

THE WANDERING EXPONENT IN FIRST PASSAGE PERCOLATION

ALICE WANG

ABSTRACT. In this expository paper, based in large part on [2], we first introduce first passage percolation (FPP), a model of fluid flow on a lattice. We discuss geodesics in this model and state the various definitions of the wandering exponent, which characterizes the scale of their deviation, and provide some bounds. We also introduce its physical applications such as the 2-dimensional Ising ferromagnet.

CONTENTS

1. Introduction	1
2. First passage percolation (FPP)	2
3. The limit shape	2
4. Variance and concentration bounds	6
4.1. The fluctuation exponent	6
5. Geodesics	7
5.1. The wandering exponent	8
6. Applications	15
6.1. 2-dimensional Ising ferromagnet	16
Acknowledgments	16
7. Bibliography	16
References	16

1. INTRODUCTION

First passage percolation (FPP) is a model of fluid flow through a medium first introduced by Hammersley and Welsh (1965). It is especially useful for its simple description of a random metric space and its connection to further branches of graph theory, ergodic theory, and stochastic calculus.

In this expository paper, based in large part on [2], we state the various definitions of the wandering exponent, which characterizes the scale of the deviation of directed "shortest-paths" in FPP. Using well-known results such as the shape theorem, we will provide finite bounds for the wandering exponent. We also introduce its physical applications in models of disease or bacterial spread, as well as the 2-dimensional Ising ferromagnet. We will provide a sketch of the field and proofs of its major theorems for the reader.

2. FIRST PASSAGE PERCOLATION (FPP)

The model of d -dimensional first passage percolation (FPP) is based on assigning random “weights” τ_e , according to a given probability distribution, to each edge e on the lattice $\mathbb{Z}^d$. We can think of each weight as the time it takes for fluid to flow to or through that given edge; hence, any path $\Gamma \subset \mathbb{R}^d$ (i.e. set of edges in $\mathbb{Z}^d$ each connected by a vertex) has a corresponding “travel time” given by the sum of its edge weights. Based on this conception, we can define the following:

Definition 2.1. For a path $\Gamma \subset \mathbb{R}^d$, the *passage time* T is given by:

$$T(\Gamma) = \sum_{e \in \Gamma} \tau_e.$$

The usefulness of FPP as a physical model comes from analyzing this metric as the lattice edge length approaches 0 (i.e. as $\mathbb{Z}^d$ becomes $\mathbb{R}^d$). Hence, for two points $x, y \in \mathbb{R}^d$, it is also useful to define the passage time as

$$T(x, y) = \inf_{\Gamma \in \mathcal{T}} T(\Gamma),$$

$$\mathcal{T} = \{\text{finite paths } \Gamma \mid x', y' \in \Gamma, x \in x' + [0, 1)^d, y \in y' + [0, 1)^d\} \text{ [2].}$$

$T(x, y)$ can be thought of as a distance metric. Note that for any vertex in $\mathbb{Z}^d$, we refer to the $2d$ independent and identically-distributed (i.i.d.) passage times of its adjacent edges as $t_1, \dots, t_{2d}$.

3. THE LIMIT SHAPE

Treating $(\mathbb{R}^d, T(\cdot, \cdot))$ as a metric space, the ball of radius t is defined as

$$B(t) := \{x \in \mathbb{R}^d \mid T(0, x) < t\}.$$

In FPP, we are interested in the *limit shape*, which can be thought of as $B(t)$ as $t \rightarrow \infty$. In this section, we reproduce two important results:

- (1) the first order growth of passage time between two points with respect to the L^1 distance between them.
- (2) the shape theorem, which states that the normalized ball $B(t)/t$ converges to a unique shape as $t \rightarrow \infty$.

To prove (1), we use the following well-known result in probability theory:

Theorem 3.1. *Subadditive Ergodic Theorem [2], [7]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with probability-preserving map $T : \Omega \rightarrow \Omega$. Let $(X_{m,n} : \Omega \rightarrow \mathbb{R})_{m,n \in \mathbb{N}}$ be a family of random variables with the following properties:*

- $X_{m+1,n+1} = X_{m,n} \circ T$ (i.e. probability distribution is preserved),
- $X_{0,n} \leq X_{0,m} + X_{n,m}$ (subadditivity).

Then there almost surely exists some $Y \in [-\infty, \infty)$, invariant of T , such that

$$\lim_{n \rightarrow \infty} \frac{X_{0,n}}{n} = Y.$$

Furthermore, if T is ergodic, Y is constant and almost surely,

$$Y = \lim_{n \rightarrow \infty} \frac{\mathbb{E}X_{0,n}}{n} = \inf \left\{ \frac{\mathbb{E}X_{0,n}}{n} \mid n \in \mathbb{N} \right\}.$$

While the proof of the above theorem is beyond the scope of this paper, one can see how a straightforward application of it is used to prove the following statement of linear growth of passage time:

Theorem 3.2. [2]. *Suppose $\mathbb{E} \min(t_1, \dots, t_{2d}) < \infty$ for all vertices in $\mathbb{Z}^d$. Then (without loss of generality) there exists some $\mu(e_1) \in [0, \infty)$ such that*

$$\lim_{n \rightarrow \infty} \frac{T(0, ne_1)}{n} = \mu(e_1).$$

Proof. Consider the family of random variables

$$X_{m,n} := T(me_1, ne_1) \geq 0.$$

We can define a probability-preserving map T that acts as a translation by e_1 . Since definitionally $T(0, x) \leq T(0, y) + T(x, y)$, the sequence is subadditive. Hence, we can straightforwardly apply Theorem 3.1 to see that $\lim_{n \rightarrow \infty} \frac{T(0, ne_1)}{n}$ exists and is positive finite. Furthermore, since the edge weights are i.i.d., $X_{m,n}$ is ergodic. Hence,

$$\mu(e_1) := \lim_{n \rightarrow \infty} \frac{\mathbb{E}T(0, ne_1)}{n} = \inf \left\{ \frac{\mathbb{E}T(0, ne_1)}{n} \mid n \in \mathbb{N} \right\}.$$

□

From here on it is necessary to introduce the dimension-dependent *bond percolation threshold* defined as

$$p_c = p_c(d) := \sup\{p \in [0, 1] \mid$$

$$\mathbb{P}(\exists \Gamma \subset \mathbb{Z}^d : \Gamma \text{ infinite, connected, and self-avoiding; } \forall e \in \Gamma, \tau_e = 0) = 0, \\ \mathbb{P}(\tau_e = 0) = p\} \text{ [2].}$$

Physically, this can be thought of as a critical point between a “non-percolated” phase and “percolated phase”, in which an infinite connected path can be formed on the lattice [8]. Analogous to the law of large numbers, $\mu(e_1) > 0$ if and only if the edge weight distribution function $F(0) < p_c(d)$ [2]. Hence, assuming the system is in this subcritical phase and extending Theorem 3.2 to arbitrary directions rather than just e_1 , we get the following metric for FPP:

Definition 3.3. The *FPP-norm* is the homogenous function $\mu : \mathbb{Q}^d \rightarrow [0, \infty)$ given by

$$\mu(x) := \lim_{n \rightarrow \infty} \frac{T(0, nx)}{n} \text{ [2].}$$

By the proof of Theorem 3.2, it follows that:

- $\forall x, y \in \mathbb{Q}^d, \mu(x + y) \leq \mu(x) + \mu(y)$.
- $\forall x \in \mathbb{Q}^d, c \in \mathbb{Q}, \mu(cx) = |c|\mu(x)$.
- μ is invariant under symmetries of $\mathbb{Z}^d$ that preserve the origin.
- Since μ is Lipschitz continuous on bounded subsets of $\mathbb{Q}^d$, there is a unique continuous extension of μ to $\mathbb{R}$.

