THE LEVELWISE FINITE GENERATION OF FREE TAMBARA FUNCTORS

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Abstract. We prove the levelwise finite generation of free polynomial G-Tambara functors in a collection of cases, most notably when G is a finite Dedekind group or when $G \cong C_p \rtimes C_q$, $p > q$ primes. In the process, we establish the permanence of various finiteness conditions under box products and norms n_H^G of Tambara functors, including a weak Hilbert Basis Theorem.

Contents

1. Introduction

1.1. **Background.** Tambara functors are equivariant analogues of commutative rings, appearing naturally as ring structures associated to systems of representations. Representation rings, Galois extensions, Burnside rings, and even commutative rings with a G-action in the simplest case all give rise to Tambara functors, which are defined as a functor sending finite G-sets to sets satisfying certain properties.

The notion of a *free polynomial Tambara functor,* first introduced in [BH18], is the equivariant analogue of a free polynomial ring; they represent the functors which send a Tambara functor to its underlying set in a chosen level. In some sense, these can be viewed as very lage Burnside rings—indeed, specializing to the generator associated to the G-set $X = \varnothing$ recovers the usual Burnside ring.

Free polynomial Tambara functors both mirror and differ from their classical counterparts in interesting ways. A result of [Bru05] shows that the bottom level of any free polynomial Tambara functor $\underline{\mathbf{A}}[X]$ is a free polynomial ring with $|X|$ generators. On the other hand, in [HMQ22], free Tambara functors are shown to be almost never flat, a striking deviation from classical algebra.

However, not much is known about the levelwise structure of polynomial Tambara functors aside from a collection of specialized cases. In [SSW25], the free polynomial Tambara functor on a top-level generator is shown to be levelwise finitely generated. A weak Hilbert basis theorem for free polynomial Tambara functors was shown for $G = C_p$ by recent work in [4DS], where a computation of their Nakaoka spectra is also given. Knowing such results in greater generality allows us to better apply the wealth of knowledge from classical commutative algebra in the equivariant setting.

1.2. **Main Results.** In this paper, we establish several finiteness results for free polynomial Tambara functors. First, a definition:

Definition (Definition 4.10)**.** *A finite group G is said to be Tambara finite if the free polynomial Tambara functor $\underline{\mathbf{A}}[X]$ is levelwise finitely generated for all finite G-sets X, i.e. $\underline{\mathbf{A}}[X](G/L)$ is a finite-type $\mathbf{Z}$-algebra for all $L \leq G$. We say that G is transitively Tambara finite or just transitively finite if $\underline{\mathbf{A}}[X]$ is levelwise finitely generated for all transitive finite G-sets X.*

The Tambara-finiteness of $G = C_p$ is established in [4DS]; our main theorem is the extension of this result to a much larger class of groups. More precisely, we prove:

Theorem A (Theorem 6.17)**.** *Let G be a finite group satisfying one of the following conditions:*

- *(a) G is a Dedekind group.*
- *(b) Every proper, nontrivial subgroup of G is maximal, and every subgroup is either normal or satisfies $N_G(H) = H$.*
- *(c) $G \cong D_8$.*

Then G is Tambara finite.

Recall that a *Dedekind group* is a group whose subgroups are all normal. In particular, the class of G to which the theorem applies includes all finite abelian groups and the quaternion group Q_8.

This is a fairly strong finiteness condition on $\underline{\mathbf{A}}[X]$. For general G, we prove a weaker theorem:

Theorem B (Theorem 6.16)**.** *Let G be a finite group and X be a G-set. Then $\underline{\mathbf{A}}[X]$ is relatively finite-dimensional, in the sense that all restriction maps Res^K_H are finite ring maps. In particular, $\underline{\mathbf{A}}[X]$ is module-Noetherian when G is Tambara finite.*

The notion of a relatively finite-dimensional Green or Tambara functor is new, due to [CW25], and it is critical to extending any results obtained for transitive G-sets to general

G-sets, as it is well-behaved under box products; we will see this in §6. We also prove a weak Hilbert Basis Theorem which generalizes the corresponding result for $G = C_p$ presented in [4DS].

Theorem C (Weak Hilbert Basis Theorem, Theorem 6.20)**.** *Let G be a Tambara finite group, X a finite G-set, and T a levelwise Noetherian, relatively finite-dimensional Tambara functor. Then $T[X] = T \boxtimes \underline{\mathbf{A}}[X]$ is also levelwise Noetherian and relatively finite-dimensional. If T is levelwise finitely generated, then so is $T[X]$.*

In particular, under the assumptions above, $T[X]$ is Noetherian, i.e. satisfies the ascending chain condition on Tambara ideals.

