

HAAR MEASURE, SPECTRAL THEORY AND THE PETER-WEYL THEOREM

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ABSTRACT. The Peter-Weyl theorem asserts the denseness of the matrix coefficients of a locally compact group G in the Hilbert space $L^2(G)$. In this paper, we first present construction of a Haar measure, then build prerequisite material concerning functional analysis and representation theory, and finally, prove the the Peter-Weyl theorem.

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1. INTRODUCTION

In 1927, Hermann Weyl and Peter Fritz proved the denseness of the matrix coefficients of a compact Lie group G in $C(G)$, as well as the reducibility of any unitary representation of G [1]. Following this, in 1933, Alfréd Haar developed the Haar measure, a translation-invariant, countably additive, nontrivial measure on a locally compact group G [2]. Haar's results had several immediate applications: John von Neumann solved Hilbert's fifth problem on compact groups in an article directly succeeding Haar's [3]. In addition, Haar's measure was used to extend the Peter-Weyl theorem to the setting of compact groups.

We begin by proving existence and uniqueness of the Haar measure up to a scalar multiple on locally compact groups. After, we develop the prerequisite knowledge in representation theory and functional analysis, proving the spectral theorem for compact operators. Finally, we bring everything together, culminating in the Peter-Weyl theorem concerning the denseness of matrix coefficients in $L^2(G)$.

This paper assumes familiarity with topology, group theory, linear algebra and measure theory. We freely use Fubini-Tonelli theorem (proven in [4]) the Arzelà-Ascoli theorem (proven in [6]), and more. In addition, the Riesz-Markov representation theorem is stated and used without proof. See [4] for a detailed proof.

2. PRELIMINARIES

2.1. Topological Preliminaries.

Definition 2.1 ($C_c(X)$). Let X be a locally compact Hausdorff space. The set of all complex (resp. real)-valued continuous functions with compact support on X is denoted by $C_c(X)$ (resp. $C_c^{\mathbb{R}}(X)$).

Definition 2.2 (Positive Cone). Let X be a locally compact Hausdorff space. The positive cone, denoted $C_c^+(X)$, is given by

$$C_c^+(X) = \{f \in C_c^{\mathbb{R}}(X) : f(x) \geq 0 \text{ for all } x \in X\}.$$

Lemma 2.3. Let X be a locally compact Hausdorff space, let $U \subseteq X$ be open, and let $K \subseteq U$ be compact. Then there exists an open set N with compact closure such that $K \subseteq N \subseteq \text{cl}_X(N) \subseteq U$.

Proof. Let Y be the one-point compactification of X . Since X is a locally compact Hausdorff space, Y is normal. Since U is open in X , U is open in Y . Since K is compact in X , it is compact in Y . Since Y is Hausdorff, K is closed. Since Y is normal, there exists an open set V in Y such that $K \subseteq V$ and $\text{cl}_Y(V) \subseteq U$. Then $V \subseteq X$, so V is open in X .

Let $x \in K$. Since X is locally compact, there exists an open neighborhood N_x of x such that $\text{cl}_X(N_x)$ is compact. Then $N_x \cap V$ is an open neighborhood of x . Furthermore, $\text{cl}_X(N_x \cap V)$ is compact and contained in U . By the compactness of K , there exists finitely many points $\{x_1, \dots, x_n\}$ such that $K \subseteq \bigcup_{i=1}^n (N_{x_i} \cap V)$. Set

$N = \bigcup_{i=1}^n (N_{x_i} \cap V)$. Then $\text{cl}_X(N) = \bigcup_{i=1}^n \text{cl}_X(N_{x_i} \cap V)$. Since $\text{cl}_X(N_{x_i} \cap V)$ is compact and contained in U for each i , $\text{cl}_X(N)$ is compact and contained in U . In addition, N is open as it is the union of open sets. Gathering, we see $\text{cl}_X(N)$ is compact and

$$K \subseteq N \subseteq \text{cl}_X(N) \subseteq U. \quad \square$$

Theorem 2.4. Let X be a locally compact Hausdorff space. Suppose $U \subseteq X$ is open and $K \subseteq U$ is compact. Then there exists a continuous function $f : X \rightarrow [0, 1]$ with compact support such that $\text{supp } f \subseteq U$ and $f(K) = \{1\}$.

Proof. Let Y be the one-point compactification of X . By the previous lemma, there is an open set N with compact closure such that

$$K \subseteq N \subseteq \text{cl}_X(N) \subseteq U.$$

Observe K and $Y \setminus N$ are disjoint closed sets in Y . By Urysohn's Lemma, there exists a continuous function $h : Y \rightarrow [0, 1]$ such that $h(Y \setminus N) = \{0\}$ and $h(K) = \{1\}$. Let $f = h|_X$. Then $f : X \rightarrow [0, 1]$ is continuous and $f(K) = \{1\}$. In addition, $\{x \in X : f(x) \neq 0\} \subseteq N$. Thus,

$$\text{supp } f \subseteq \text{cl}_X(N) \subseteq U.$$

Since $\text{cl}_X(N)$ is compact and $\text{supp } f$ is closed, f has compact support. $\square$

Definition 2.5 (Topological Group). A topological group G is a group that is also a T_1 topological space such that the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous.

A remarkable fact of topological groups we will not prove is that simply by assuming they are T_1 topological spaces, we can obtain they are regular (T_3) topological spaces. By further assuming local compactness, we obtain that they are normal (T_4) spaces, allowing use of Urysohn's lemma and related tools.

Definition 2.6. Let G be a topological group and let $A, B \subseteq G$. Then $AB = \{ab : a \in A, b \in B\}$ and $A^{-1} = \{a^{-1} : a \in A\}$.

Proposition 2.7. Let G be a topological group and let U be a neighborhood of e . Then there exists a symmetric neighborhood V (that is, $V = V^{-1}$) of e such that $V^2 \subseteq U$.

Proof. Let m be the multiplication map. Since U is open and G is a topological group, $m^{-1}(U)$ is open in $G \times G$. Then there exists a basic open set $A \times B \subseteq m^{-1}(U)$ such that $(e, e) \in A \times B$. Let $W = A \cap B$. Then W is open. Note $e \in W$.

Set $V = W \cap W^{-1}$. Again, $e \in V$. In addition, V is open as it is the intersection of open sets. Furthermore, if $v \in V$, then $v \in W \cap W^{-1}$. Thus, $v^{-1} \in W^{-1} \cap W$, so $v^{-1} \in V$. Thus, $V \subseteq V^{-1}$.

Similarly, if $x \in V^{-1}$, then $x = v^{-1}$ for some $v \in V = W \cap W^{-1}$. Then $x \in W^{-1} \cap W$, so $x \in V$. Thus, $V^{-1} \subseteq V$. Thus, $V = V^{-1}$. Thus, V is symmetric. Observe $m(V \times V) \subseteq m(W \times W) \subseteq m(A \times B) \subseteq U$, so $V^2 \subseteq U$. $\square$

Proposition 2.8. Let G be a topological group and $K \subseteq G$ be compact. Let U be an open set containing K . Then there exists a neighborhood V of e such that $KV \cup VK \subseteq U$.

Proof. For each $x \in K$, let $V_{x,1}$ be a neighborhood of e such that $V_{x,1}^2 \subseteq x^{-1}U$. Then $xV_{x,1}^2 \subseteq U$. Similarly, let $V_{x,2}$ be a neighborhood of e such that $V_{x,2}^2 x \subseteq U$. Set $V_x = V_{x,1} \cap V_{x,2}$. Then $xV_x^2 \subseteq U$.

By compactness, there exists $x_1, \dots, x_n$ such that $K \subseteq \bigcup_{i=1}^n x_i V_{x_i}$ and $K \subseteq \bigcup_{i=1}^n V_{x_i} x_i$.

Set $V = \bigcap_{i=1}^n V_{x_i}$. Then $KV \subseteq \bigcup_{i=1}^n x_i V_{x_i} V \subseteq \bigcup_{i=1}^n x_i V_{x_i}^2 \subseteq U$. Similarly, $VK \subseteq \bigcup_{i=1}^n V V_{x_i} x_i \subseteq \bigcup_{i=1}^n V_{x_i}^2 x_i \subseteq U$. $\square$

2.2. The functional $(\cdot : g)$. Our construction of the Haar measure will be done in 3 steps. In each step, we will investigate the property of a certain family of functionals. We start with the functional $(\cdot : g)$, defined in Definition 2.12. Throughout the construction of the Haar measure, G refers to a locally compact topological group.

Definition 2.9 ($\Omega_n(f : g)$). Let $n \in \mathbb{N}$ and $f, g \in C_c^+(G) \setminus \{0\}$. For each $n \in \mathbb{N}$, we define

$$\Omega_n(f : g) = \left\{ (c_1, \dots, c_n, s_1, \dots, s_n) \in (\mathbb{R}^+)^n \times G^n : \forall x \in G \left[f(x) \leq \sum_{i=1}^n c_i g(s_i x) \right] \right\}.$$

Definition 2.10 ($\Omega(f : g)$). Let $f, g \in C_c^+(G) \setminus \{0\}$. Then

$$\Omega(f : g) = \bigcup_{n=1}^{\infty} \Omega_n(f : g).$$

Lemma 2.11. Suppose $f, g \in C_c^+(G) \setminus \{0\}$. Then $\Omega(f : g)$ is nonempty.

Proof. Since $g \neq 0$, we see that $\|g\| > 0$. Let $\alpha \in \mathbb{R}$ with $0 < \alpha < \|g\|$. Then there exists $a \in G$ such that $g(a) > \alpha$. By continuity, there exists a neighborhood U of a such that $g(x) > \alpha$ for all $x \in U$. Set $V = a^{-1}U$. Then V is a neighborhood of e such that $g(x) > \alpha$ for all $x \in aV$.

