)^2 = |\partial S| \left(\frac{1}{\mathrm{vol}(S)} + \frac{1}{\mathrm{vol}(V \setminus S)}\right)^2.$$

Then, we compute the denominator, utilizing the definition of volume.

$$\sum_{v \in V} \mathrm{d}(v)f(v)^2 = \mathrm{vol}(S)\left(\frac{1}{\mathrm{vol}(S)}\right)^2 + \mathrm{vol}(V \setminus S)\left(\frac{1}{\mathrm{vol}(V \setminus S)}\right)^2 = \frac{1}{\mathrm{vol}(S)} + \frac{1}{\mathrm{vol}(V \setminus S)}.$$

Finally, substituting our simplifications and applying the Courant-Fischer characterization of λ_2, we have

$$\lambda_2 \leq \frac{|\partial S|\left(\frac{1}{\mathrm{vol}(S)} + \frac{1}{\mathrm{vol}(V \setminus S)}\right)^2}{\frac{1}{\mathrm{vol}(S)} + \frac{1}{\mathrm{vol}(V \setminus S)}} \leq 2\frac{|\partial S|}{\min\{\mathrm{vol}(S), \mathrm{vol}(V \setminus S)\}} = 2\varphi(G).$$

Thus, we have

$$\frac{\lambda_2}{2} \leq \varphi(G).$$

Concerning the upper bound, first let $g : V \to \mathbb{R}$ realize the Courant-Fischer characterization of λ_2, and order the vertices $V = \{v_1, \ldots, v_n\}$ such that

$$g(v_1) \geq g(v_2) \geq \ldots \geq g(v_n).$$

For each $i \in [n]$, let $S_i = \{v_1, \ldots, v_i\}$, and note that by definition of $\varphi(G)$, we have $|\partial S_i| \geq \varphi(G)\min\{\mathrm{vol}(S_i), \mathrm{vol}(\bar{S}_i)\}$.

Choose r such that $\mathrm{vol}(S_r) \leq \frac{1}{2}\mathrm{vol}(V) \leq \mathrm{vol}(S_{r+1})$, and define

$$h(v) = g(v) - g(v_{r+1}), \quad p(v) = \max\{h(v), 0\}.$$

Note that the Rayleigh Quotient, $R(p)$, must be less than or equal to λ_2, as

$$\sum_{\{u,v\} \in E} (p(u) - p(v))^2 \leq \sum_{\{u,v\} \in E} (g(u) - g(v))^2$$

and

$$\sum_{v\in V} \mathrm{d}(v)(p(v))^2 \leq \sum_{v\in V} \mathrm{d}(v)(g(v))^2.$$

By the Cauchy-Schwarz inequality, we have

$$\left(\sum_{\{u,v\}\in E}(p(u)-p(v))^2\right)\left(\sum_{\{u,v\}\in E}(p(u)+p(v))^2\right) \geq \left(\sum_{\{u,v\}\in E}(p(u)^2-p(v)^2)^2\right).$$

Then, as $p \geq 0$, each term $(p(u)+p(v))^2 \leq 2(p^2(u)+p^2(v))$. Summing over all edges, and noting that each vertex v is included in exactly $\mathrm{d}(v)$ edges gives

$$\sum_{\{u,v\}\in E}(p(u)-p(v))^2 \geq \frac{1}{2}\frac{\left(\sum_{\{u,v\}\in E} p(u)-p(v)\right)^2}{\sum_{v\in V}\mathrm{d}(v)(p(v))^2}.$$

The function p inherits an ordering from g, so we have $p(v_1) \geq \ldots \geq p(v_r) > 0$ and $p(v_{r+1}) = \ldots = p(v_n) = 0$. Thus the numerator of $R(p)$ telescopes, and for each S_i, the only nonzero contributions are from $\{u,v\} \in \partial S_i$ and are exactly

$$p(u) - p(v) = p(v_i)^2 - p(v_{i+1})^2$$

Hence, grouping by the cut each edge crosses, we have

$$\sum_{\{u,v\}\in E}(p(u)-p(v))^2 = \sum_{i=1}^{n-1}\left(p(v_i)^2 - p(v_{i+1})^2\right)|\partial S_i|$$