We can further view the FPP-norm $\mu(x)$ as a continuous function of F [2]. Letting $\mathcal{M}$ be the set of “subcritical” Borel probability measures such that $\mathbb{E} \min\{t_1^d, \dots, t_{2d}^d\} < \infty$ and $F(0) < p_c(d)$, we can begin characterizing the limit shape.

Theorem 3.4. *Cox-Durrett Shape Theorem [2]. For any $\nu \in \mathcal{M}$, there exists a deterministic, convex, and compact set $\mathcal{B}$ such that for each $\epsilon > 0$,*

$$\mathbb{P} \left((1 - \epsilon)\mathcal{B} \subset \frac{B(t)}{t} \subset (1 + \epsilon)\mathcal{B} \text{ for all large enough } t \right) = 1.$$

Furthermore, $\mathcal{B}$ has a non-empty interior and has the same axes of symmetry as $\mathbb{Z}^d$ preserving the origin.

Proof. We can equivalently restate the theorem as almost surely

$$(3.5) \quad \limsup_{\|x\|_1 \rightarrow \infty} \frac{|T(0, x) - \mu(x)|}{\|x\|_1} = 0.$$

First note that as a consequence of the Borel-Cantelli Lemma¹, there must exist some $\kappa > 0$ such that for all $x \in \mathbb{Z}^d$,

$$(3.6) \quad \mathbb{P} \left(\sup_{y \in \mathbb{Z}^d, y \neq x} \frac{T(x, y)}{\|x - y\|_1} < \kappa \right) > 0.$$

We call some $x \in \mathbb{Z}^d$ that satisfies the inequality in (3.6) “good”. Furthermore, for any $\zeta \in \mathbb{Z}^d \setminus \{0\}$, consider the sequence $(n_k) \subset \mathbb{N}$ such that $n_k \zeta$ are “good” vertices, and let B_m be the event that $m\zeta$ is “good”. Then by the ergodic theorem,

$$\begin{aligned} \frac{k}{n_k} &= \frac{1}{n_k} \sum_{m=1}^{n_k} \mathbf{1}_{B_m} \rightarrow \mathbb{P}(x \in (n_k \zeta) \text{ is “good”}) = 1, \\ &\implies \frac{n_{k+1}}{n_k} = \frac{n_{k+1}}{k+1} \frac{k+1}{k} \frac{k}{n_k} \rightarrow 1. \end{aligned}$$

Assume for the sake of contradiction that (3.5) does not hold. Then for some edge weight configuration there exists some $\delta > 0$ such that for infinitely many $x \in \mathbb{Z}^d$,

$$(3.7) \quad |T(0, x) - \mu(x)| > \delta \|x\|_1.$$

Since the L_1 unit sphere in $\mathbb{Z}^d$ is compact, there exists some $(x_i) \subset \mathbb{Z}^d$ with $(x_i / \|x_i\|_1) \rightarrow y \in \mathbb{Z}^d$ and $\|y\|_1 = 1$ by Bolzano-Weierstrauss. Hence, for any $\epsilon > 0$, there is a large enough $N \in \mathbb{N}$ such that for $n > N$,

$$\begin{aligned} \|x_n / \|x_n\|_1 - y\|_1 &< \epsilon, |\mu(x_n) - \|x_n\|_1 \mu(y)| < \delta \|x_n\|_1 / 2 \\ &\implies |T(0, x_n) - \|x_n\|_1 \mu(y)| > \delta \|x_n\|_1 / 2. \end{aligned}$$

Additionally, there must exist some $z = x/M \in \mathbb{Q}^d$ with $x \in \mathbb{Z}^d$ and $\|z\|_1 = 1$ such that $\|z - y\|_1 < \epsilon$. We have shown that there is some infinite $(n_k) \subset \mathbb{N}$ such that each $n_k x = n_k M z$ is “good”, with $n_{k+1}/n_k \rightarrow 1$. Then there exists sufficiently large $n > N$, $K \in \mathbb{N}$ such that for $k > K$,

¹It’s more complicated than that- since there are $2d$ disjoint paths between x and z when $\mathbb{E}\tau_e^d < \infty$, the likelihood that all have passage times greater than κ is very small. We can extend this to the general case by “renormalizing” the lattice to larger edge lengths [2].

$$\begin{aligned}
n_{k+1} &< (1 + \epsilon)n_k, \left| \frac{T(0, n_k M z)}{n_k M} - \mu(z) \right| < \epsilon, n_k M \leq \|x_n\|_1 \leq n_{k+1} M, \\
\implies |T(0, x_n) - T(0, n_k M z)| &\leq T(x_n, n_k M z) \leq \kappa \|x_n - n_k M z\|_1 < 3\kappa\epsilon \|x_n\|_1.
\end{aligned}$$

Note that by the triangle inequality,

$$\begin{aligned}
\left| \frac{T(0, x_n)}{\|x_n\|_1} - \mu(y) \right| &\leq \left| \frac{T(0, x_n) - T(0, n_k M z)}{\|x_n\|_1} \right| + \frac{T(0, n_k M z)}{n_k M} \left(1 - \frac{n_k M}{\|x_n\|_1} \right) \\
&\quad + \left| \frac{T(0, n_k M z)}{n_k M} - \mu(z) \right| + |\mu(z) - \mu(y)| \\
&\leq 3\kappa\epsilon + (\mu(z) + \epsilon) \left(1 - \frac{1}{1 + \epsilon} \right) + \epsilon + C\epsilon^2,
\end{aligned}$$

where C is a constant. Since ϵ is arbitrary, this contradicts (3.7). Therefore, (3.5) holds and we can define

$$\mathcal{B} := \{x \in \mathbb{R}^d \mid \mu(x) \leq 1\}^3$$

since the limit shape has a “normalized” radius. $\square$

To further explore properties of the limit shape, for any unit vector $\hat{u} \in \mathbb{R}^d$, it is useful to define H_0 as the hyperplane in $\mathbb{R}^d$ containing the origin such that $\hat{u} + H_0$ is the supporting hyperplane of $\mu(\hat{u})\mathcal{B}$ [2]. This can be thought of as the tangent plane to the limit shape of a fluid traveling in the direction $\hat{u}$.

Definition 3.8. Suppose $\partial\mathcal{B}$ is differentiable. The *curvature exponent* $\kappa(\hat{u})$ in the direction of unit vector $\hat{u} \in \mathbb{R}^d$ is the unique real such that for any $0 < c_1 < c_2, \epsilon > 0$ and $z \in H_0$ such that $\|z\|_2 < \epsilon$,

$$c_1 \|z\|_2^{\kappa(\hat{u})} \leq \mu(\hat{u} + z) - \mu(\hat{u}) \leq c_2 \|z\|_2^{\kappa(\hat{u})}.$$

We can think of this as the boundary of the limit shape in the direction $\hat{u}$ following the curve $z \mapsto z^{\kappa(\hat{u})}$ for some z in the tangent plane to it. Furthermore, there is a notion of *uniform curvature* that occurs when Definition 3.8 is satisfied for uniform values of ϵ and c . It is conjectured (but unproven) that in FPP, $\mathcal{B}$ is uniformly curved for all continuous F [2].

Proposition 3.9. Suppose $\mathcal{B}$ is uniformly curved. Then it is strictly convex, i.e. $\kappa(\hat{u}) > 0$ for all unit vectors $\hat{u} \in \mathbb{R}^d$ [2].