By the compactness of $\text{supp } f$, there exist $x_1, \dots, x_n \in G$ such that $\text{supp } f \subseteq \bigcup_{k=1}^n x_k V$. Set $s_k = ax_k^{-1}$ for $1 \leq k \leq n$. Set $c_k = \frac{\|f\|}{\alpha}$ for $1 \leq k \leq n$. Suppose $x \in \text{supp } f$. Then there exists k with $1 \leq k \leq n$ such that $x \in x_k V$. Then $s_k x \in ax_k^{-1}x_k V = aV$. Thus, $g(s_k x) > \alpha$. Thus,

$$f(x) \leq \|f\| \leq \frac{\|f\|}{\alpha} g(s_k x) \leq \sum_{i=1}^n c_i g(s_i x).$$

Suppose $x \notin \text{supp } f$. Then

$$f(x) = 0 \leq \sum_{i=1}^n c_i g(s_i x).$$

Thus, $(c_1, \dots, c_n, s_1, \dots, s_n) \in \Omega_n(f : g)$. Thus, $\Omega(f : g)$ is nonempty. $\square$

Definition 2.12 ($(f : g)$). Let $f, g \in C_c^+(G) \setminus \{0\}$. Then we define $(f : g)$ by

$$(f : g) = \inf \sum_{i=1}^n c_i,$$

where the infimum is taken over all $(c_1, \dots, c_n, s_1, \dots, s_n) \in \Omega(f : g)$.

In the following lemmas, we prove some properties of $(f : g)$. These properties will be instrumental in demonstrating similar properties in the following functionals, and eventually, the Haar measure.

Lemma 2.13 (Left-Translation Invariance). Let $f, g \in C_c^+(G) \setminus \{0\}$. Fix $s \in G$. Define $f_s(x) = f(sx)$ for all $x \in G$. Then

$$(f_s : g) = (f : g).$$

Proof. Suppose $(c_1, \dots, c_n, s_1, \dots, s_n) \in \Omega(f : g)$. Then

$$f_s(x) = f(sx) \leq \sum_{i=1}^n c_i g(s_i sx)$$

for all $x \in G$. Thus, $(c_1, \dots, c_n, s_1 s, \dots, s_n s) \in \Omega(f_s : g)$. Thus,

$$(f : g) \leq (f_s : g).$$

However, since $f = (f_s)_{s^{-1}}$,

$$(f_s : g) \leq ((f_s)_{s^{-1}} : g) = (f : g).$$

Thus, $(f_s : g) = (f : g)$. $\square$

Lemma 2.14. Let $f, g \in C_c^+(G) \setminus \{0\}$. Then for all $c > 0$,

$$(cf : g) = c(f : g).$$

Proof. Let $\epsilon > 0$. Suppose $(c_1, \dots, c_n, s_1, \dots, s_n) \in \Omega(f : g)$ be such that $\sum_{i=1}^n c_i < (f : g) + \epsilon$. Then

$$(cf)(x) \leq \sum_{i=1}^n cc_i g(s_i x)$$

for all $x \in X$. Thus,

$$(cf : g) \leq \sum_{i=1}^n cc_i < c(f : g) + c\epsilon.$$

Since ϵ is arbitrary, $(cf : g) \leq c(f : g)$. Using this, we obtain

$$(f : g) = \left(\frac{1}{c} cf : g \right) \leq \frac{1}{c} (cf : g),$$

so $c(f : g) \leq (cf : g)$. Thus, $(cf : g) = c(f : g)$. $\square$

Lemma 2.15. Let $f, g \in C_c^+(G) \setminus \{0\}$. Then $\frac{\|f\|}{\|g\|} \leq (f : g)$.

Proof. Since $\text{supp } f$ is compact, there exists $x_0 \in G$ such that $f(x_0) = \|f\|$ (by the Extreme Value Theorem). Then for any $(c_1, \dots, c_n, s_1, \dots, s_n) \in \Omega(f : g)$,

$$\|f\| = f(x_0) \leq \sum_{i=1}^n c_i g(s_i x_0) \leq \|g\| \sum_{i=1}^n c_i.$$

Thus, $\frac{\|f\|}{\|g\|} \leq \sum_{i=1}^n c_i$. Thus, $\frac{\|f\|}{\|g\|} \leq (f : g)$. $\square$

Lemma 2.16. Let $f_1, f_2, g \in C_c^+(G) \setminus \{0\}$. If $f_1 \leq f_2$, then $(f_1 : g) \leq (f_2 : g)$.

Proof. Suppose $(c_1, \dots, c_n, s_1, \dots, s_n) \in \Omega(f_2 : g)$. Then for all $x \in G$,

$$f_1(x) \leq f_2(x) \leq \sum_{i=1}^n c_i g(s_i x).$$

Thus, $(c_1, \dots, c_n, s_1, \dots, s_n) \in \Omega(f_1 : g)$. Thus, $\Omega(f_2 : g) \subseteq \Omega(f_1 : g)$. Thus, $(f_1 : g) \leq (f_2 : g)$. $\square$

Lemma 2.17. Let $f_1, f_2, g \in C_c^+(G) \setminus \{0\}$. Then

$$(f_1 + f_2 : g) \leq (f_1 : g) + (f_2 : g).$$

Proof. Let $\epsilon > 0$. By definition of $(f_1 : g)$ and $(f_2 : g)$, there exist $(c_1, \dots, c_n, s_1, \dots, s_n) \in \Omega(f_1 : g)$ and $(d_1, \dots, d_m, t_1, \dots, t_m) \in \Omega(f_2 : g)$ such that

$$\sum_{i=1}^n c_i < (f_1 : g) + \frac{\epsilon}{2} \quad \text{and} \quad \sum_{j=1}^m d_j < (f_2 : g) + \frac{\epsilon}{2}.$$

Set $a_k = c_k$ and $r_k = s_k$ for $1 \leq k \leq n$ and set $a_k = d_{k-n}$ and $r_k = t_{k-n}$ for $n < k \leq n+m$. Then for all $x \in G$,

$$f_1(x) + f_2(x) \leq \sum_{i=1}^n c_i g(s_i x) + \sum_{j=1}^m d_j g(t_j x) = \sum_{k=1}^{n+m} a_k g(r_k x).$$

Thus, $(a_1, \dots, a_{n+m}, r_1, \dots, r_{n+m}) \in \Omega(f_1 + f_2 : g)$. Thus,

$$(f_1 + f_2 : g) \leq \sum_{k=1}^{n+m} a_k = \sum_{i=1}^n c_i + \sum_{j=1}^m d_j < (f_1 : g) + (f_2 : g) + \epsilon.$$

Since ϵ is arbitrary, $(f_1 + f_2 : g) \leq (f_1 : g) + (f_2 : g)$. $\square$

Lemma 2.18. Let $f, g, h \in C_c^+(G) \setminus \{0\}$. Then

$$(f : h) \leq (f : g)(g : h).$$

Proof. Suppose $(c_1, \dots, c_n, s_1, \dots, s_n) \in \Omega(f : g)$ and $(d_1, \dots, d_m, t_1, \dots, t_m) \in \Omega(g : h)$. Then for all $x \in G$,

$$f(x) \leq \sum_{i=1}^n c_i g(s_i x) \leq \sum_{i=1}^n c_i \left(\sum_{j=1}^m d_j h(t_j s_i x) \right) = \sum_{i=1}^n \sum_{j=1}^m c_i d_j h(t_j s_i x).$$

Thus, $(c_1 d_1, \dots, c_n d_m, t_1 s_1, \dots, t_m s_n) \in \Omega(f : h)$. Thus,

$$(f : h) \leq \sum_{i=1}^n \sum_{j=1}^m c_i d_j = \left(\sum_{i=1}^n c_i \right) \left(\sum_{j=1}^m d_j \right).$$

Taking the infimum on RHS, $(f : h) \leq (f : g)(g : h)$. $\square$

2.3. The functional $\Lambda_{g,h}$. We now begin the second step of our construction, turning our attention to a new class of functionals, $\Lambda_{g,h}$.

Definition 2.19 ($\Lambda_{g,h}$). Let $g, h \in C_c^+(G) \setminus \{0\}$. Then we define $\Lambda_{g,h} : C_c^+(G) \setminus \{0\} \rightarrow \mathbb{R}$ by

$$\Lambda_{g,h}(f) = \frac{(f : h)}{(g : h)}$$

for all $f \in C_c^+(G) \setminus \{0\}$.

Note. Observe that we may view $\Lambda_{g,h}$ as an element of

$$\mathbb{R}^{C_c^+(G) \setminus \{0\}} = \prod_{f \in C_c^+(G) \setminus \{0\}} \mathbb{R}$$

by viewing $\Lambda_{g,h}(f)$ as the f th coordinate of λ .

Lemma 2.20. Let $g, h \in C_c^+(G) \setminus \{0\}$. Then

$$\Lambda_{g,h} \in \prod_{f \in C_c^+(G) \setminus \{0\}} \left[\frac{1}{(g : f)}, (f : g) \right].$$

Proof. Let $f \in C_c^+(G) \setminus \{0\}$. By Lemma 2.18, $\Lambda_{g,h}(f) \leq (f : g)$. Using Lemma 2.18 again,

$$(g : f) \Lambda_{g,h}(f) = (g : f) \frac{(f : h)}{(g : h)} \geq \frac{(g : h)}{(g : h)} \geq 1.$$

Thus,

$$\frac{1}{(g : f)} \leq \Lambda_{g,h}(f) \leq (f : g).$$

Thus,

$$\Lambda_{g,h} \in \prod_{f \in C_c^+(G) \setminus \{0\}} \left[\frac{1}{(g : f)}, (f : g) \right].$$

□

We note some properties of the functional $\Lambda_{g,h}$ which follow immediately from the properties proven about $(f : g)$.

Lemma 2.21. Let $g, h \in C_c^+(G) \setminus \{0\}$. Then

- (1) For all $s \in G$ and $f \in C_c^+(G) \setminus \{0\}$, if f_s is given by $f_s(x) = f(sx)$, then $\Lambda_{g,h}(f) = \Lambda_{g,h}(f_s)$.
- (2) For all $c > 0$ and $f \in C_c^+(G) \setminus \{0\}$, $\Lambda_{g,h}(cf) = c\Lambda_{g,h}(f)$.
- (3) For all $f_1, f_2 \in C_c^+(G) \setminus \{0\}$, if $f_1 \leq f_2$, then $\Lambda_{g,h}(f_1) \leq \Lambda_{g,h}(f_2)$.
- (4) For all $f_1, f_2 \in C_c^+(G) \setminus \{0\}$, $\Lambda_{g,h}(f_1 + f_2) \leq \Lambda_{g,h}(f_1) + \Lambda_{g,h}(f_2)$.