By construction, we have $|\partial S_i| \geq \varphi(G)\min\{\mathrm{vol}(S_i), \mathrm{vol}(\bar{S}_i)\}$. Additionally, $i \leq r$ implies that $\mathrm{vol}(S) \leq \mathrm{vol}(\bar{S}_i)$ and $i > r$ implies that $p(v_i)^2 - p(v_{i+1})^2 = 0$, which means that

$$\sum_{\{u,v\}\in E}(p(u)) - p(v))^2 \geq \varphi(G)\sum_{i=1}^{r}\left(p(v_i)^2 - p(v_{i+1})^2\right)\mathrm{vol}(S_i).$$

Furthermore, from $\mathrm{vol}(S_i) - \mathrm{vol}(S_{i-1}) = \mathrm{d}(v_i)$, we know that

$$\sum_{i=1}^{r}\left(p(v_i)^2 - p(v_{i+1})^2\right)\mathrm{vol}(S_i) = \sum_{i=1}^{r} p(v_i)^2\left(\mathrm{vol}(S_i) - \mathrm{vol}(S_{i-1})\right) = \sum_{i=1}^{r}\mathrm{d}(v_i)p(v_i)^2$$

Then, as p vanishes after index r, we know that this final sum is equal to the denominator of $R(p)$. Thus, combining these inequalities, we have

$$\sum_{\{u,v\}\in E}(p(u)-p(v))^2 \geq \frac{1}{2}\frac{\varphi(G)\sum_{v\in V}\mathrm{d}(v)(p(v))^2}{\sum_{v\in V}\mathrm{d}(v)(p(v))^2} = \frac{\varphi(G)^2}{2}\sum_{v\in V}\mathrm{d}(v)(p(v))^2$$

This shows that $R(p) \geq \frac{\varphi(G)^2}{2}$. From before, we know that $R(p) \leq \lambda_2$, which implies that $\lambda_2 \geq \frac{\varphi^2}{2}$ or equivalently,

$$\varphi(G) \leq \sqrt{2\lambda_2}$$

□

Thus, we have the bound $\frac{\lambda_2}{2} \leq \varphi(G) \leq \sqrt{2\lambda_2}$. These inequalities are valuable because computing λ_2 of the Laplacian is a polynomial-time numerical task, whereas minimizing Cheeger Constant directly is combinatorially hard in general; hence spectral quantities serve as efficient proxies for detecting bottlenecks in networks. Concretely, the lower bound guarantees that a nontrivial second laplacian eigenvalue rules out very sparse cuts. Together these statements explain why eigenvalue-based algorithms are both theoretically justified and practically effective for tasks such as clustering and estimating mixing times of random walks.

Having examined how Laplacian eigenvalues control expansion, we now turn to another classical application of spectral methods: relating adjacency-matrix eigenvalues to purely combinatorial invariants such as the clique number. The techniques differ in detail, as we now work with the quadratic form of the adjacency matrix on the probability simplex rather than the Laplacian Rayleigh quotient, but the guiding idea is the same: translate a combinatorial extreme into a matrix quadratic form and use spectral information to obtain bounds.

5. Spectral Bounds on Clique and Co-Clique Numbers

In this section we work with the adjacency matrix A to bound the clique number $\omega(G)$ and the co-clique number $\alpha(G)$. The basic object is the quadratic form $x^\top Ax$. For a subset $S \subseteq V = \{1, \ldots, n\}$, let $\mathbf{1}_S \in \{0,1\}^n$ be its indicator vector, with $(\mathbf{1}_S)_i = 1$ if $i \in S$ and 0 otherwise. Additionally, we write $\mathbf{1} = \mathbf{1}_V$ for the all-ones vector. Then $\mathbf{1}_S^\top A\, \mathbf{1}_S$ equals twice the number of edges in the subgraph induced by S. In one of the bounds below, we restrict x to the standard simplex by requiring $x_i \geq 0$ and $\sum_i x_i = 1$,so that the quadratic form represents a weighted average over vertex pairs rather than a raw edge count.