Proof. A general definition for uniform curvature is that there exists some $C > 0$ such that for all $x_1, x_2 \in \partial\mathcal{B}$, $\lambda \in [0, 1]$, and $x = (1 - \lambda)x_1 + \lambda x_2$:

$$\begin{aligned}
1 - \mu(x) &\geq C \min\{\mu(x - x_1), \mu(x - x_2)\}^2 [2], \\
\implies \mu(x) &\leq 1 - C \min\{\mu(x - x_1), \mu(x - x_2)\}^2.
\end{aligned}$$

Then for $\lambda \in (0, 1)$, since $\mu(x_1) = \mu(x_2) = 1$, $\mu(x) < (1 - \lambda)\mu(x_1) + \lambda\mu(x_2)$. $\square$

³To fully see why this is true, we need the notion of Gromov-Hausdorff convergence of metric spaces. The shape theorem can be restated as $(\frac{1}{n}\mathbb{Z}^d, \frac{1}{n}T(nx, ny)) \rightarrow (\mathbb{R}^d, \mu(x - y))$ if $F(0) < p_c$ and $\mathbb{E}e^{\alpha\tau_e} < \infty$ for some $\alpha > 0$ [2].

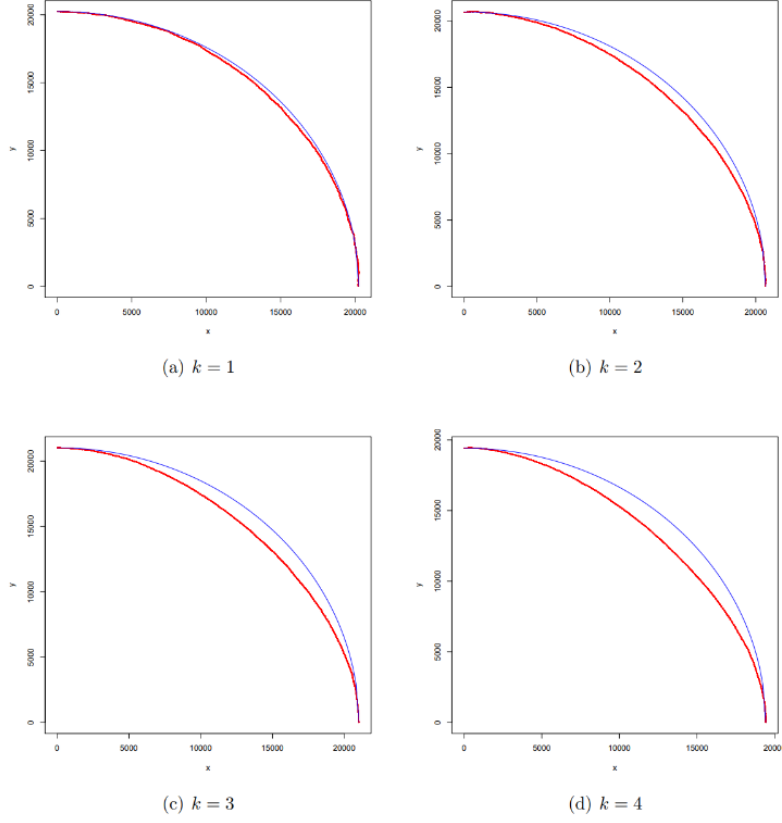


FIGURE 1. The sphere $\partial B(t = 2000)$ (red) compared to the L_2 sphere $\{x \in \mathbb{R}^2 \mid \|x\|_2 = 2000\}$ (blue) for edge weight distributions given by $\Gamma(k, k)$ [1].

4. VARIANCE AND CONCENTRATION BOUNDS

As a result of the shape theorem, we observe that for $x \in \mathbb{R}^d$,

$$T(0, x) = \mu(x) + o(\|x\|_1).$$

Hence, the fluctuation $T(0, x) - \mu(x)$ is of particular interest. We can further split it into the terms

$$o(\|x\|_1) = (T(0, x) - \mathbb{E}T(0, x)) + (\mathbb{E}T(0, x) - \mu(x)).$$

By Definition 3.3 and the Subadditive Ergodic Theorem, the variations in $\mathbb{E}T(0, x) - \mu(x)$ are non-random and can be bounded by the previous term. Hence, in this paper, we focus our attention on $\text{Var}T(0, x) := T(0, x) - \mathbb{E}T(0, x)$, which does fluctuate randomly [2].

4.1. The fluctuation exponent. From simulations and scaling theory, we predict that there is some *fluctuation exponent* $\chi \in \mathbb{R}$ for each dimension, independent of

edge weight distribution (as long as the limit shape has no flat edges⁴), such that $\text{Var}T(0, x) = \|x\|_2^{2\chi+o(1)}$. More specifically, for any $p > 0$ and a subadditive ergodic sequence $X_{m,n} \in \mathbb{N}$,

$$\begin{aligned}\bar{\chi}_p &:= \limsup_{n \in \mathbb{N}} \frac{\log \|X_{0,n} - \mathbb{E}X_{0,n}\|_p}{\log n}, \\ \underline{\chi}_p &:= \liminf_{n \in \mathbb{N}} \frac{\log \|X_{0,n} - \mathbb{E}X_{0,n}\|_p}{\log n}.\end{aligned}$$

In the case of FPP, we can then define the following for a given $x \in \mathbb{R}^d$:

$$\chi := \limsup_{n \in \mathbb{N}} \frac{\log |T(0, nx) - \mathbb{E}T(0, n)|}{\log n} = \liminf_{n \in \mathbb{N}} \frac{\log |T(0, nx) - \mathbb{E}T(0, n)|}{\log n} \quad [2].$$

5. GEODESICS

Definition 5.1. A (finite) *geodesic* in FPP is any path Γ between any two points $x, y \in \mathbb{Z}^d$ such that $T(\Gamma) = T(x, y)$. A geodesic between any $x \in \mathbb{Z}^d$ and set $A \subset \mathbb{R}^d$ is a geodesic between x and $y \in \hat{A}$, where $\hat{A} = \{x' \mid x' \in A\} \subset \mathbb{Z}^d$ as in Definition 2.1, such that $T(x, y) = \min_{y \in \hat{A}} T(x, y)$ [2]. We write $\mathcal{M}(x, y)$ (or $\mathcal{M}(x, A)$) to refer to the set of geodesics between x and y (or x and A).

Definition 5.2. For any $x, y \in \mathbb{Z}^d$, $\text{GEO}(x, y)$ is the self-avoiding geodesic between x and y with the maximum possible number of edges. We also find it useful to define the following paths:

$$\begin{aligned}\overline{\text{GEO}}(x, y) &:= \bigcup_{\Gamma \in \mathcal{M}(x, y)} \Gamma, \\ \underline{\text{GEO}}(x, y) &:= \bigcap_{\Gamma \in \mathcal{M}(x, y)} \Gamma.\end{aligned}$$

For any $x \in \mathbb{Z}^d$ and $A \subset \mathbb{R}^d$, $\text{GEO}(x, A)$, $\overline{\text{GEO}}(x, A)$, and $\underline{\text{GEO}}(x, A)$ are paths defined analogously to the above [2].

Through inspection, we observe that x, y are uniquely geodesic if and only if $\text{GEO}(x, y) = \overline{\text{GEO}}(x, y) = \underline{\text{GEO}}(x, y)$. The FPP metric space can be called *uniquely geodesic* if this holds true for any two points $x, y \in \mathbb{Z}^d$.