Our proof of Theorem A proceeds by first establishing transitive Tambara finiteness; technical results on the box product of two Tambara functors are developed in §6 to show that transitively Tambara finite groups are also Tambara finite. In particular, a simple (noninductive) formula for the box product of an arbitrary number of G-Mackey functors for a finite group G is developed in §6.

Theorem D (Theorem 6.5)**.** *Let G be a finite group, $M_1, \ldots, M_N$ be a collection of G-Mackey functors and write $M = M_1 \boxtimes \cdots \boxtimes M_N$. Fix a subgroup $L \leq G$. For any subgroup $H \leq L$, define*

$$S_H^L = M_1(G/H) \otimes \cdots \otimes M_N(G/H).$$

Then we have

$$M(G/L) = \left(\bigoplus_{H \leq L} S_H^L \right) / F,$$

where F is the submodule generated by Frobenius and Weyl relations, defined in §6.

This generalizes previously known formulas for the box product appearing in the literature for when G is a cyclic p-group; for instance, an inductive formula when $G = C_{p^n}$ is described in [Maz13]. The formula also has the advantage of having an easy-to-describe ring structure when the M_i are Tambara functors. We use this to prove the following:

Theorem E. *Let G be a finite group and let T, R be two levelwise finitely generated Tambara functors. Then $T \boxtimes R$ is levelwise finitely generated.*

This is a surprisingly delicate result, as it fails for Green functors—a simple counterexample when $G = C_p$ is given in §6.

(Bi)incomplete Tambara functors are not considered in this paper, which constitutes a direction for further investigation. While we suspect that Theorem A holds for all Tambara functors and all finite groups, it fails for Green functors in general (in the sense that not all free Green functors for a finite group G are levelwise finitely generated). In light of this fact, one might ask: what conditions must one impose on the indexing system of an incomplete Tambara functor to ensure levelwise finite generation?

We note moreover that a better understanding of the levelwise ring structure of the free polynomial functors $\underline{\mathbf{A}}[X]$ can be achieved if more explicit information were known about the ring structure of the norms n_H^G of Tambara functors, $H \leq G$. The norm functor n_H^G is the left adjoint to the restriction functor Res_H^G which sends a G-Tambara functor to an H-Tambara functor via precomposition by induction of H-sets—see [HM19] and [Hoy14]. In fact, our results show:

Theorem F (Corollary 7.2, Theorem 7.3)**.** *The norm functors n_H^G preserve levelwise finite generation over $\mathbf{Z}$ for all finite $H \leq G$ iff all finite groups are Tambara finite. If $H \leq G$ and G is a finite group satisfying the hypotheses of Theorem A, then n_H^G preserves levelwise finite generation.*

This leads to the natural followup question:

Question. *Let T be an H-Tambara functor which is levelwise Noetherian/relatively finite-dimensional. Is the same true for $n_H^G T$?*

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2. Review of Mackey and Tambara Functors

2.1. **The Polynomial Category.** We briefly review the notions of equivariant algebra we will need and establish some conventions for the rest of this text. For details, we refer the reader to [Str12]. Fix a finite group G. We use $\mathcal{P}_G$ to denote the *category of bispans of finite G-sets* or *category of polynomials of finite G-sets,* where objects are finite G-sets and morphisms are isomorphism classes of *polynomials* $[X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y]$, and $\mathcal{P}_G^+$ the *category of spans of finite G-sets,* the subcategory of $\mathcal{P}_G^+$ containing all the objects and where morphisms are polynomials above such that q is an isomorphism. Here, an isomorphism of polynomials is described by a commutative diagram of the form

$$\begin{tikzcd}
 & A \arrow[r] \arrow[dl] \arrow[dd, "\cong"] & B \arrow[dr] \arrow[dd, "\cong"] & \\
X & & & Y \\
 & A' \arrow[r] \arrow[ul] & B' \arrow[ur] &
\end{tikzcd}$$

A composition of polynomials $[X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y]$ and $[Y \xleftarrow{p'} A' \xrightarrow{q'} B' \xrightarrow{r'} Z]$ is given by $[X \leftarrow A'' \to B'' \to Z]$ in the diagram

$$\begin{tikzcd}
 & & A'' \arrow[r] \arrow[dl] & W \arrow[r] \arrow[dl] \arrow[dr] & B'' \arrow[dr] & & \\
 & A \arrow[r] \arrow[dl] & B \arrow[dr] & & A' \arrow[r] \arrow[dl] & B' \arrow[dr] & \\
X & & & Y & & & Z
\end{tikzcd}$$

where $B'' = \{(b', s) \mid s : {q'}^{-1}(b') \to B, rs = p'\}$,

$$W = B'' \times_{B'} A' = \{(a', s) \mid s : {q'}^{-1}(q'(a')) \to B, rs = p'\},$$

$W \to B$ is given by $(a', s) \mapsto s(a')$, and $A'' = W \times_B A$ (see [Str12]).