First, the Motzkin–Straus theorem identifies the exact maximum of $x^T Ax$ under that normalization and expresses it directly in terms of $\omega(G)$; evaluating $x^T Ax$ along an eigenvector direction of A then yields a lower bound for $\omega(G)$. Second, for the co-clique number, a simple re-centering of A enforces nonnegativity on the subspace orthogonal to $\mathbf{1}$; in the k–regular case this gives the classical ratio bound for $\alpha(G)$. On d–regular graphs the relation $\mathcal{L} = I - \frac{1}{d}A$ carries these adjacency–based statements to the normalized–Laplacian framework developed earlier without changing their combinatorial interpretation.

Definition 5.1. The *Clique number* of a graph G, denoted $\omega(G)$ is defined as the size of the largest completely connected subset of the vertices. Similarly, the *co-Clique number*, denoted $\alpha(G)$, is defined as the size of the largest subset of the vertices such that no two constituent vertices share an edge

5.1. **Clique Number.**

Theorem 5.2 (Motzkin–Straus, 1965)**.** *Define the classical probability simplex,*

$$\Delta = \{x \in \mathbb{R}^n \mid x_i \geq 0 \;\forall i, \quad \sum_i^n x_i = 1\},$$

and let A be the adjacency matrix for some graph G. Then

$$\max_{x \in \Delta} x^T Ax = 1 - \frac{1}{\omega(G)}$$

View the simplex as the set of probability assignments on the vertices. Each assignment induces an "edge hit rate", the chance that two independent vertex selections constitute an edge. Any probability placed outside a fully connected block inevitably creates non-edge pairs and lowers this rate. Even within a clique, a skewed assignment wastes potential pairs by oversampling some vertices and undersampling others. The rate is maximized by putting all mass on a clique and spreading it evenly there. Among all cliques, the largest one gives the best rate, so the optimizer is the uniform point on a maximum clique and zero elsewhere; the continuous search thus rediscovers the combinatorial object. A complete proof can be found in [6]. Wilf uses this theorem to produce a lower bound for the clique number.

The leading eigenvector points toward the most cohesive part of the graph. After normalizing it to a distribution, its average adjacency cannot be too small. Translating that average into a clique size yields a quick lower bound: when a single direction explains a lot of the connectivity, a reasonably large clique must be present.

Theorem 5.3 (Wilf, 1985)**.** *Let S be the sum of the entries of the normalized smallest eigenvector of G. Additionally let A be the corresponding adjacency matrix with eigenvalues*

$$\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$$

then,

$$\omega(G) \geq \frac{\mu_n}{\mu_n - S^2}$$

Proof. Let u be the normalized principle eigenvector of the adjacency matrix, and define

$$x = \frac{u}{S}$$

As all entries of u are non-negative by the Perron-Frobenius Theorem ([9, p. 65]), and both $\sum_i x_i = \frac{1}{S}$, $\sum_i u_i = 1$ hold, the vector x is contained in Δ.

Evaluating the adjacency matrix quadratic form at x, we see that

$$x^T Ax = \left(\frac{u}{S}\right)^T A \left(\frac{u}{S}\right) = \frac{1}{S^2}\left(u^T Au\right) = \frac{\mu}{S^2} u^T u = \frac{\mu}{S^2}$$

Then as $x \in \Delta$ we know that this value cannot exceed the Motzkin-Straus maximum. Thus we know that

$$\frac{\mu_n}{S^2} \leq 1 - \frac{1}{\omega(G)},$$

or equivalently

$$\omega(G) \geq \frac{\mu_n}{\mu_n - S^2}$$

□

To treat the co-Clique number $\alpha(G)$ it is convenient to recenter A so that the resulting matrix is nonnegative on the subspace orthogonal to $\mathbf{1}$. In the k–regular case this leads to the classical ratio bound, which we now state and prove.