Definition 2.22 ($C_V^+(G)$). Let V be a neighborhood of e . Then

$$C_V^+(G) = \{h \in C_c^+(G) : \text{supp } h \subseteq V\}.$$

Definition 2.23 ($\Sigma_{V,g}$). Let $g \in C_c^+(G) \setminus \{0\}$. For each neighborhood V of e , the set $\Sigma_{V,g}$ is the closure of

$$\{\Lambda_{g,h} : h \in C_V^+(G) \setminus \{0\}\}$$

in

$$\prod_{f \in C_c^+(G) \setminus \{0\}} \left[\frac{1}{(g : f)}, (f : g) \right].$$

The following lemma will be used to obtain a functional on $C_c^+(G)$ which will be extended to all of $C_c(G)$, giving us the Haar functional.

Lemma 2.24. Let $g \in C_c^+(G) \setminus \{0\}$. Let $\mathcal{V}$ be the collection of all open neighborhoods of e . Then

$$\bigcap_{V \in \mathcal{V}} \Sigma_{V,g} \neq \emptyset.$$

Proof. Set

$$\Delta = \prod_{f \in C_c^+(G) \setminus \{0\}} \left[\frac{1}{(g : f)}, (f : g) \right].$$

By Tychonoff's Theorem, Δ is compact. Fix $V \in \mathcal{V}$. By Theorem 2.4, there exists a continuous function $h : G \rightarrow [0, 1]$ with compact support such that $\text{supp } h \subseteq V$ and $f(e) = 1$. Thus, $h \in C_V^+(G) \setminus \{0\}$, so $\Lambda_{g,h} \in \Sigma_{V,g}$.

Suppose $V_1, \dots, V_n \in \mathcal{V}$ and $V = \bigcap_{i=1}^n V_i$. Then V is a neighborhood of e , so $V \in \mathcal{V}$. Suppose $h \in C_V^+(G) \setminus \{0\}$. Then $\text{supp } h \subseteq V \subseteq V_i$ for all $1 \leq i \leq n$, so $h \in \bigcap_{i=1}^n C_{V_i}^+(G)$. Thus, $\Sigma_{V,g} \subseteq \bigcap_{i=1}^n \Sigma_{V_i,g}$, so $\bigcap_{i=1}^n \Sigma_{V_i,g} \neq \emptyset$ (since $\Sigma_{V,g}$ is nonempty by the previous argument).

Thus, $\{\Sigma_{V,g} : V \in \mathcal{V}\}$ is a family of closed subsets of the compact space Δ having the finite intersection property. Thus,

$$\bigcap_{V \in \mathcal{V}} \Sigma_{V,g} \neq \emptyset.$$

□

We now prove a lemma asserting the uniform continuity of a compactly supported function. This will see use in proving the uniqueness of the Haar functional and several lemmas for the Peter-Weyl theorem.

Lemma 2.25. Let $f \in C_c(G)$. Then for all $\epsilon > 0$, there exists a neighborhood V of e such that

$$|f(x) - f(y)| < \epsilon$$

for all $x, y \in G$ such that $y^{-1}x \in V$.

Proof. Fix $\epsilon > 0$. Let $K = \text{supp } f$. For each $x \in K$, let U_x be a neighborhood of x such that $|f(x) - f(y)| < \epsilon/2$ for all $y \in U_x$ (this is guaranteed by continuity). Then $x^{-1}U_x$ is a neighborhood of e . By Proposition 2.7, there exists a symmetric neighborhood W_x of e such that $W_x^2 \subseteq x^{-1}U_x$. Clearly, $\{xW_x\}_{x \in K}$ is an open cover of K . By compactness, it admits a finite subcover $\{x_iW_{x_i}\}_{i=1}^n$.

Set $V = \bigcap_{i=1}^n W_{x_i}$. Clearly, V is a neighborhood of e . Suppose that $x, y \in G$ such that $y^{-1}x \in V$. If $x \notin K$ and $y \notin K$, then $|f(x) - f(y)| = 0 < \epsilon$.

Suppose that $x \in K$. Then there exists i with $1 \leq i \leq n$ such that $x \in x_iW_{x_i}$. Thus, $x_i^{-1}x \in W_{x_i}$. Since $y^{-1}x \in V \subseteq W_{x_i}$ and W_{x_i} is symmetric, $x^{-1}y \in W_{x_i}$. Since $W_{x_i}^2 \subseteq x_i^{-1}U_{x_i}$, we see $x_i^{-1}y = x_i^{-1}xx^{-1}y \in x_i^{-1}U_{x_i}$. Thus, $y \in U_{x_i}$. Furthermore, since $x_i^{-1}x \in W_{x_i}$, we see $x_i^{-1}x = ex_i^{-1}x \in x_i^{-1}U_{x_i}$, so $x \in U_{x_i}$. Thus, since $x, y \in U_{x_i}$,

$$|f(x) - f(y)| \leq |f(x) - f(x_i)| + |f(x_i) - f(y)| < \epsilon.$$

A similar argument holds when $y \in K$. Thus, V is a neighborhood of e such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in G$ such that $y^{-1}x \in V$. $\square$

Lemma 2.26. Let $g, f_1, f_2 \in C_c^+(G) \setminus \{0\}$. Let $\epsilon > 0$. Then there exists a neighborhood V of e such that

$$\Lambda_{g,h}(f_1) + \Lambda_{g,h}(f_2) \leq \Lambda_{g,h}(f_1 + f_2) + \epsilon$$

for all $h \in C_V^+(G) \setminus \{0\}$.

Proof. Set $f = f_1 + f_2$. Note $f \in C_c^+(G) \setminus \{0\}$. Thus, $\text{supp } f$ is compact. By Theorem 2.4, there exists a compactly supported function $k : G \rightarrow [0, 1]$ such that $k(\text{supp } f) = \{1\}$. Let

$$\delta = \frac{\epsilon}{4(k : g)} \quad \text{and} \quad \eta = \min \left\{ \frac{\epsilon}{4(f : g)}, \frac{1}{2} \right\}.$$

Then $2\eta(f : g) \leq \epsilon/2$, $2\eta \leq 1$, and $2\delta(k : g) \leq \epsilon/2$. Let $F = f + \delta k$. Set $h_1 = f_1/F$ and $h_2 = f_2/F$ (for points where $F = 0$, set $h_1 = h_2 = 0$). Then h_1 and h_2 are continuous functions, and clearly have compact support. By the previous lemma, for $i \in \{1, 2\}$, there exists a neighborhood V_i of e such that

$$|h_i(x) - h_i(y)| < \eta$$

whenever $x, y \in G$ such that $y^{-1}x \in V_i$. Set $V = V_1 \cap V_2$. Let $h \in C_V^+(G) \setminus \{0\}$. Let $(c_1, \dots, c_n, s_1, \dots, s_n) \in \Omega(F : h)$ and $x \in G$. Let $j \in \{1, \dots, n\}$ such that $h(s_j x) \neq 0$, then $s_j x \in V$, so $|h_i(x) - h_i(s_j^{-1})| < \eta$ for $i \in \{1, 2\}$. Thus,

$$h_i(x) \leq |h_i(x) - h_i(s_j^{-1})| + h_i(s_j^{-1}) < h_i(s_j^{-1}) + \eta$$

for $i \in \{1, 2\}$. For each $1 \leq i \leq 2$ and $1 \leq j \leq n$, set $c_{i,j} = c_j[h_i(s_j^{-1}) + \eta]$. Thus, for $i \in \{1, 2\}$,

$$f_i(x) = F(x)h_i(x) \leq \sum_{j=1}^n c_j h(s_j x) h_i(x) \leq \sum_{j=1}^n c_{i,j} h(s_j x).$$

Thus, $(f_i : h) \leq \sum_{j=1}^n c_{i,j}$ for $i \in \{1, 2\}$. Since $h_1 + h_2 = f/F \leq 1$,

$$(f_1 : h) + (f_2 : h) \leq \sum_{j=1}^n c_{1,j} + c_{2,j} \leq \sum_{j=1}^n c_j(1 + 2\eta).$$

Taking the infimum over all $(c_1, \dots, c_n, s_1, \dots, s_n) \in \Omega(F : h)$,

$$\begin{aligned} (f_1 : h) + (f_2 : h) &\leq (F : h)(1 + \eta) \leq [(f : h) + \delta(k : h)](1 + 2\eta) \\ &= (f : h) + 2\eta(f : h) + \delta(1 + 2\eta)(k : h). \end{aligned}$$

Dividing by $(g : h)$, we obtain

$$\Lambda_{g,h}(f_1) + \Lambda_{g,h}(f_2) \leq \Lambda_{g,h}(f) + 2\eta\Lambda_{g,h}(f) + \delta(1 + 2\eta)\Lambda_{g,h}(k).$$

Since $2\eta\Lambda_{g,h}(f) \leq 2\eta(f : g) \leq \epsilon/2$ and $\delta(1 + 2\eta)\Lambda_{g,h}(k) \leq 2\delta(k : g) \leq \epsilon/2$,

$$\Lambda_{g,h}(f_1) + \Lambda_{g,h}(f_2) \leq \Lambda_{g,h}(f_1 + f_2) + \epsilon. \quad \square$$

3. EXISTENCE AND UNIQUENESS OF THE HAAR MEASURE

Now, we begin the final step of the construction of the Haar measure, turning our attention to the final functional, the (left) Haar functional Λ . Once we have the Haar functional, we use the Riesz-Markov representation theorem, stated below, to transform the left Haar functional to a left Haar measure.