It is only proven that geodesics exist with probability one for a few cases- including when $d = 2$. To prove this, we define the *passage time to infinity* as

$$\rho := \lim_{n \rightarrow \infty} T(0, \partial B(n)), B(n) := [-n, n]^d \quad [2].$$

Lemma 5.3. $\rho = \inf\{T(\Gamma) \mid \Gamma \text{ is an infinite self-avoiding path from } 0\}$ [2].

Proof. Let (Γ_n) be a sequence of self-avoiding geodesics from 0 to $\partial B(n)$. There must exist some $(\Gamma_{n_k}) \subset (\Gamma_n)$ such that $\Gamma_{n_k} \rightarrow \Gamma$, that is for any $N \in \mathbb{N}$, there is some $K \in \mathbb{K}$ such that for all $k > K$, Γ_{n_k} and Γ share the first N steps. Denoting these steps by $\hat{\Gamma}_N$, we can write

⁴This is known to occur for at least one critical/supercritical case and no subcritical cases thus far [2], so I'm going to pretend it's never going to occur.

$$T(0, \partial B(n_k))_{k > K} \geq T(\hat{\Gamma}_N) \geq T(0, \partial B(N)).$$

Under $k \rightarrow \infty$, $T(0, \partial B(n_k)) \rightarrow \rho$, and under $N \rightarrow \infty$, $T(\hat{\Gamma}_N) \rightarrow T(\Gamma)$ and $T(0, \partial B(N)) \rightarrow \rho$. Then by the squeeze theorem, $\rho = T(\Gamma)$. $\square$

Theorem 5.4. (Wierman-Reh) [2]. *For any edge weight distribution function F , there almost surely exists a geodesic between any two points $x, y \in \mathbb{Z}^2$.*

Proof. For our purposes, assume $F(0) < p_c$ ⁵. Then there exists some $\delta > 0$ such that $F(\delta) < p_c$, so there is no infinite, connected, and self-avoiding path containing only $\tau_e \leq \delta$. Rather, we can see that since any Γ from 0 as in Lemma 5.3 has infinitely many infinite subpaths, it must contain infinitely many edges of weight $\tau_e > \delta$. Hence, $\rho = \inf T(\Gamma) = \infty$.

Let $x, y \in \mathbb{Z}^2$ with some path Γ between them. Then there exists $n \in \mathbb{N}$ such that $x \in B(n)$ and $T(x, \partial B(n)) > T(\Gamma)$, and by convexity, for any Γ' from x not contained in $B(n)$, $T(\Gamma') \geq T(x, \partial B(n)) \geq T(\Gamma)$. Hence,

$$\begin{aligned} T(x, y) &= \inf \{T(\Gamma) \mid \Gamma \text{ connecting } x \text{ and } y\} \\ &= \min \{T(\Gamma) \mid \Gamma \subset B(n), \Gamma \text{ connecting } x \text{ and } y\} \end{aligned}$$

so geodesics exist. $\square$

5.1. The wandering exponent. Section 3 and especially Proposition 3.9 suggest that for most edge weight distributions we are interested in, the limit shape will be convex. Additionally, we know that the length of a geodesic between 0 and x will be on the order of $\|x\|_1$. Hence, for large $\|x\|_1$, we expect $\overline{\text{GEO}}(0, x)$ to follow the straight line between 0 and x , that is,

$$L_x := \{mx \mid m \in \mathbb{R}\} \subset \mathbb{R}^d,$$

with some variation. Hence, we loosely define the *wandering exponent* ξ as the real number such that

$$\max \{\|y - z\|_2 \mid y \in \overline{\text{GEO}}(0, x), z \in L_x\} \sim \|x\|_2^\xi.$$

It is additionally useful to define the cylinder of radius $m > 0$ about axis L_x as

$$\mathcal{C}(x, m) := \{z \in \mathbb{R}^d \mid \min_{x \in L_x} \|z - x\|_2 \leq m\},$$

as we can imagine a geodesic bounded within it. There are several different variously "weak" or "strong" definitions for ξ , each with different proven (lower) bounds, which we now discuss further. It is still unproven how or if these definitions are equivalent [2].

The first two definitions are based on the upper bound of $\alpha \geq 0$ such that a set of possible geodesics is contained within $\mathcal{C}(x, \|x\|_2^\alpha)$. Hence, they are particularly suited for bounding ξ from below [6].

Definition 5.5. The first definition is based on "point-to-point" geodesics:

$$\xi^{(0)} := \sup \left\{ \alpha \geq 0 \mid \limsup_{\|x\|_2 \rightarrow 0} \mathbb{P}(\overline{\text{GEO}}(0, x) \subseteq \mathcal{C}(x, \|x\|_2^\alpha)) < 1 \right\} \quad [2].$$

⁵The cases $F(0) > p_c$ and $F(0) = p_c$ are also known to be geodesic but are harder to prove and lie outside the scope of this paper [2].

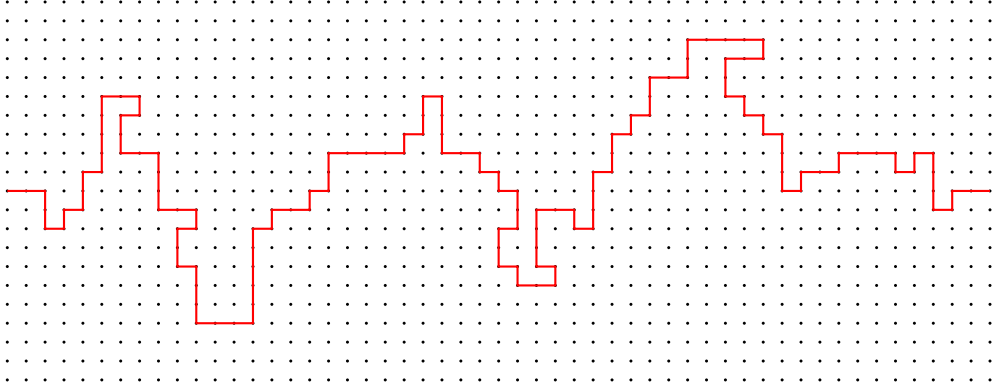


FIGURE 2. Depiction of geodesic wandering (in red) for $d = 2$. Given the horizontal length of the box n , the vertical length can be thought of as n^ξ .

The remaining definitions utilize “point-to-hyperplane” or “hyperplane-to-hyperplane” distance, so for any $L > 0$ and unit vector $\hat{u} \in \mathbb{R}^d$, we define the set

$$\Lambda(\hat{u}, L) := \{x \in \mathbb{Z}^d \mid x \cdot \hat{u} < L\}.$$

We can think of this as the vertices in the half-space containing the origin and bounded by the hyperplane perpendicular to $L\hat{u} \in \mathbb{R}^d$ [6].

Definition 5.6. The second definition is based on “point-to-plane” geodesics:

$$\xi^{(1)} := \sup\{\alpha \geq 0 \mid \limsup_{L \rightarrow \infty} \sup_{\hat{x} \in \mathbb{R}^d, \|\hat{x}\|_2=1} \sup_{0 \neq x \in \mathbb{R}^d} \mathbb{P}(\mathcal{M}(0, \partial\Lambda(\hat{x}, L)) \subseteq \mathcal{C}(x, \|x\|_2^\alpha)) < 1\} \text{ [6].}$$

The next two definitions are “weaker” as instead of geodesics, they invoke “near-geodesics”, that is, for $M > 0$ and $x, y \in \mathbb{Z}^d$,

$$\overline{\text{GEO}}(x, y; M) := \bigcup \{\text{paths } \Gamma \text{ between } x \text{ and } y \text{ such that } T(\Gamma) \leq T(x, y) + M\}.$$

Furthermore, for any path $\Gamma \subset \mathbb{R}^d$ and $M > 0$, let

$$R(\Gamma, M) := \{x \in \Gamma \mid T(0, x) \leq T(0, \Gamma) + M\},$$

where $T(0, \Gamma) = \min\{T(0, y) \mid y \in \Gamma\}$ [6].