5.2. **Co-Clique Number.** The co-clique number asks how large a vertex set can be while carrying no internal edges. Instead of a combinatorial search over all subsets, we use a spectral surrogate that penalizes any tendency to place probability mass on adjacent vertices: pick a graph–dependent linear operator whose quadratic form is nonnegative on natural fluctuations, so it cannot "see" an edge in those directions. Any independent set then determines a principal block that must inherit this nonnegativity. Testing that block with a simple vector (e.g., the uniform vector on the set) converts "no internal edges" into a short inequality governed by global spectral parameters.

Theorem 5.4. *Let G be a simple k-regular graph on n vertices with adjacency matrix A, having eigenvalues,*

$$\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$$

If $\mu_1 < 0$ then the co-clique number $\alpha(G)$ satisfies

$$\alpha(G) \leq \frac{-\mu_1}{k-\mu_1}\, n.$$

Proof. Set $J = \mathbf{1}\mathbf{1}^T$, and define

$$M = A - \mu_1 I - \frac{k-\mu_1}{n} J.$$

Since $A\mathbf{1} = k\mathbf{1}$ and $J\mathbf{1} = n\mathbf{1}$ we have

$$M\left(\frac{\mathbf{1}}{\sqrt{n}}\right) = \big(k - \mu_1 - (k-\mu_1)\big)\frac{\mathbf{1}}{\sqrt{n}} = 0.$$

If $v \perp \mathbf{1}$ then $Jv = 0$ and hence $Mv = (A - \mu_1 I)v$, so the eigenvalues of M on $\mathbf{1}^\perp$ are $\mu_i - \mu_1 \geq 0$. Thus all eigenvalues of M are nonnegative.

Let $S \subseteq V(G)$ be an independent set of size $\alpha = |S|$ and order the vertices so the first α coordinates correspond to S. The principal $\alpha \times \alpha$ submatrix of A indexed by S is the zero matrix, so the corresponding principal submatrix of M is

$$M_S = -\mu_1 I_\alpha - \frac{k-\mu_1}{n} J_\alpha,$$

where I_α and J_α are the $\alpha \times \alpha$ identity and all-ones matrices. Since all eigenvalues of M are nonnegative, the same is true for M_S. The eigenvalues of J_α are α, with eigenvector $\mathbf{1}_\alpha$, and 0, with multiplicity $\alpha - 1$, hence the eigenvalues of M_S are $-\mu_1$ with multiplicity $\alpha - 1$ and

$$-\mu_1 - \frac{k-\mu_1}{n}\,\alpha$$

on the $\mathbf{1}_\alpha$-direction. Positivity of M_S forces the last quantity to be nonnegative, so

$$-\mu_1 - \frac{k-\mu_1}{n}\,\alpha \geq 0.$$

Rearranging yields

$$\alpha \leq \frac{-\mu_1}{k-\mu_1}\, n.$$

□

This section shows that adjacency spectra provide a direct and interpretable route to extremal subgraph parameters. Motzkin–Straus expresses the clique number as a clean maximization over Δ, Wilf's inequality follows by evaluating that maximization at an eigen-direction associated with the largest adjacency eigenvalue, and a mild spectral recentering of A gives the co-clique bound in the regular case. Because $L = I - \frac{1}{d}A$ on d–regular graphs, these adjacency-based statements immediately inform the normalized–Laplacian viewpoint developed earlier, and together they offer a coherent framework connecting linear–algebraic structure with combinatorial complexity.

6. Conclusion and Future Work

In this paper we established a set of spectral inequalities tying eigenvalue information of A and $\mathcal{L}$ to classical combinatorial graph parameters. The presentation prioritizes concise, modular arguments and, wherever possible, explicit, approachable proofs that reduce reliance on heavy machinery. This formulation makes it straightforward to extend the arguments to related matrices or to strengthen individual estimates when additional graph structure is available.

Spectral tests are computationally inexpensive and often provide useful connectivity information when exact combinatorial computations are out of reach. A particularly attractive avenue for future research is the development of spectral bounds for Ramsey numbers. Because eigenvalue computations scale far better than exhaustive combinatorial enumeration, spectral bounds are especially promising as a source of usable heuristics for otherwise intractable instances. Pursuing this line appears likely to produce practically useful insights and new theoretical bounds for Ramsey-type problems.

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