Theorem 3.1 (Riesz-Markov Representation Theorem). Let (X, τ) be a locally compact Hausdorff space and let $\phi : C_c(X) \rightarrow \mathbb{C}$ be a positive linear functional. Let $(X, \mathcal{M}, \mu)$ be the measure space induced by ϕ . Then for all $f \in C_c(X)$,

$$(*) \quad \phi(f) = \int_X f d\mu$$

In addition:

- (1) $\mathcal{B}(X) \subseteq \mathcal{M}$, where $\mathcal{B}(X)$ are the Borel subsets of X .
- (2) If $\mathcal{K}$ is the collection of compact subsets of X , then μ is finite on $\mathcal{K}$.
- (3) $\mu(E) = \inf_{E \subseteq V \in \tau} \mu(V)$ for all $E \in \mathcal{M}$.
- (4) (a) $\mu(E) = \sup_{\{K \in \mathcal{K} : K \subseteq E\}} \mu(K)$ for all $E \in \tau$.
 (b) $\mu(E) = \sup_{\{K \in \mathcal{K} : K \subseteq E\}} \mu(K)$ for all $E \in \mathcal{M}$ with $\mu(E) < \infty$.
- (5) The measure space $(X, \mathcal{M}, \mu)$ is complete.

Furthermore, μ is uniquely determined on $\mathcal{M}$ by equation (*), (2), (3), and (4a).

We now prove the existence of the Haar functional.

Theorem 3.2 (Existence of the Haar Functional). Let $g \in C_c^+(G) \setminus \{0\}$, and let $\mathcal{V}$ be the collection of neighborhoods of e . Let

$$\Lambda \in \bigcap_{V \in \mathcal{V}} \Sigma_{V,g},$$

and extend Λ to $C_c(G)$ by setting

- (1) $\Lambda(0) = 0$.
- (2) $\Lambda(f) = \Lambda(f^+) - \Lambda(f^-)$ for $f \in C_c^\mathbb{R}(G)$.
- (3) $\Lambda(u + iv) = \Lambda(u) + i\Lambda(v)$ for $u, v \in C_c^\mathbb{R}(G)$.

Then Λ is a nonzero, left-translation invariant, positive linear functional on $C_c(G)$.

Proof. (Nonzero, Positivity) Set

$$\Delta = \prod_{f \in C_c^+(G) \setminus \{0\}} \left[\frac{1}{(g:f)}, (f:g) \right].$$

Note that for any $f \in C_c^+(G) \setminus \{0\}$, $\Lambda(f) \geq \frac{1}{(g:f)}$, so Λ is positive. In addition, Λ is not identically zero.

(Homogeneity) We first prove that for $f \in C_c^+(G) \setminus \{0\}$ and $c > 0$, we have $\Lambda(cf) = c\Lambda(f)$. Fix $f \in C_c^+(G) \setminus \{0\}$ and $c > 0$. Consider

$$N = \{\Phi \in \Delta : |\Phi(f) - \Lambda(f)| < \epsilon \text{ and } |\Phi(cf) - \Lambda(cf)| < \epsilon\}.$$

Clearly, N is a neighborhood of Λ . Since $\Lambda \in \Sigma_{G,g}$, there exists $h \in C_G^+(G) \setminus \{0\}$ such that $\Lambda_{g,h} \in N$ (by definition of $\Sigma_{G,g}$). Then

$$|\Lambda(cf) - c\Lambda(f)| \leq |\Lambda(cf) - \Lambda_{g,h}(cf)| + |c\Lambda_{g,h}(f) - c\Lambda(f)| < (1+c)\epsilon.$$

Since ϵ is arbitrary, $\Lambda(cf) = c\Lambda(f)$.

This homogeneity clearly extends immediately to the case that $c = 0$ or $f = 0$ since $\Lambda(0) = 0$.

Suppose $f \in C_c^\mathbb{R}(G)$ and $c \in \mathbb{R}$. If $c = 0$, linearity holds immediately as stated prior. Suppose $c > 0$. Then

$$\Lambda(cf) = \Lambda((cf)^+) - \Lambda((cf)^-) = c\Lambda(f^+) - c\Lambda(f^-) = c(\Lambda(f^+) - \Lambda(f^-)) = c\Lambda(f).$$

Note

$$\Lambda(-f) = \Lambda((-f)^+) - \Lambda((-f)^-) = \Lambda(f^-) - \Lambda(f^+) = -\Lambda(f).$$

Thus, for $c < 0$,

$$\Lambda(cf) = (-c)\Lambda(-f) = c\Lambda(f).$$

Finally, suppose $f \in C_c(G)$ and $c \in \mathbb{C}$. Let $u = \Re f$ and $v = \Im f$. Let $a = \Re c$ and $b = \Im c$. Then

$$\Lambda(cf) = \Lambda((a+bi)(u+iv)) = \Lambda(au - bv + i(av + bu))$$

$$a\Lambda(u) - b\Lambda(v) + ia\Lambda(v) + ib\Lambda(u) = (a+bi)(\Lambda(u) + i\Lambda(v)) = c\Lambda(f).$$

(Additivity) Suppose $f_1, f_2 \in C_c^+(G) \setminus \{0\}$ and $\epsilon > 0$. By Lemma 2.26, there exists $V \in \mathcal{V}$ such that for any $h \in C_V^+(G) \setminus \{0\}$,

$$|\Lambda_{g,h}(f_1 + f_2) - \Lambda_{g,h}(f_1) - \Lambda_{g,h}(f_2)| \leq \epsilon.$$

Consider

$$N = \{\Phi \in \Delta : |\Phi(f_1) - \Lambda(f_1)| < \epsilon, |\Phi(f_2) - \Lambda(f_2)| < \epsilon \\ \text{and } |\Phi(f_1 + f_2) - \Lambda(f_1 + f_2)| < \epsilon\}.$$

As before, N is a neighborhood of Λ . Since $\Lambda \in \Sigma_{V,g}$, there exists $h \in C_V^+(G) \setminus \{0\}$ such that $\Lambda_{g,h} \in N$.

Set $f = f_1 + f_2$. Then

$$\begin{aligned} |\Lambda(f) - \Lambda(f_1) - \Lambda(f_2)| &\leq |\Lambda(f) - \Lambda_{g,h}(f)| + |\Lambda_{g,h}(f) - \Lambda_{g,h}(f_1) - \Lambda_{g,h}(f_2)| \\ &\quad + |\Lambda_{g,h}(f_1) - \Lambda(f_1)| + |\Lambda_{g,h}(f_2) - \Lambda(f_2)| < 4\epsilon. \end{aligned}$$

Since ϵ is arbitrary, $\Lambda(f) = \Lambda(f_1 + f_2)$.

Let $f_1, f_2 \in C_c^\mathbb{R}(G)$. Set $f = f_1 + f_2$. Then $f^+ - f^- = (f_1^+ - f_1^-) + (f_2^+ - f_2^-)$. Thus, $f^+ + f_1^- + f_2^- = f^- + f_1^+ + f_2^+$. Thus,

$$\begin{aligned} \Lambda(f^+) + \Lambda(f_1^-) + \Lambda(f_2^-) &= \Lambda(f^+ + f_1^- + f_2^-) \\ &= \Lambda(f^- + f_1^+ + f_2^+) = \Lambda(f^-) + \Lambda(f_1^+) + \Lambda(f_2^+). \end{aligned}$$

Thus,

$$\Lambda(f) = \Lambda(f^+) - \Lambda(f^-) = (\Lambda(f_1^+) - \Lambda(f_1^-)) + (\Lambda(f_2^+) - \Lambda(f_2^-))\Lambda(f_1) + \Lambda(f_2).$$

Finally, let $f_1, f_2 \in C_c(G)$, and set $f = f_1 + f_2$. Let $u = \Re f$ and $v = \Im f$. For $i \in \{1, 2\}$, let $u_i = \Re f_i$ and $v_i = \Im f_i$. Then

$$\begin{aligned} \Lambda(f) &= \Lambda(u + iv) = \Lambda(u_1 + u_2) + i\Lambda(v_1 + v_2) \\ &= \Lambda(u_1) + \Lambda(u_2) + i\Lambda(v_1) + i\Lambda(v_2) = \Lambda(f_1) + \Lambda(f_2). \end{aligned}$$

(Translation Invariance) For a function $f \in C_c(G)$ and $s \in G$, let $f_s(x) = f(sx)$. Fix $f \in C_c^+(G) \setminus \{0\}$ and $s \in G$. Consider

$$N = \{\Phi \in \Delta : |\Phi(f) - \Lambda(f)| < \epsilon \text{ and } |\Phi(f_s) - \Lambda(f_s)| < \epsilon\}.$$

Clearly, N is a neighborhood of Λ . Since $\Lambda \in \Sigma_{G,g}$, there exists $h \in C_G^+(G) \setminus \{0\}$ such that $\Lambda_{g,h} \in N$ (by definition of $\Sigma_{G,g}$). Then

$$|\Lambda(f_s) - \Lambda(f)| \leq |\Lambda(f_s) - \Lambda_{g,h}(f_s)| + |\Lambda_{g,h}(f_s) - \Lambda(f)| < 2\epsilon.$$

Since ϵ is arbitrary, $\Lambda(f_s) = \Lambda(f)$.