Definition 5.7. The third definition is based on “point-to-plane” distance:

$$\xi^{(2)}(M) := \sup\{\alpha \geq 0 \mid \exists(\hat{\theta}_n) \subset \mathbb{R}^d, (J_n) \subset \mathbb{R} \text{ such that } J_n \rightarrow \infty \text{ and}$$

$$\exists \text{ deterministic } A_n \text{ with } L^2 \text{ diam}(A_n) \leq J_n^\alpha \text{ such that}$$

$$P(R(\partial\Lambda(\hat{\theta}_n, J_n), M)) \subseteq A_n \rightarrow 1\}$$

for $M > 0$ and unit vectors $(\hat{\theta}_n)$ [6].

Despite being the vaguest, this definition allows us to limit our discussion of “wandering” to the endpoints of the geodesic [2]. Its definition is discussed more in depth in Theorem 5.16.

Definition 5.8. Again, the fourth definition is based on “point-to-plane” geodesics:

$$\xi^{(3)}(M) := \sup\{\alpha \geq 0 \mid \exists(\hat{\theta}_n) \subset \mathbb{R}^d, (J_n) \subset \mathbb{R} \text{ such that } J_n \rightarrow \infty \text{ and}$$

$$! \exists \text{ deterministic } x_n \in \mathbb{R}^d \text{ such that}$$

$$P(\overline{\text{GEO}}(0, \partial\Lambda(\hat{\theta}_n, J_n); M) \subseteq \mathcal{C}(x_n, J_n^\alpha)) \rightarrow 1\}$$

for $M > 0$ and unit vectors $(\hat{\theta}_n)$ [6].

This definition can be viewed as a combination of Definitions 5.6 and 5.7, since it looks for near-geodesics bounded by a cylinder [2].

5.1.1. *Upper bounds.* Bounding ξ upward should be invariant of the definitions given in the previous section, as for $\epsilon > 0$, we expect

$$\begin{aligned} \lim_{\|x\|_2 \rightarrow \infty} \mathbb{P}(\overline{\text{GEO}}(0, x) \subseteq \mathcal{C}(x, \|x\|_2^{\xi+\epsilon})) &= 1, \\ \lim_{\|x\|_2 \rightarrow \infty} \mathbb{P}(\overline{\text{GEO}}(0, x) \subseteq \mathcal{C}(x, \|x\|_2^{\xi-\epsilon})) &= 0, \end{aligned}$$

implying ξ should be the same in all directions. Then for some unit vector $\hat{\theta} \in \mathbb{R}^d$, this allows us to define the wandering and fluctuation exponents as⁶

$$\begin{aligned} \xi_{\hat{\theta}} &:= \inf \left\{ \alpha > 0 \mid \liminf_{n \in \mathbb{N}} \mathbb{P}(\overline{\text{GEO}}(0, n\hat{\theta}) \subseteq \mathcal{C}(\hat{\theta}, n^\alpha)) > 0 \right\}, \\ (5.9) \quad \chi' &:= \inf\{\kappa \in \mathbb{R} \mid \text{for large } t, \text{ almost surely } (t - t^\kappa)\mathcal{B} \subseteq B(t) \subseteq (t + t^\kappa)\mathcal{B}\}, \\ \chi_{\hat{\theta}} &:= \sup\{\alpha > 0 \mid \exists C > 0 : \forall n \in \mathbb{N}, n \geq 1, \text{Var}T(0, n\hat{\theta}) \geq Cn^{2\alpha}\}. \end{aligned}$$

Furthermore, we call $\hat{\theta}$ a *direction of curvature for $\mathcal{B}$* if it satisfies the lower bound on $\mu(\hat{\theta} + z) - \mu(\hat{\theta})$ in Definition 3.8 for $\kappa = 2$ [2].

Theorem 5.10. [2]. Assume $F(0) < p_c$, $d \geq 2$, and $\mathbb{E}\tau_e^2 < \infty$. If $\hat{\theta} \in \mathbb{R}^n$ is a direction of curvature,

$$\xi_{\hat{\theta}} \leq \frac{1 + \chi'}{2}.$$

Proof. For simplicity’s sake, write $\chi = \chi' < 1$. Since the actual proof is quite complicated, in order to avoid having to make further arguments involving concentration of measure, we assume this property of χ (after which the steps of the actual proof generally follow):

$$(5.11) \quad \exists c > 0 : \forall x \in \mathbb{Z}^d \text{ and } \lambda \geq 0, \mathbb{P}(|T(0, x) - \mu(x)| \geq \lambda \|x\|_2^\chi) \leq e^{-c\lambda}.$$

Without loss of generality, pick $\hat{\theta} = e_1$. Let $\xi' = \frac{1+\chi}{2} + \epsilon < 1$ for some $\epsilon > 0$ and $L_n \subset \mathbb{R}^d$ be the line segment connecting 0 and ne_1 for $n \in \mathbb{N}$. Define

$$V := \{v \in \mathbb{Z}^d \mid \exists z \in L_n : \|z - v\|_2 \in [n^{\xi'}, 2n^{\xi'}]\}.$$

We want to show that for $n \rightarrow \infty$,

⁶Note that χ' below is not the same as the actual fluctuation exponent as in Section 4, since it includes non-random variation [2].

$$\begin{aligned}
(5.12) \quad & \sum_{v \in V} \mathbb{P}(\exists \Gamma \in \mathcal{M}(0, ne_1) : v \in \Gamma) \\
&= \sum_{v \in V} \mathbb{P}(T(0, ne_1) = T(0, v) + T(v, ne_1))
\end{aligned}$$

converges to 0, in which case $\xi_{\hat{\theta}} \leq \xi'$. To do this, we need to prove the following:

Lemma 5.13. $\exists C > 0 : \forall v \in V, \mu(v) + \mu(ne_1 - v) - \mu(ne_1) \geq Cn^{\chi+2\epsilon}$.

Proof. Let H_0 be the hyperplane as in Definition 3.8 and $w \in \mathbb{R}^d$ be the projection of v onto the e_1 -axis along H_0 , so $\mu(w) + \mu(ne_1 - w) = \mu(ne_1)$. Note that by convexity of the limit shape, $\mu(v) - \mu(w), \mu(ne_1 - v) - \mu(ne_1 - w) \geq 0$. Hence:

$$\mu(v) + \mu(ne_1 - v) - \mu(ne_1) = \mu(v) - \mu(w) - \mu(ne_1 - w) + \mu(ne_1 - v) \geq \mu(v) - \mu(w).$$

Furthermore, by the definition of direction of curvature,

$$\mu(v) - \mu(w) = \|w\|_2 \left(\mu\left(\frac{v-w}{\|w\|_2} + \frac{w}{\|w\|_2}\right) - \mu\left(\frac{w}{\|w\|_2}\right) \right) \geq C\|w\|_2 \|v-w\|_2^2.$$

We now have three cases based on the position of w :

Case 1: Let $w \in L_n$. Without loss of generality, assume that $\|v\|_2 \geq n/2$ (since either v or $ne_1 - v$ must have an L_2 magnitude of at least $n/2$). Then since $\|v-w\|_2 \geq n^{\xi'}$:

$$\mu(v) + \mu(ne_1 - v) - \mu(ne_1) \geq C\|w\|_2 \|v-w\|_2^2 \geq Cn^{2\xi'-1} = Cn^{\chi+2\epsilon}.$$