There are three distinguished types of morphisms in $\mathcal{P}_G$. For any map $f : X \to Y$ of finite G-sets, we write

$$R_f = [Y \xleftarrow{f} X \xrightarrow{1} X \xrightarrow{1} X], \quad N_f = [X \xleftarrow{1} X \xrightarrow{f} Y \xrightarrow{1} Y], \quad T_f = [X \xleftarrow{1} X \xrightarrow{1} X \xrightarrow{f} Y].$$

These are called *restriction, norm,* and *transfer* along f, respectively.

Let $g \in G$. for any $H \leq G$, there is a G-isomorphism $f_g : G/H \to G/gHg^{-1}$ defined by $xH \mapsto xHg^{-1} = xg^{-1}(gHg^{-1})$. Transfer along f_g is denoted by C_g, and in fact

$$C_g = T_{f_g} = \operatorname{Res}_{f_g}^{-1}.$$

2.2. **Mackey and Tambara Functors.**

Definition 2.1. A *G-semi-Mackey functor* is a product-preserving functor $\mathcal{P}_G^+ \to \mathbf{Set}$. A *$G$-semi-Tambara functor* is a product-preserving functor $\mathcal{P}_G \to \mathbf{Set}$.

It is well-known that every semi-Mackey functor takes values in commutative monoids, and that every semi-Tambara functor takes values in commutative semirings.

Definition 2.2. A *G-Mackey functor* is a semi-Mackey functor which takes values in (abelian) groups. A *G-Tambara functor* is a semi-Tambara functor which takes values in (commutative) rings.

In particular, the group completion of a semi-Mackey functor is a Mackey functor, and the additive completion of a semi-Tambara functor is a Tambara functor.

There is an equivalent description of Mackey and Tambara functors using only the transitive G-sets, which we now describe.

Definition 2.3. A *G-Mackey functor M* consists of the following data:

(a) For each subgroup $H \leq G$, an abelian group $M(G/H)$.
(b) For each $g \in G$ and $H \leq G$, an isomorphism $c_g : M(G/H) \to M(G/gHg^{-1})$ (the dependence on H is supressed in the notation) such that such that $c_h = \mathrm{id}$ for all $h \in H$, $c_g c_{g'} = c_{gg'}$ for all $g, g' \in G$.
(c) For each subgroup inclusion $H \leq K$, group homomorphisms $\operatorname{Res}_H^K : M(G/K) \to M(G/H)$ and $\operatorname{Tr}_H^K : M(G/H) \to M(G/K)$ such that
   (i) $\operatorname{Res}_H^H = \operatorname{Tr}_H^H = \mathrm{id}$ for all H.
   (ii) $\operatorname{Res}_H^K \operatorname{Res}_K^L = \operatorname{Res}_H^L$ and $\operatorname{Tr}_K^L \operatorname{Tr}_H^K = \operatorname{Tr}_H^L$ whenever $H \leq K \leq L$.
   (iii) For all $g \in G$,
   $$c_g \operatorname{Res}_H^K = \operatorname{Res}_{gHg^{-1}}^{gKg^{-1}} c_g, \quad c_g \operatorname{Tr}_H^K = \operatorname{Tr}_{gHg^{-1}}^{gKg^{-1}} c_g$$
   whenever $H \leq K$.
   (iv) (Double coset formula) Whenever $H, K \leq L$,
   $$\operatorname{Res}_K^L \operatorname{Tr}_H^L = \sum_{g \in K \backslash L / H} \operatorname{Tr}_{K \cap gHg^{-1}}^K c_g \operatorname{Res}_{H \cap g^{-1}Kg}^H.$$

A semi-Mackey functor is obtained by relaxing the condition that each $M(G/H)$ is an abelian group, requiring only a commutative monoid instead.

Definition 2.4. A *G-Tambara functor T* consists of the following data:

(a) For each subgroup $H \leq G$, a commutative ring $T(G/H)$.

(b) For each subgroup inclusion $H \leq K$, maps $\mathrm{Res}_H^K : T(G/K) \to T(G/H)$, $\mathrm{Tr}_H^K : T(G/H) \to T(G/K)$, and $\mathrm{Nm}_H^K : T(G/H) \to T(G/K)$, and for each $g \in G$, isomorphisms $c_g : T(G/H) \to T(G/gHg^{-1})$ such that
  (i) The family of maps $(\{\mathrm{Res}_H^K\}, \{\mathrm{Tr}_H^K\}, \{c_g\})$ turns T into a Mackey functor with respect to the additive group structure.
  (ii) The family of maps $(\{\mathrm{Res}_H^K\}, \{\mathrm{Nm}_H^K\}, \{c_g\})$ turns T into a semi-Mackey functor under the multiplicative monoid structure.