This immediately extends to the case $f = 0$, since then $f_s = 0$ for any $s \in G$. Now, let $f \in C_c^\mathbb{R}(G)$ and $s \in G$. Set

$$\begin{aligned} f_s^{++}(x) &= \begin{cases} f_s^+(x) & f(x) \geq 0 \\ 0 & f(x) < 0 \end{cases}, & f_s^{+-}(x) &= \begin{cases} f_s^+(x) & f(x) < 0 \\ 0 & f(x) \geq 0 \end{cases}, \\ f_s^{-+}(x) &= \begin{cases} f_s^-(x) & f(x) \geq 0 \\ 0 & f(x) < 0 \end{cases}, & f_s^{--}(x) &= \begin{cases} f_s^-(x) & f(x) < 0 \\ 0 & f(x) \geq 0 \end{cases}. \end{aligned}$$

In addition, set

$$\begin{aligned} f^{++}(x) &= \begin{cases} f^+(x) & f_s(x) \geq 0 \\ 0 & f_s(x) < 0 \end{cases}, & f^{+-}(x) &= \begin{cases} f^+(x) & f_s(x) < 0 \\ 0 & f_s(x) \geq 0 \end{cases}, \\ f^{-+}(x) &= \begin{cases} f^-(x) & f_s(x) \geq 0 \\ 0 & f_s(x) < 0 \end{cases}, & f^{--}(x) &= \begin{cases} f^-(x) & f_s(x) < 0 \\ 0 & f_s(x) \geq 0 \end{cases}. \end{aligned}$$

Then

$$\begin{aligned} \Lambda(f_s) &= \Lambda(f_s^+) - \Lambda(f_s^-) = \Lambda(f_s^{++}) + \Lambda(f_s^{+-}) - \Lambda(f_s^{-+}) - \Lambda(f_s^{--}) \\ &= \Lambda(f^{++}) + \Lambda(-f^{-+}) - \Lambda(-f^{+-}) - \Lambda(f^{--}) \\ &= \Lambda(f^{++}) + \Lambda(f^{+-}) - \Lambda(f^{-+}) - \Lambda(f^{--}) = \Lambda(f^+) - \Lambda(f^-) = \Lambda(f). \end{aligned}$$

Finally, let $f \in C_c(G)$ and $s \in G$. Set $u = \Re f$ and $v = \Im f$. Then

$$\Lambda(f_s) = \Lambda(u_s + iv_s) = \Lambda(u_s) + i\Lambda(v_s) = \Lambda(u) + i\Lambda(v) = \Lambda(f).$$

Thus, Λ is a nonzero, left-translation invariant, positive linear functional on $C_c(G)$. $\square$

Theorem 3.3 (Uniqueness of the Haar Functional). Let Λ_1 and Λ_2 be left-translation invariant positive linear functionals on $C_c(G)$ with Λ_2 nonzero. Then $\Lambda_1 = c\Lambda_2$ for some $c \geq 0$.

Proof. If $\Lambda_1 = 0$, take $c = 0$. Thus, we may assume $\Lambda_1 \neq 0$. By linearity, Λ_1 and Λ_2 are determined by their values on $C_c^+(G) \setminus \{0\}$. Thus, it suffices to show that $\frac{\Lambda_1}{\Lambda_2}$ is constant on $C_c^+(G) \setminus \{0\}$.

Let $f, g \in C_c^+(G) \setminus \{0\}$. Let $K = \text{supp } f$. Since G is locally compact, by Lemma 2.3 that there exists an open set W with compact closure so $K \subseteq W$. Let $\mathcal{V}$ be the collection of neighborhoods of e . By Proposition 2.8, there exists $V_1 \in \mathcal{V}$ such that $KV_1 \cup V_1K \subseteq W$. Let $\epsilon > 0$. By Lemma 2.25 (it is clear from the proof we may take $xy^{-1} \in V$ as the condition rather than $y^{-1}x \in V$), there exist $V_2, V_3 \in \mathcal{V}$ such that for all $x \in G$,

$$|f(x) - f(xt^{-1})| < \epsilon/2 \quad \text{for all } t \in V_2$$

and

$$|f(x) - f(s^{-1}x)| < \epsilon/2 \quad \text{for all } s \in V_3.$$

Set $U = V_1 \cap V_2 \cap V_3$ and $V = U \cap U^{-1}$. Then $KV \cup VK \subseteq W$, $V^{-1} = V$ and for all $x \in G$, $s, t \in V$,

$$|f(sx) - f(xt)| < \epsilon.$$

By Theorem 2.4, there exists a function $k : G \rightarrow [0, 1]$ with compact support such that $f(\text{cl}(W)) = \{1\}$. Fix $x \in G$ and $s \in V$. If $x \in W$, then $f(sx) = f(sx)k(x)$ and $f(xs) = f(xs)k(x)$.

If $x \notin W$, then $x \notin KV \cup VK$. If $sx \in K$, then $s^{-1} \in V$, so $x = s^{-1}(sx) \in VK$, a contradiction. Thus, $sx \notin K$. Thus, $f(sx) = 0$, so $f(sx) = f(sx)k(x)$. A similar argument shows $f(xs) = 0$, so $f(xs) = f(xs)k(x)$.

Thus, for all $x \in G$ and $s \in V$

$$|f(sx) - f(xs)| = |k(x)| |f(sx) - f(xs)| \leq \epsilon |k(x)|.$$

Fix $h' \in C_V^+(G) \setminus \{0\}$ and set $h(x) = h'(x) + h'(x^{-1})$ for all x . Let μ_1 be the measure induced by Λ_1 and μ_2 be the measure induced by Λ_2 . Since $h(x^{-1}y)f(y) \in C_c(G \times G) \subseteq L^1(\mu_1 \otimes \mu_2)$, by Fubini's Theorem and the fact $h(x^{-1}y) = h(y^{-1}x)$, we see

$$\int \int h(y^{-1}x)f(y) d\mu_1(x) d\mu_2(y) = \int \int h(x^{-1}y)f(y) d\mu_2(y) d\mu_1(x).$$

By the left-translation invariance of Λ_1 , Fubini's Theorem, and the Riesz-Markov Representation Theorem,

$$\int \int h(y^{-1}x)f(y) d\mu_1(x) d\mu_2(y) = \int \left(\int h(y^{-1}x) d\mu_1(x) \right) f(y) d\mu_2(y)$$

$$= \int \left(\int h(x) d\mu_1(x) \right) f(y) d\mu_2(y) = \Lambda_1(h) \Lambda_2(f).$$

By the left-translation invariance of Λ_2 ,

$$\int \int h(x^{-1}y) f(y) d\mu_2(y) d\mu_1(x) = \int \int h(y) f(xy) d\mu_2(y) d\mu_1(x).$$

Thus,

$$\Lambda_1(h) \Lambda_2(f) = \int \left(\int h(y) f(xy) d\mu_2(y) \right) d\mu_1(x).$$

Now, observe by the left-translation invariance of Λ_1 and the Riesz-Markov Representation theorem

$$\begin{aligned} \int \left(\int h(y) f(yx) d\mu_1(x) \right) d\mu_2(y) &= \int h(y) \left(\int f(yx) d\mu_1(x) \right) d\mu_2(y) \\ &= \int h(y) \left(\int f(x) d\mu_1(x) \right) d\mu_2(y) = \Lambda_1(f) \Lambda_2(h). \end{aligned}$$

Since $\text{supp } h \subseteq V$,

$$\begin{aligned} |\Lambda_1(h) \Lambda_2(f) - \Lambda_1(f) \Lambda_2(h)| &= \left| \int_{x \in G} \int_{y \in V} h(y) [f(xy) - f(yx)] d\mu_2(y) d\mu_1(x) \right| \\ &\leq \epsilon \int_{x \in G} \int_{y \in V} h(y) k(x) d\mu_2(y) d\mu_1(x) = \epsilon \Lambda_2(h) \Lambda_1(k). \end{aligned}$$

A similar argument produces a function j (associated to g as k is to f) such that

$$|\Lambda_1(h) \Lambda_2(g) - \Lambda_1(g) \Lambda_2(h)| \leq \epsilon \Lambda_2(h) \Lambda_1(j).$$

Dividing these inequalities (by $\Lambda_2(h) \Lambda_2(f)$ and $\Lambda_2(h) \Lambda_2(g)$ respectively), we obtain

$$\left| \frac{\Lambda_1(h)}{\Lambda_2(h)} - \frac{\Lambda_1(f)}{\Lambda_2(f)} \right| \leq \epsilon \frac{\Lambda_1(k)}{\Lambda_2(f)} \quad \text{and} \quad \left| \frac{\Lambda_1(h)}{\Lambda_2(h)} - \frac{\Lambda_1(g)}{\Lambda_2(g)} \right| \leq \epsilon \frac{\Lambda_1(j)}{\Lambda_2(g)}.$$

By the triangle inequality, we obtain

$$\left| \frac{\Lambda_1(f)}{\Lambda_2(f)} - \frac{\Lambda_1(g)}{\Lambda_2(g)} \right| \leq \epsilon \left[\frac{\Lambda_1(k)}{\Lambda_2(f)} + \frac{\Lambda_1(j)}{\Lambda_2(g)} \right].$$

Since ϵ is arbitrary, $\frac{\Lambda_1(f)}{\Lambda_2(f)} = \frac{\Lambda_1(g)}{\Lambda_2(g)}$, so $\frac{\Lambda_1}{\Lambda_2}$ is constant on $C_c^+(G) \setminus \{0\}$. Thus, $\Lambda_1 = c \Lambda_2$ for some $c \geq 0$. $\square$

Definition 3.4 ((Left) Haar Functional). Let G be a locally compact topological group. A (nonzero) left-translation invariant positive linear functional Λ on $C_c(G)$ is called a (left) Haar functional of G .

Definition 3.5 ((Left) Haar Measure Space). Let G be a locally compact topological group and Λ a (left) Haar functional on G . The measure space $(X, \mathcal{M}, \mu)$ induced by Λ by the Riesz-Markov Representation theorem is known as a (left) Haar measure space for G .

Note. Note the Haar functionals are unique up to a constant. If G is compact, since the Haar measure is finite (by the Riesz-Markov Representation Theorem), we may normalize the Haar measure μ so $\mu(G) = 1$ (since $\mu(G)$ is finite).

4. THE SPECTRAL THEOREM

We now turn to proving the spectral theorem, the second main ingredient in proving the Peter-Weyl theorem.

Definition 4.1 (Self-Adjoint Operator). Let V be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $T : V \rightarrow V$ be a linear operator. We say T is self-adjoint if for all $u, v \in V$,

$$\langle T(u), v \rangle = \langle u, T(v) \rangle.$$

Lemma 4.2. Let V be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $T : V \rightarrow V$ be a self-adjoint operator. Then

$$\|T\| = \sup_{x \neq 0} \frac{|\langle Tx, x \rangle|}{\langle x, x \rangle}.$$

Proof. Suppose $x \in V$ with $x \neq 0$. Then

$$|\langle Tx, x \rangle| \leq \|Tx\| \|x\| \leq \|T\| \|x\|^2 = \|T\| \langle x, x \rangle.$$

Thus, $\frac{|\langle Tx, x \rangle|}{\langle x, x \rangle} \leq \|T\|$. Thus, $\sup_{x \neq 0} \frac{|\langle Tx, x \rangle|}{\langle x, x \rangle} \leq \|T\|$.