Case 2: Let $w \notin L_n$ be L_1 closer to ne_1 than 0, i.e. $n \leq \|w\|_2 \leq n + 2n^{\xi'}$. Then by the definition of V ,

$$n^{2\xi'} \leq \|v - ne_1\|_2^2 \leq \|w - ne_1\|_2^2 + \|v - w\|_2^2 = (\|w\|_2 - n)^2 + \|v - w\|_2^2.$$

If $\|v - w\|_2^2 \geq n^{2\xi'}/2$, we have the same as Case 1. If $(\|w\|_2 - n)^2 \geq n^{2\xi'}/2$, then by the direction of curvature, for sufficiently large n :

$$\begin{aligned}
\|w\|_2 - n \geq n^{\xi'}/\sqrt{2} &\implies \mu(v) + \mu(ne_1 - v) - \mu(ne_1) \geq \mu(w) - \mu(ne_1) \\
&= n \left(\mu\left(\frac{w - ne_1}{n} + e_1\right) - \mu(e_1) \right) \geq Cn^{2\xi'}/2 \geq Cn^{2\xi'-1} = Cn^{\chi+2\epsilon}.
\end{aligned}$$

Case 3: Let $w \notin L_n$ be L_1 closer to 0 than ne_1 , i.e. $0 \leq \|w\|_2 \leq 2n^{\xi'}$. Analogously to the previous case,

$$n^{2\xi'} \leq \|v\|_2^2 \leq \|v - w\|_2^2 + \|w\|_2^2.$$

If $\|v - w\|_2^2 \geq n^{2\xi'}/2$, we again have the same as Case 1. If $\|w\|_2^2 \geq n^{2\xi'}/2$, then $n^{\xi'}/\sqrt{2} \leq \|w\|_2 \leq \|ne_1 - w\|_2$, so we have the same as Case 2. $\square$

Combining Lemma 5.13 with the event in (5.12), we have

$$(T(0, ne_1) - \mu(ne_1)) - (T(0, v) - \mu(v)) - (T(v, ne_1) - \mu(ne_1 - v)) \geq Cn^{\chi+2\epsilon}.$$

Hence, for some $x \in \{ne_1, v, ne_1 - v\}$, $|T(0, x) - \mu(x)| \geq (C/3)n^{\chi+2\epsilon} \geq An^b\|x\|_2^\chi$ for some $A, b > 0$ ⁷. Applying (5.11) gives

$$\mathbb{P}(\exists \Gamma \in \mathcal{M}(0, ne_1) : v \in \Gamma) \leq e^{-cA}e^{-n^b} \rightarrow 0$$

for all $v \in V$ and $n \rightarrow \infty$. Then since

$$\forall \epsilon > 0, \xi_{\hat{\theta}} \leq \frac{1+\chi}{2} + \epsilon,$$

we have the desired result. $\square$

Theorem 5.14. [2]. Assume $F(0) < p_c$ and $\mathbb{E}e^{\alpha\tau_e} < \infty$ for some $\alpha > 0$. Let $\hat{\theta} \in \mathbb{R}^d$ be a direction of curvature for $\mathcal{B}$. Then:

$$\xi_{\hat{\theta}} \leq 3/4,$$

and if $d = 2$,

$$\chi_{\hat{\theta}} \geq 1/8.$$

Proof. It has been shown that our simplifying assumption of (5.11) works for any $\chi = 1/2 + \delta$, $\delta > 0$ [2], giving us $\xi_{\hat{\theta}} \leq 3/4$. The $d = 2$ case comes from the Wehr-Aizenmann bound $\chi_{\hat{\theta}} \geq \frac{1-(d-1)\xi_{\hat{\theta}}}{2}$ [2], the proof of which is beyond the scope of this paper. $\square$

5.1.2. *Lower bounds.* The following bounds are proven by Licea, Newman, and Piza in [6] for the different definitions in Section 5.1.

Theorem 5.15. Let $d \geq 2$ and $\mathbb{E}\tau_e^2 < \infty$. Then

$$\xi^{(0)}, \xi^{(1)} \geq \frac{1}{d+1}.$$

Proof. We only show this for $\xi^{(0)}$ since the proof for $\xi^{(1)}$ follows almost identically. Let $\alpha > \xi^{(0)}$ and $(x_n) \subset \mathbb{Z}^d$ such that

$$\|x_n\|_2 \rightarrow \infty, \mathbb{P}(\overline{\text{GEO}}(0, x_n) \subseteq \mathcal{C}(x_n, \|x_n\|_2^\alpha)) \rightarrow 1$$
⁸.

Define $(\hat{y}_n) \subset \mathbb{R}^d$ such that for each $n \in \mathbb{N}$, $\|\hat{y}_n\|_2 = 1$ and $\hat{y}_n \perp x_n$. For a fixed $C > 0$, let $0', x'_n \in \mathbb{Z}^d$ correspond to $C\|x_n\|_2^\alpha \hat{y}_n, x_n + C\|x_n\|_2^\alpha \hat{y}_n \in \mathbb{R}^d$ respectively, as in Definition 2.1, with Γ_1 connecting $0'$ to 0 and Γ_2 connecting x'_n to x_n . Furthermore, let $\Delta T := T(0, x_n) - T(0', x'_n)$. Then:

$$\begin{aligned} |\Gamma_1|, |\Gamma_2| &\leq C\|x_n\|_2^\alpha, T(0, x_n) \leq T(0, 0') + T(0', x_n), T(0', x'_n) \leq T(0', x_n) + T(x_n, x'_n), \\ &\implies |\Delta T| \leq T(0, 0') - T(x_n, x'_n) \leq T(\Gamma_1) + T(\Gamma_2) \lesssim C\|x_n\|_2^\alpha \end{aligned}$$

since this upper bound is limited by its end behavior. However, we can find an exact bound for $\text{Var}\Delta T \leq C\|x_n\|_2^{2\alpha}$ using a technique originated by Newman and Piza

⁷We can write $T(v, ne_1) \sim T(0, ne_1 - v)$ since for large enough n they should not differ meaningfully.

⁸This assumption is what changes for bounding $\xi^{(1)}$ as opposed to $\xi^{(0)}$.

[2]. Let Σ_j be the sigma algebra generated by $\{\tau_{e_{k_1}}, \dots, \tau_{e_{k_j}}\}$ for some collection of edges $\{e_{k_1}, \dots, e_{k_j}\}$ ⁹. Then since $\{\Delta T \mid \Sigma_j\}$ is a martingale:

$$\text{Var} \Delta T = \sum_{j=1}^{\infty} \mathbb{E} (\mathbb{E}\{\Delta T \mid \Sigma_j\} - \mathbb{E}\{\Delta T \mid \Sigma_{j-1}\})^2.$$

Define the events

$$F_j := \{\tau_{e_{k_j}} = 1 \text{ and } e_{k_j} \in \overline{\text{GEO}}(0, x_n)\},$$

$$G_j := \{e_{k_j} \in \overline{\text{GEO}}(0, x_n), e_{k_j} \notin \overline{\text{GEO}}(0', x'_n)\}.$$

Hence, for some $C_1 > 0$ (dependent on the distribution F):

$$\mathbb{E} (\mathbb{E}\{\Delta T \mid \Sigma_j\} - \mathbb{E}\{\Delta T \mid \Sigma_{j-1}\})^2 \geq C_1 \mathbb{P}(G_j)^2 \implies \text{Var} \Delta T \geq C_1 \sum_{j=1}^{\infty} \mathbb{P}(G_j)^2.$$