Suppose $x \in V$ with $x \neq 0$. Then

$$\frac{|\langle Tx, x \rangle|}{\langle x, x \rangle} = \frac{|\langle Tx, x \rangle|}{\|x\|^2} = \left| \left\langle T \left(\frac{x}{\|x\|} \right), \frac{x}{\|x\|} \right\rangle \right|.$$

Thus, for all $x \in V$ with $x \neq 0$, there exists $y \in V$ with $\|y\| = 1$ such that

$$\frac{|\langle Tx, x \rangle|}{\langle x, x \rangle} = |\langle Ty, y \rangle|.$$

Thus, $\sup_{x \neq 0} \frac{|\langle Tx, x \rangle|}{\langle x, x \rangle} = \sup_{\|y\|=1} |\langle Ty, y \rangle|$ (equality holds since $\|y\|^2 = \langle y, y \rangle = 1$). Set

$M = \sup_{\|y\|=1} |\langle Ty, y \rangle|$. Let $x, y \in V$. Then

$$\langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle = 2\langle Tx, y \rangle + 2\langle Ty, x \rangle = 4\langle Tx, y \rangle.$$

Let $x, y \in V$ with $\|x\| = \|y\| = 1$. Then, applying the Parallelogram Identity,

$$|\langle Tx, y \rangle| \leq \frac{1}{4} |\langle T(x+y), x+y \rangle + \langle T(x-y), x-y \rangle| \leq \frac{\alpha(\|x+y\|^2 + \|x-y\|^2)}{4} \leq M.$$

Finally, setting $y = \frac{Tx}{\|Tx\|}$, for all $x \in V$ with $\|x\| = 1$,

$$\langle Tx, y \rangle = \frac{\|Tx\|^2}{\|Tx\|} = \|Tx\|.$$

Thus, $\|T\| = \sup_{\|x\| \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\| \leq \sup_{\|y\|=1} |\langle Ty, y \rangle|$. Thus, $\|T\| = \frac{|\langle Tx, x \rangle|}{\langle x, x \rangle}$. $\square$

Definition 4.3 (Compact Operator). Let V be a normed vector space and let $T : V \rightarrow V$ be a linear operator. We say T is compact if T is bounded and for every sequence $\{x_i\}_{i=1}^\infty$ of points in V , the sequence $\{T(x_i)\}_{i=1}^\infty$ has a convergent subsequence.

Lemma 4.4. Let V be a nonzero Hilbert space and let $T : V \rightarrow V$ be a compact, self-adjoint, nonzero operator. Then T has an eigenvector with nonzero eigenvalue.

Proof. By Lemma 4.2, there exists a sequence $\{x_i\}_{i=1}^\infty \subseteq V$ such that $\|x_i\| = 1$ for all $i \in \mathbb{N}$ and $|\langle Tx_i, x_i \rangle| = \|T\|$. By self-adjointness and the sesquilinearity of the inner product, for all $x \in V$,

$$\langle Tx, x \rangle = \langle x, Tx \rangle = \overline{\langle Tx, x \rangle}.$$

Thus, $\langle Tx_i, x_i \rangle$ is real for all $i \in \mathbb{N}$. Set $\lambda = \|T\|$, and note $\lambda \neq 0$. Then $\lim_{i \rightarrow \infty} |\langle Tx_i, x_i \rangle| = \lambda$. WLOG, there exists some subsequence $\{\langle Tx_{i_j}, x_{i_j} \rangle\}_{j=1}^\infty$ converging to λ . Since T is a compact operator, there exists a further subsequence $\{Tx_{i_{j_k}}\}_{k=1}^\infty$ converging to some vector $v \in V$. For the sake of notation, we set $y_k = x_{i_{j_k}}$ for all $k \in \mathbb{N}$.

Fix $k \in \mathbb{N}$. Then, by the Cauchy-Schwarz inequality,

$$|\langle Ty_k, y_k \rangle| \leq \|Ty_k\| \|y_k\| = \|Ty_k\| \leq \|T\| \|y_k\| = \lambda.$$

Since $\lim_{k \rightarrow \infty} \langle Ty_k, y_k \rangle = \lambda$, $\lim_{k \rightarrow \infty} \|Ty_k\| = \lambda$. Since T is self-adjoint, for all $k \in \mathbb{N}$

$$\|\lambda y_k - Ty_k\|^2 = \langle \lambda y_k - Ty_k, \lambda y_k - Ty_k \rangle = \lambda^2 \|y_k\|^2 + \|Ty_k\|^2 - 2\lambda \langle Ty_k, y_k \rangle.$$

Since $\lim_{k \rightarrow \infty} \|Ty_k\|^2 = \lambda^2$ and $\lim_{k \rightarrow \infty} \langle Ty_k, y_k \rangle = \lambda$,

$$\lim_{k \rightarrow \infty} \lambda^2 \|y_k\|^2 + \|Ty_k\|^2 - 2\lambda \langle Ty_k, y_k \rangle = \lambda^2 + \lambda^2 - 2\lambda^2 = 0.$$

Thus, $\lim_{k \rightarrow \infty} \lambda y_k - Ty_k = 0$. Since $\lim_{k \rightarrow \infty} Ty_k = v$, we see $\lim_{k \rightarrow \infty} \lambda y_k = v$. Thus, $\lim_{k \rightarrow \infty} y_k = \lambda^{-1}v$. Since T is a compact operator, it is bounded, and thus, continuous. Thus, $v = \lim_{k \rightarrow \infty} Ty_k = \lambda^{-1}Tv$. Thus, $\lambda v = Tv$, so v is an eigenvector of T with eigenvalue λ . $\square$

Lemma 4.5. Let V be an inner product space, let $T : V \rightarrow V$ be a self-adjoint operator, and let v be an eigenvector of T with eigenvalue $\lambda \neq 0$. Then $v \in (\ker T)^\perp$.

Proof. Since v is an eigenvector of T , there exists λ such that $Tv = \lambda v$. Suppose $w \in \ker T$. Then

$$\lambda \langle v, w \rangle = \langle \lambda v, w \rangle = \langle Tv, w \rangle = \langle v, Tw \rangle = \langle v, 0 \rangle = 0.$$

Thus, $\langle v, w \rangle = 0$, so $v \in (\ker T)^\perp$. $\square$

Theorem 4.6 (Spectral Theorem For Compact Operators). Let V be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $T : V \rightarrow V$ be a compact self-adjoint operator. Let $\mathfrak{N} = \ker T$. Then dimension of $\mathfrak{N}^\perp$ is countable, and for any positive eigenvalue, the dimension of the corresponding eigenspace is finite. In addition, $\mathfrak{N}^\perp$ has a orthonormal basis $\{\phi_i\}$ of eigenvectors of T so that $T(\phi_i) = \lambda_i \phi_i$. If $\mathfrak{N}^\perp$ is not finite-dimensional, then $\lim_{i \rightarrow \infty} \lambda_i = 0$.

Proof. If $T = 0$, then the statement clearly holds, since $\mathfrak{N}^\perp = \{0\}$.

Suppose T is nonzero. Suppose $x \in \mathfrak{N}^\perp$. Let $y \in \mathfrak{N}$. Then $\langle Tx, y \rangle = \langle x, Ty \rangle = \langle x, 0 \rangle = 0$, so $Tx \in \mathfrak{N}^\perp$. Thus, $T(\mathfrak{N}^\perp) \subseteq \mathfrak{N}^\perp$. Let Σ be the collection of all orthonormal subsets of $\mathfrak{N}^\perp$ whose elements are eigenvectors of T , partially ordered by inclusion. By Lemma 4.4, T has an eigenvector v with nonzero eigenvalue. By Lemma 4.5, $v \in \mathfrak{N}^\perp$, so Σ is nonempty.

Let $\{U_\alpha\}_{\alpha \in J}$ be a chain in Σ . Take $U = \bigcup_{\alpha \in J} U_\alpha$. Then U is an orthonormal subset of $\mathfrak{N}^\perp$ whose elements are eigenvectors of T such that $U_\alpha \subseteq U$ for all α .

By Zorn's Lemma, Σ has a maximal element S . Set $W = \text{cl}(\text{span } S)$. Set $\mathfrak{H} = W^\perp$. Suppose $x \in \mathfrak{N}$. Since $W \subseteq \mathfrak{N}^\perp$, we see $\langle x, w \rangle = 0$ for all $w \in W$. Thus, $x \in \mathfrak{H}$. Thus, $\mathfrak{N} \subseteq \mathfrak{H}$.

Let $x \in \mathfrak{H}$ such that $Tx \neq 0$. Then T is a nonzero operator on the nonzero Hilbert space $\mathfrak{H}$. Thus, by Lemma 4.4, T has an eigenvector $v \in \mathfrak{H}$. But then, by Lemma 4.5, $S \cup \{v\} \in \Sigma$, contradicting the maximality of S since $v \notin S$. Thus, for all $x \in \mathfrak{H}$, we see $Tx = 0$. Thus, $\mathfrak{H} \subseteq \mathfrak{N}$, so $\mathfrak{H} = \mathfrak{N}$. Thus, $W = \mathfrak{N}^\perp$.

Thus, $S = \{\phi_\alpha\}_{\alpha \in J}$ is an orthonormal basis of $\mathfrak{N}^\perp$ consisting of eigenvectors of T (since S is an orthonormal basis of W). For each $\alpha \in J$, let λ_α be the eigenvalue corresponding to ϕ_α .

Fix $\epsilon > 0$. Suppose there exist infinitely many $\alpha \in J$ such that $|\lambda_\alpha| > \epsilon$. Then there exists a sequence $\{\phi_{\alpha_i}\}_{i=1}^\infty$ such that $\|\lambda_{\alpha_i} \phi_{\alpha_i}\| = \|T\phi_{\alpha_i}\| > \epsilon$ for all $i \in \mathbb{N}$. Fix $i \neq j$. Since $\langle \lambda_{\alpha_i} \phi_{\alpha_i}, \lambda_{\alpha_j} \phi_{\alpha_j} \rangle = 0$,

$$\begin{aligned} \|T\phi_{\alpha_i} - T\phi_{\alpha_j}\|^2 &= \|\lambda_{\alpha_i} \phi_{\alpha_i} - \lambda_{\alpha_j} \phi_{\alpha_j}\|^2 = \langle \lambda_{\alpha_i} \phi_{\alpha_i} - \lambda_{\alpha_j} \phi_{\alpha_j}, \lambda_{\alpha_i} \phi_{\alpha_i} - \lambda_{\alpha_j} \phi_{\alpha_j} \rangle \\ &= \langle \lambda_{\alpha_i} \phi_{\alpha_i}, \lambda_{\alpha_i} \phi_{\alpha_i} \rangle + \langle \lambda_{\alpha_j} \phi_{\alpha_j}, \lambda_{\alpha_j} \phi_{\alpha_j} \rangle = \lambda_{\alpha_i}^2 \|\phi_{\alpha_i}\|^2 + \lambda_{\alpha_j}^2 \|\phi_{\alpha_j}\|^2 = \lambda_{\alpha_i}^2 + \lambda_{\alpha_j}^2. \end{aligned}$$

Thus, for $i \neq j$, $\|T\phi_{\alpha_i} - T\phi_{\alpha_j}\| \geq \epsilon\sqrt{2}$. Thus, $T\phi_{\alpha_i}$ has no convergent subsequence, contradicting the compactness of T . Thus, there exist finitely many $\alpha \in J$ such that $|\lambda_\alpha| > \epsilon$. In particular, for any positive eigenvalue $\lambda > 0$, the dimension of the eigenspace corresponding to λ is finite. In addition, $\mathfrak{N}^\perp$ is countable-dimensional, and $\lim_{i \rightarrow \infty} \lambda_{\alpha_i} = 0$. $\square$

5. THE PETER-WEYL THEOREM

5.1. Representations and Matrix Coefficients.

Definition 5.1 (Representation). Let V be a finite-dimensional vector space over $\mathbb{C}$ and let G be a topological group. A representation of G is a continuous group homomorphism $\rho : G \rightarrow \text{GL}(V)$.

Definition 5.2 (Matrix Coefficients). Let V be a finite-dimensional vector space over $\mathbb{C}$ and let G be a topological group. Suppose $\rho : G \rightarrow \text{GL}(V)$ is a representation of G and $L : V \rightarrow \mathbb{C}$ is a linear functional. For each $v \in V$, the function $\phi_v : G \rightarrow \mathbb{C}$ given by

$$\phi_v(g) = L(\rho(g)(v))$$

is called a matrix coefficient of ρ . The matrix coefficients of a topological group G are the collection all matrix coefficients of all representations of G .

Note. Let V be a finite-dimensional complex vector space. Let $\{e_1, \dots, e_n\}$ be a basis of V and let $\rho : G \rightarrow \text{GL}(V)$ be a representation of the group G . Let

$v = \sum_{i=1}^n v_i e_i$. Then

$$\rho(g)v = \begin{pmatrix} \rho_{11}(g) & \cdots & \rho_{1n}(g) \\ \vdots & & \vdots \\ \rho_{n1}(g) & \cdots & \rho_{nn}(g) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix},$$

where each of the functions ρ_{ij} is a matrix coefficient of ρ given by $\rho_{ij}(g) = L_i(\pi(g)(e_j))$, where $L_i\left(\sum_{j=1}^n v_j e_j\right) = v_i$. This is why we use the name matrix coefficient.

Proposition 5.3. If G is a topological group, the matrix coefficients of G are continuous functions.

Proof. Let $\phi : G \rightarrow \mathbb{C}$ be a matrix coefficient. Then there exists a finite dimensional topological complex vector space V , a representation $\rho : G \rightarrow \text{GL}(V)$, and a linear functional $L : V \rightarrow \mathbb{C}$, and a point $v \in V$ such that

$$\phi(g) = L(\rho(g)(v))$$

for all $g \in G$. Note L is continuous since it is a linear functional on a finite-dimensional complex vector space. Thus, ϕ is continuous, as it is composed from continuous maps. $\square$

Definition 5.4 (r_G). Let G be a group. We define the action r_G of G on the space of complex valued functions on G by

$$(r_G(g)f)(x) = f(xg)$$

for all $x, g \in G$ and functions $f : G \rightarrow \mathbb{C}$.

Theorem 5.5. Let G be a topological group and let $f : G \rightarrow \mathbb{C}$ be a continuous function. Then f is a matrix coefficient of G if and only if the functions $\{r_G(g)f\}_{g \in G}$ span a finite-dimensional complex vector space.

Proof. ($\Rightarrow$) Suppose f is a matrix coefficient of G . Then there exists a representation $\rho : G \rightarrow \text{GL}(V)$, where V is some finite-dimensional complex vector space such that f is a matrix coefficient of ρ . Then there exists some (continuous) linear functional $L : V \rightarrow \mathbb{C}$ and some $v \in V$ such that $f(g) = L(\rho(g)(v))$ for all $g \in G$. Fix $h \in G$. Then for all $x \in G$,

$$(r_G(g)f)(x) = f(xg) = L(\rho(xg)(v)) = L(\rho(x)\rho(g)(v)).$$

Set $u = \rho(g)v$. Then for all $x \in G$,

$$(r_G(g)f)(x) = L(\rho(x)(u)).$$

Thus, $r_G(g)f$ is a matrix coefficient of the representation $\rho : G \rightarrow \text{GL}(V)$ for all $g \in G$. Let $n = \dim V$. Since the matrix coefficients of ρ span a vector space of dimension n^2 (n dimensions in the choice of L , n dimensions in the choice of v), we see that $\{r_G(g)f\}_{g \in G}$ spans a finite-dimensional vector space.

($\Leftarrow$) Suppose the functions $\{r_G(g)f\}_{g \in G}$ span a finite-dimensional F -vector space V . Then $r_G : G \rightarrow \text{GL}(V)$ is a representation of G . Define $L : V \rightarrow \mathbb{C}$ by $L(\phi) = \phi(e)$ for all $\phi \in V$. Then $L(r_G(g)f) = f(g)$ for all $g \in G$, so f is a matrix coefficient. $\square$

5.2. The Peter-Weyl Theorem. We are now at the final steps of proving the Peter-Weyl theorem. Throughout this subsection, G is a compact group and μ is the left Haar measure on G normalized so $\mu(G) = 1$.

Definition 5.6 (Convolution). Suppose $f_1, f_2 \in C(G)$. The convolution of f_1 and f_2 , denoted $f_1 * f_2$, is given by

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1})f_2(h) d\mu(h).$$

Using substitution of variables $h \mapsto h^{-1}g$, we see

$$(f_1 * f_2) = \int_G f_1(h)f_2(h^{-1}g) d\mu(h).$$

In addition, we sometimes perform the convolution of non-continuous functions, such as if $f_1 \in L^\infty(G)$ and $f_2 \in L^1(G)$. For $\phi \in C(G)$, we define T_ϕ by $T_\phi(f) = \phi * f$.

Proposition 5.7. For $\phi \in C(G)$, T_ϕ is a bounded linear operator on $L^1(G)$ and if $f \in L^1(G)$, then

$$\|T_\phi(f)\|_\infty \leq \|\phi\|_\infty \|f\|_1.$$

Proof. Suppose $f_1, f_2 \in L^1(G)$ and $\alpha_1, \alpha_2 \in \mathbb{C}$. Then

$$\begin{aligned} T_\phi(\alpha_1 f_1 + \alpha_2 f_2) &= \int_G \phi(gh^{-1})[(\alpha_1 f_1 + \alpha_2 f_2)(h)] d\mu(h) \\ &= \alpha_1 \int_G \phi(gh^{-1})f_1(h) d\mu(h) + \alpha_2 \int_G \phi(gh^{-1})f_2(h) d\mu(h) = \alpha_1 T_\phi(f_1) + \alpha_2 T_\phi(f_2). \end{aligned}$$

Suppose $f \in L^1(G)$. Then

$$\|T_\phi(f)\|_\infty = \sup_{g \in G} \left| \int_G \phi(gh^{-1})f(h) d\mu(h) \right| \leq \|\phi\|_\infty \int_G \|f(h)\| d\mu(h) = \|\phi\|_\infty \|f\|_1.$$

Thus, T_ϕ is a bounded linear operator on $L^1(G)$. $\square$

Lemma 5.8. The following holds: $L^\infty(G) \subseteq L^2(G) \subseteq L^1(G)$.

Proof. Suppose $f \in L^\infty(G)$. Then

$$\|f\|_2 = \left(\int_G |f|^2 d\mu \right)^{1/2} \leq \left(\int_G \|f\|_\infty^2 d\mu \right)^{1/2} = \|f\|_\infty < \infty.$$

Thus, $f \in L^2(G)$.

Suppose $f \in L^2(G)$. Let $I : G \rightarrow \mathbb{C}$ be given by $I(g) = 1$ for all $g \in G$. By Hölder's Inequality with $p = q = 2$,

$$\|f\|_1 = \|f \cdot I\|_1 \leq \|f\|_2 \cdot \|I\|_2 = \|f\|_2 < \infty.$$

Thus, $f \in L^1(G)$. $\square$

Proposition 5.9. For $\phi \in C(G)$, T_ϕ is a bounded operator on $L^2(G)$ and

$$\|T_\phi\| \leq \|\phi\|_\infty.$$

In addition, T_ϕ is a compact operator on $L^2(G)$ and if $\phi(g^{-1}) = \overline{\phi(g)}$, then T_ϕ is self-adjoint.

Proof. By the previous lemma, $L^\infty(G) \subseteq L^2(G) \subseteq L^1(G)$. In addition, for all $f \in L^2(G)$,

$$\|T_\phi f\|_2 \leq \|T_\phi f\|_\infty \leq \|\phi\|_\infty \|f\|_1 \leq \|\phi\|_\infty \|f\|_2.$$

Thus, T_ϕ is a bounded operator on $L^2(G)$ with $\|T_\phi\| \leq \|\phi\|_\infty$.