Furthermore, since we are only concerned with $j \in \mathcal{J} := \{j \mid e_{k_j} \text{ in } \mathcal{C}(x_n, \|x_n\|_2^\alpha)\}$:

$$\begin{aligned} \sum_{j \in \mathcal{J}} \mathbb{P}(G_j) &\geq \mathbb{E} |\overline{\text{GEO}}(0, x_n) \cap \mathcal{C}(x_n, \|x_n\|_2^\alpha)| - \mathbb{E} |\overline{\text{GEO}}(0', x'_n) \cap \mathcal{C}(x_n, \|x_n\|_2^\alpha)| \\ &\rightarrow \mathbb{E} |\overline{\text{GEO}}(0, x_n)| \geq C_2 \|x_n\|_2 \end{aligned}$$

for large enough n and some $C_2 > 0$. Then by Jensen's inequality:

$$\begin{aligned} \sum_{j \in \mathcal{J}} \mathbb{P}(G_j) &\geq C_2 \|x_n\|_2 \implies \sum_{j \in \mathcal{J}} \mathbb{P}(G_j)^2 \geq \frac{1}{|\mathcal{J}|} \left(\sum_{j \in \mathcal{J}} \mathbb{P}(G_j) \right)^2 \\ &= \frac{1}{|\mathcal{C}(x_n, \|x_n\|_2^\alpha)|} \left(\sum_{j \in \mathcal{J}} \mathbb{P}(G_j) \right)^2 \geq \frac{C_2^2 \|x_n\|_2^2}{|\mathcal{C}(x_n, \|x_n\|_2^\alpha)|}, \\ \implies \text{Var} \Delta T &\geq \frac{C_1 C_2^2 \|x_n\|_2^2}{|\mathcal{C}(x_n, \|x_n\|_2^\alpha)|} \geq \frac{C_1 C_2^2 \|x_n\|_2^2}{\|x_n\|_2 \|x_n\|_2^{\alpha(d-1)}} \geq C \|x_n\|_2^{1+\alpha-\alpha d}. \end{aligned}$$

Hence, we have:

$$\begin{aligned} \|x_n\|_2^{1+\alpha-\alpha d} \leq \|x_n\|_2^{2\alpha} &\implies 1 + \alpha(1-d) \leq 2\alpha, \\ \implies \alpha &\geq 1/(1+d) \implies \xi^{(0)} \geq \frac{1}{1+d}. \end{aligned}$$

□

Theorem 5.16. *Let $d \geq 2$, $P(\tau_e = 0) < p_c$, and $M \geq 0$ such that $P(\tau_e \leq M) > 0$. Then*

$$\xi^{(2)}(M) \geq 1/2.$$

⁹We can imagine each Σ_j as adding an edge weight to the previous configuration, where Σ_0 corresponds to a lattice with all $\tau_e = 0$.

Proof. Fix $M > 0$ and suppose for the sake of contradiction that $\xi^{(2)} < 1/2$, so we can choose some $\alpha \in (\xi^{(2)} + \epsilon, 1/2)$. Without loss of generality, let $\{\hat{\theta}_i\}_{1 \leq i \leq n} = (\cos \theta_i e_1 + \sin \theta_i e_2)_{1 \leq i \leq n} \subset \mathbb{R}^d$ be unit vectors in the $e_1 \times e_2$ plane such that for $L > 0$, we define the “ n -gon barrel”

$$P_L := \bigcap_{1 \leq i \leq n} \Lambda(\hat{\theta}_i, L_i) \times \mathbb{R}^{d-2}$$

where for each $1 \leq i \leq n$, $L_i \sim L$. We can effectively treat this as a 2D n -gon with sides S_i corresponding to $\hat{\theta}_i$. Let $(\hat{\theta}_i)_{1 \leq i \leq n}$ and $(L_i)_{1 \leq i \leq n}$ be chosen such that:

$$\begin{aligned} \exists a_1, a_2, a_3, a_4 \in \mathcal{N} : 1 \leq a_1, a_2, a_3, a_4 \leq n, \\ \{\hat{\theta}_{a_1}, \hat{\theta}_{a_2}, \hat{\theta}_{a_3}, \hat{\theta}_{a_4}\} = \{(\pm 1, 0), (0, \pm 1)\}, L_{a_1} = L_{a_2} = L_{a_3} = L_{a_4} = L, \\ \forall i \notin \{a_1, a_2, a_3, a_4\}, S_i = D > 0. \end{aligned}$$

Furthermore, for some general $\hat{\theta} = \cos \theta e_1 + \sin \theta e_2$, let some $A(\theta, L) \subseteq \partial \Lambda(\hat{\theta}, L)$ be called *good* if

$$\mathbb{P}(R(\partial \Lambda(\hat{\theta}, L), M) \subseteq A(\theta, L)) \geq 1 - \epsilon,$$

in other words, if it has a high probability of containing endpoints of geodesics to it. For some $D' > 0$, define the slab

$$S(\theta, \gamma, D') := \{x = (x_1, \dots, x_d) \in \mathbb{Z}^d \mid -D'/2 \leq x_1 \sin \theta + x_2 \cos \theta - \gamma \leq D'/2\}.$$

Then there exists some $\bar{\gamma} = \bar{\gamma}(\theta, L)$ such that, for each $0 \leq \theta \leq \pi/2$ and $L > 0$,

$$(5.17) \quad \bar{A}(\theta, L) := S(\theta, \bar{\gamma}(\theta, L), \hat{D}_L(\theta)) \cap \partial \Lambda(\hat{\theta}, L),$$

is good, where

$$\hat{D}_L(\theta) := 5/2 + \sqrt{2} + \inf\{D \mid \exists \gamma' > 0 : S(\theta, \gamma', D) \cap \partial \Lambda(\hat{\theta}, L) \text{ is good}\}.$$

Our goal in defining these terms is to construct the sequences $(\hat{\theta}_i)_{1 \leq i \leq n}$ and $(L_i)_{1 \leq i \leq n}$ such that for $1 \leq i \leq n$, $|S_i| \sim L^\alpha$, and each “face” of the n -gon barrel contains a good subset $A_i := \bar{A}(\theta_i, L_i)$ as in (5.17) containing or close to the line bisecting that face.

First, let $\hat{\theta}_1 = e_1$ and $L_1 = L$ with $|S_1| = L^\alpha$. Let $\hat{\theta}_n = \frac{1}{\sqrt{2}}(e_1 + e_2)$. For $1 \leq i \leq n$, let $z_i \in \mathbb{R}^d$ be the counterclockwise endpoint of S_i . Then we can fill in the remaining points inductively as follows, for $1 \leq j \leq n-1$:

- If the L_2 distance between z_j and the line $\{a\hat{\theta}_n \mid a \in \mathbb{R}\}$ is at most $o(L^\alpha)$, we can choose $\hat{\theta}_{j+1} = \hat{\theta}_n$ and $L_{j+1} > 0$ such that $z_j \in \partial \Lambda(\hat{\theta}_{j+1}, L_{j+1})$.
- Otherwise, choose $|S_{j+1}| = L^\alpha$ and $\hat{\theta}_{j+1} \in \mathbb{R}^d$ such that $\theta_{j+1} \in (\theta_j, \pi/4)$ and we can choose an appropriate A_{j+1} . Note that the choice of such a θ_j is possible because of the continuity of our choice of $S(\theta_i, \gamma, D')$ (and its position relative to $\bar{A}(\theta_i, L)$ with respect to θ_i [6]).