Let $\mathfrak{B} = \{T_\phi(f) : f \in L^1(G), \|f\|_1 \leq 1\}$. By the previous proposition, $\mathfrak{B}$ is bounded in $L^\infty(G)$. By Lemma 2.25, since ϕ is continuous and G is compact, for all $\epsilon > 0$, there exists a neighborhood V of e such that

$$|\phi(kg) - \phi(g)| < \epsilon$$

for all g with $k \in V$. Suppose $f \in L^1(G)$ and $\|f\|_1 \leq 1$. Then for all $g \in G$,

$$|(\phi * f)(kg) - (\phi * f)(g)| = \left| \int_G [\phi(kgh^{-1}) - \phi(gh^{-1})] f(h) d\mu(h) \right|$$

$$\int_G |\phi(kgh^{-1}) - \phi(gh^{-1})| |f(h)| d\mu(h) < \epsilon \|f\|_1 \leq \epsilon$$

when $k \in V$. Thus, $\mathfrak{B}$ is equicontinuous. In addition, we immediately see $\mathfrak{B} \subseteq C(G)$ and $\mathfrak{B}$ is closed. Thus, by the Arzelà-Ascoli Theorem, for any sequence $\{f_i\}_{i=1}^\infty$ in $\mathfrak{B}$, there is a subsequence $\{f_{i_k}\}_{i_k=1}^\infty$ which converges uniformly (that is, with respect to the maximum norm). Set $\mathfrak{U} = \{T_\phi(f) : f \in L^2(G), \|f\|_2 \leq 1\}$. Since $\mathfrak{U}$ is a closed subset of $\mathfrak{B}$, $\mathfrak{U}$ is sequentially compact with respect to the maximum norm, and thus, $\mathfrak{U}$ is sequentially compact with respect to the $L^2(G)$ norm. Thus, T_ϕ is a compact operator.

Suppose $\phi(g^{-1}) = \overline{\phi(g)}$ for all $g \in G$. Then

$$\begin{aligned} \langle T_\phi f_1, f_2 \rangle &= \int_G \int_G \phi(gh^{-1}) f_1(h) \overline{f_2(g)} dg dh \\ &= \langle f_1, T_\phi f_2 \rangle = \int_G \int_G f_1(h) \overline{\phi(hg^{-1}) f_2(g)} dg dh = \langle f_1, T_\phi f_2 \rangle. \end{aligned}$$

Thus, in this case, T_ϕ is self-adjoint. $\square$

Proposition 5.10. Let $\phi \in C(G)$ and let λ be an eigenvalue of T_ϕ . Then the λ -eigenspace

$$V(\lambda) = \{f \in L^2(G) : T_\phi(f) = \lambda f\}$$

is invariant under r_G for all $g \in G$.

Proof. Suppose $T_\phi f = \lambda f$. Then

$$(T_\phi(r_G(g)f))(x) = \int_G \phi(xh^{-1}) f(hg) d\mu(h).$$

Applying the change of variables $h \rightarrow hg^{-1}$,

$$\int_X \phi(xh^{-1}) f(hg) d\mu(h) = \int_G \phi(xgh^{-1}) f(h) d\mu(h) = r_G(g)(T_\phi f) = \lambda(r_G(g)f)(x).$$

Thus, $r_G f \in V(\lambda)$, so $V(\lambda)$ is invariant under r_G . $\square$

Lemma 5.11. For any open set U , there exists $\phi \in C_U^+(G)$ such that $\phi(g) = \phi(g^{-1})$ for all $g \in G$ and

$$\int_G f d\mu = 1.$$

Proof. Since U is open, $\mu(U) > 0$. By the inner regularity of μ on open sets, there exists $K \subseteq U$ such that $0 < \mu(K) \leq \mu(U)$. By Theorem 2.4, there exists $\phi^* : G \rightarrow [0, 1]$ such that $\text{supp } \phi^* \subseteq U$ and $\phi^*(K) = \{1\}$. Then

$$0 < \int_G \phi^* d\mu < \infty.$$

Setting $\phi^\dagger(g) = \phi^\dagger(g) + \phi^\dagger(g^{-1})$ for all $g \in G$, $\phi^\dagger(g) = \phi^\dagger(g^{-1})$ and

$$0 < \int_G \phi^\dagger d\mu < \infty.$$

Thus, by multiplying by a positive real number, we obtain a positive real-valued function $\phi \in C_U^+(G)$ such that $\phi(g) = \phi(g^{-1})$ for all $g \in G$ and

$$\int_G \phi d\mu = 1. \quad \square$$

Theorem 5.12 (Peter-Weyl Theorem). The matrix coefficients of G are dense in $C(G)$.

Proof. Let $f \in C(G)$ and fix $\epsilon > 0$. Since G is compact, by Lemma 2.25, there exists a neighborhood U of e such that if $g \in U$, then

$$|f(xg) - f(x)| < \epsilon/2.$$

for all $x \in G$. Thus, for $g \in U$,

$$\|r_G(g)f - f\|_\infty < \epsilon/2.$$

Let μ be the left Haar measure on G normalized so $\mu(G) = 1$. By Lemma 5.11, there exists $\phi \in C_U^+(G)$ such that $\phi(g) = \phi(g^{-1})$ for all $g \in G$ and

$$\int_G \phi d\mu = 1.$$

Define $T_\phi : L^2(G) \rightarrow L^2(G)$ by $T_\phi(f) = \phi * f$. By Proposition 5.2, T_ϕ is a self-adjoint compact operator on $L^2(G)$. Let $h \in G$. Then

$$\begin{aligned} |(\phi * f)(h) - f(h)| &= \left| \int_G \phi(g)f(g^{-1}h) d\mu - f(h) \int_G \phi(g) d\mu(g) \right| \\ &= \left| \int_G \phi(g)f(g^{-1}h) - \phi(g)f(h) d\mu(g) \right| \leq \int_U \phi(g) |f(g^{-1}h) - f(h)| d\mu(g) \\ &\quad + \int_{U^c} \phi(g) \|r_G(g)f - f\| d\mu(g) \leq \int_U \phi(g)(\epsilon/2) d\mu(g) + \int_{U^c} \phi(g)(\epsilon/2) d\mu(g) = \frac{\epsilon}{2}. \end{aligned}$$

Thus, for all $f \in L^2(G)$, $\|T_\phi f - f\|_\infty < \epsilon/2$. If λ is an eigenvalue of T_ϕ , let $V(\lambda)$ be the λ -eigenspace. By the spectral theorem (Theorem 4.6), $V(\lambda)$ is finite-dimensional for all $\lambda \neq 0$, and are mutually orthogonal and span $L^2(G)$. For each eigenvalue λ , let f_λ be the projection of f on $V(\lambda)$. By orthogonality, $\sum_\lambda \|f_\lambda\|_2^2 = \|f\|_2^2 < \infty$. By the Cauchy criterion for convergent series, there exists some $q > 0$ such that

$$\sqrt{\sum_{0 < |\lambda| < q} \|f_\lambda\|_2^2} < \frac{\epsilon}{2\|\phi\|_\infty}.$$

Set $f^* = \sum_{|\lambda| \geq q} f_\lambda$ and $f' = T_\phi(f^*)$. By the spectral theorem (Theorem 4.6), there are finitely many eigenvalues λ such that $|\lambda| \geq q$, each with finite-dimensional

corresponding eigenspace. Thus, $V = \bigoplus_{|\lambda| \geq q} V(\lambda)$ is finite-dimensional. Clearly, for any eigenvalue λ of T_ϕ , $T_\phi(V(\lambda)) \subseteq V(\lambda)$. Thus, since $f^* \in V$, we see that $f' = T_\phi(f^*) \in V$. By Proposition 5.10, V is invariant under r_G , so $\{r_G(g)f'\}_{g \in G} \subseteq V$. Since V is finite-dimensional, $\{r_G(g)f'\}_{g \in G}$ spans a finite-dimensional vector space. Thus, by Theorem 5.5, f' is a matrix coefficient of G . Observe

$$T_\phi(f - f^*) = T_\phi\left(f_0 + \sum_{0 \leq |\lambda| < q} f_\lambda\right) = T_\phi\left(\sum_{0 \leq |\lambda| < q} f_\lambda\right).$$

By Proposition 5.7 and Lemma 5.8,

$$\|T_\phi(f - f^*)\|_\infty \leq \|\phi\|_\infty \|f - f^*\|_1 \leq \|\phi\|_\infty \|f - f^*\|_2 = \|\phi\|_\infty \sqrt{\sum_{0 < |\lambda| < q} \|f_\lambda\|_2^2} < \frac{\epsilon}{2}.$$

Thus,

$$\|f - f'\|_\infty = \|f - T_\phi f + T_\phi(f - f^*)\|_\infty \leq \|f - T_\phi f\|_\infty + \|T_\phi(f - f^*)\|_\infty < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, the matrix coefficients of G are dense in $C(G)$. $\square$

Corollary 5.13. The matrix coefficients of G are dense in $L^2(G)$.

Proof. By the Peter-Weyl Theorem, the matrix coefficients of G are dense in $C(G)$. Since $C(G)$ is dense in $L^2(G)$, the matrix coefficients of G are dense in $L^2(G)$. $\square$

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REFERENCES

- [1] Fritz Peter and Hermann Weyl. “Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe”. In: *Mathematische Annalen* 97.1 (1927), pp. 737–755.
- [2] Alfred Haar. “Der Massbegriff in der Theorie der kontinuierlichen Gruppen”. In: *Annals of Mathematics* 34.1 (1933), pp. 147–169. ISSN: 0003486X, 19398980. URL: <http://www.jstor.org/stable/1968346>.
- [3] John von Neumann. “Die Einführung Analytischer Parameter in Topologischen Gruppen”. In: *Annals of Mathematics* 34.1 (1933), pp. 170–190. ISSN: 0003486X, 19398980. URL: <http://www.jstor.org/stable/1968347>.
- [4] Shmuel Kantorovitz. *Introduction to Modern Analysis*. Oxford University Press, 2003.
- [5] Anton Deitmar and Siegfried Echterhoff. *Principles of harmonic analysis*. Springer, 2009.
- [6] H.L. Royden and P. Fitzpatrick. *Real Analysis*. 4th ed. Prentice Hall, 2010.
- [7] Daniel Bump. *Lie Groups*. 2nd ed. Springer, 2013.
- [8] James Munkres. *Topology*. Pearson, 2017.