Using this construction, we have a sequence of $(A_i)_{1 \leq i \leq n}$ such that each A_i is within an L_2 distance of $(\sqrt{d}+1)/2$ from the line bisecting each “face” of the n -gon barrel. Additionally, for each $1 \leq j \leq n-1$, $\theta_{j+1} - \theta_j$ is at most order $o(L^{\alpha-1})$, and

$\partial\Lambda(\hat{\theta}_j, L_j)$ is a distance of $o(L^{2\alpha-1}) \leq o(1)$ from the aforementioned bisecting line of the face associated with S_{j+1} . However, this implies that there is a path between $\partial\Lambda(\hat{\theta}_j, L_j)$ and the face S_{j+1} with an $O(1)$ number of edges (extending to passage time), meaning there is a “near-geodesic” from 0 to $\partial\Lambda(\hat{\theta}_j, L_j)$ that intersects S_{j+1} , which contradicts our construction¹⁰.

Therefore, we can conclude that $\xi^{(2)} \geq 1/2$. $\square$

Theorem 5.18. *Let $d = 2$, $\mathbb{E}\tau_e^2 < \infty$, and $M > 0$ such that $P(\tau_e \leq M) > 0$. Then*

$$\xi^{(3)}(M) \geq 3/5.$$

Proof. Let $\alpha > \xi^{(3)}(M)$. We can construct the n -sided polygonal barrel P_L as in the proof of Theorem 5.15. For $1 \leq j \leq n-1$, define (as in the proof of Theorem 5.15)

$$\Delta T := T(0, \partial\Lambda(\hat{\theta}_{j+1}, L_{j+1})) - T(0, \partial\Lambda(\hat{\theta}_j, L_j)).$$

As in the proof of Theorem 5.16, we have that for $1 \leq j \leq n-1$, the distance between $\partial\Lambda(\hat{\theta}_j, L_j)$ and the face corresponding to S_{j+1} is roughly $o(L^{2\alpha-1})$. Again, this implies that there is a path between $\partial\Lambda(\hat{\theta}_j, L_j)$ to $\partial\Lambda(\hat{\theta}_{j+1}, L_{j+1})$ with passage time less than or similar to $L^{2\alpha-1}$. Hence,

$$\text{Var}\Delta T \lesssim L^{4\alpha-2}.$$

Additionally, as in the proof of Theorem 5.15, we can find the same lower bound for $\text{Var}\Delta T$ using martingale approximation, which gives

$$\text{Var}\Delta T \gtrsim CL^{1+\alpha-\alpha d} = CL^{1-\alpha}$$

for some fixed $C > 0$. Hence, we have

$$\begin{aligned} L^{1-\alpha} &\leq L^{4\alpha-2} \implies 1-\alpha \leq 4\alpha-2, \\ \implies \alpha &\geq 3/5 \implies \xi^{(3)} \geq 3/5. \end{aligned}$$

$\square$

5.1.3. *The scaling relation.* In FPP, it is conjectured that for $d \geq 2$,

$$(5.19) \quad \chi = 2\xi - 1;$$

in other words, there is a linear relationship between the variation between $T(0, x)$ and $\mathbb{E}T(0, x)$ (χ) and the variation between some $\Gamma \in \overline{\text{GEO}}(0, x)$ and L_x (ξ)¹¹. From the proof of Theorem 5.10, we can reason that $\chi \geq 2\xi - 1$. However, the other direction remains unproven- moreover, an edge weight configuration yielding (5.19) has not yet been determined [2].

6. APPLICATIONS

Beyond fluid flow, there are several important physical applications of FPP- most obviously, it is related to several models of disease spread. Additionally, the study of geodesics is applicable to the 2-dimensional Ising ferromagnet in condensed matter theory [2].

¹⁰Since we assume that each A_j contains with high likelihood geodesics “localized” to that side.

¹¹This is the famous KPZ relation [3].

6.1. 2-dimensional Ising ferromagnet. We can extend our notion of geodesics between two points to *geodesic lines*, which extend infinitely in some two given directions, and are defined such that between any two points on the geodesic line is a finite geodesic. The 2-dimensional Ising ferromagnet can be modeled as a dual lattice $\mathbb{Z}_*^2$, with a *spin configuration* given by $\sigma \in \{\pm 1\}^{\mathbb{Z}_*^2}$. As in FPP, we assign independent and identically-distributed random variables to the edges ε_*^2 of the dual lattice, notated by $(J_{x,y})_{x,y \in \varepsilon_*^2}$. The (Hamiltonian) energy for a given spin configuration and finite $S \subset \mathbb{Z}_*^2$ is given by

$$(6.1) \quad H_S(\sigma) := - \sum_{x,y \in \varepsilon_*^2, x \in S} J_{x,y} \sigma_x \sigma_y.$$

A *ground state* is a spin configuration σ such that for all $\rho \in \{\pm 1\}^{\mathbb{Z}_*^2}$ with $\rho_x = \sigma_x$ for all x outside of a finite set, $H_S(\sigma) \leq H_S(\rho)$ for all finite $S \subset \mathbb{Z}_*^2$. We are interested in non-trivial ground states. In these configurations, we can see that there cannot be a finite $S \subset \mathbb{Z}_*^2$ such that $\sigma_x = 1$ for all $x \in S$ and $\sigma_x = -1$ for all $x \in \partial S$, or vice versa. Then, a non-trivial ground state is either constant, or there is an infinite, circuitless collection of edges dividing the lattice into disjoint sections S_1 and S_2 , with $\sigma_x = 1$ for $x \in S_1$ and $\sigma_x = -1$ for $x \in S_2$. Such a collection of edges is given by a geodesic line in FPP with $\tau_e = J_{x,y}$ for $\{x,y\}$ dual to e . Hence, the problem of existence of geodesic lines has a physical significance [2].

ACKNOWLEDGMENTS

It is a pleasure to thank my mentor, DeVon Ingram, for his dedication proof-reading my paper and providing me with important resources.

7. BIBLIOGRAPHY

REFERENCES

- [1] S. Alm, M. Deijfen. First passage percolation on $\mathbb{Z}^2$ — a simulation study. arXiv. 2014. <https://arxiv.org/pdf/1412.5924>.
- [2] A. Auffinger, M. Damron, J. Hanson. 50 Years of First-Passage Percolation. American Mathematical Society. 2010.
- [3] S. Chatterjee. The universal relation between scaling exponents in first-passage percolation. Annals of Mathematics. 2013. <https://annals.math.princeton.edu/wp-content/uploads/annals-v177-n2-p07-p.pdf>.
- [4] S. Ganguly. Random metric geometries on the plane and Kardar-Parisi-Zhang universality. arXiv. 2021.
- [5] G. Lawler. Notes on Probability. 2016. <https://www.math.uchicago.edu/~lawler/probnotes.pdf#page=1&zoom=auto,-74,388>.
- [6] C. Licea, C.M. Newman, M.S.T. Piza. Superdiffusivity in first-passage percolation. Probability Theory and Related Fields. 1996. <https://link.springer.com/article/10.1007/s004400050075>.
- [7] J. Pitman, S. Oh. Lecture 12: Subadditive Ergodic Theorem (UC Berkeley Statistics 205b: Probability Theory). 2003. <https://www.stat.berkeley.edu/~pitman/s205s03/lecture12.pdf>.
- [8] C. Stover and E. W. Weinstein. "Percolation Threshold." From MathWorld—A Wolfram Resource. <https://mathworld.wolfram.com/PercolationThreshold.html>.
- [9] W. van Saarloos, V. Vitelli, Z. Zeravcic. Soft Matter: Concepts, Phenomena, and Applications. Princeton University Press. 2024.