(c) (Exponential formula) For any exponential diagram

$$\begin{tikzcd} X \arrow[d] & A \arrow[l] & X \times_Y \Pi_f A \arrow[l] \arrow[d] \\ Y & & \Pi_f A \arrow[ll] \end{tikzcd}$$

of finite G-sets, the following diagram commutes:

$$\begin{tikzcd} T(X) \arrow[d, "\mathrm{Nm}"'] & T(A) \arrow[l, "\mathrm{Tr}"'] \arrow[r, "\mathrm{Res}"] & T(X \times_Y \Pi_f A) \arrow[d, "\mathrm{Nm}"] \\ T(Y) & & T(\Pi_f A) \arrow[ll, "\mathrm{Tr}"] \end{tikzcd}$$

Remark 2.5. The maps $\mathrm{Res}_H^K, \mathrm{Tr}_H^K, \mathrm{Nm}_H^K$ are the maps induced by restriction, transfer, and norm along the projection $G/H \to G/K$, while c_g is the map induced by C_g. Note that the maps c_g induce an action of $W_H := N(H)/H$ on $T(G/H)$, where $N(H)$ is the normalizer of H in G; W_H is the *Weyl group* of H.

Remark 2.6. Only (c) is not expressed precisely in terms of the data we have given, as there is no succinct way to express the exponential formula using only transitive G-sets. When X is not transitive, say $X = \coprod_i G/H_i$, we extend $T(X) = \prod_i T(G/H_i)$; there is also a way to extend the transfer and norm maps using only the data given on the transitive sets. A consequence of the exponential formula we will use repeatedly is Frobenius reciprocity,

$$x \,\mathrm{Tr}_H^K(y) = \mathrm{Tr}_H^K(\mathrm{Res}_H^K(x)y)$$

whenever defined.

Remark 2.7. In most examples, we will represent the data of a Mackey or Tambara functor via its *Lewis diagram,* a diagram of all the $T(X)$ for transitive X with only the maps $\mathrm{Res}_H^K, \mathrm{Tr}_H^K, c_g$ labeled.

Remark 2.8. We will occasionally reference the notion of a (commutative) *Green functor,* a Mackey functor taking values in commutative rings such that the Res_H^K and c_g are ring maps, and such that transfer and restriction satisfy the Frobenius reciprocity relations above. In particular, the data of a Green functor does not come with norm maps Nm_H^K.

3. Free Polynomial Tambara Functors

3.1. **Basic Properties of Free Tambara Functors.** For any two finite G-sets X, Y, the set of morphisms $\mathcal{P}_G(X, Y)$ has a semiring structure, with addition given by

$$[X \leftarrow A \to B \to Y] + [X \leftarrow A' \to B' \to Y] = [X \leftarrow A \sqcup A' \to B \sqcup B' \to Y],$$

multiplication given by

$$[X \leftarrow A \to B \to Y] \cdot [X \leftarrow A' \to B' \to Y] = [X \leftarrow (A \times_Y B') \sqcup (A' \times_Y B) \to B \times_Y B' \to Y],$$

and additive and multiplicative identities given by

$$[X \leftarrow \varnothing \to \varnothing \to Y], \quad [X \leftarrow \varnothing \to Y \to Y]$$

respectively (see [HMQ22, Theorem 2.14]).

Definition 3.1. Given a finite G-set X, the *free polynomial Tambara functor* or the *polynomial Tambara functor* on X is the additive completion of the representable functor

$$\mathcal{P}_G(X, -) : \mathcal{P}_G \to \mathbf{Set}.$$

In particular, we have a natural isomorphism

$$\mathrm{Hom}_{G\mathbf{Tamb}}(\underline{\mathbf{A}}[X], T) \cong T(X)$$

for any G-Tambara functor T and finite G-set X. When $X = \varnothing$, $\underline{\mathbf{A}} := \underline{\mathbf{A}}[\varnothing]$ is the *Burnside Tambara functor,* the initial object in the category of Tambara functors.

We cite some basic facts about the structure maps in a polynomial Tambara functor.

Lemma 3.2 ([HMQ22, Proposition 3.32])**.** *Let $H \leq G$, $f : Y \to Z$ be a map of finite G-sets. Then T_f sends*

$$[G/H \leftarrow A \to B \xrightarrow{r} Y] \mapsto [G/H \leftarrow A \to B \xrightarrow{f \circ r} Z],$$

while R_f sends

$$[G/H \leftarrow A \to B \to Z] \mapsto [G/H \leftarrow A \times_Z Y \to B \times_Z Y \to Y].$$

Let H be a subgroup of G. In order to understand $\underline{\mathbf{A}}[G/H]$, it suffices to understand the structure of $\underline{\mathbf{A}}[G/H](G/L) = \mathcal{P}_G(G/H, G/L)$ for $L \leq G$. By splitting up the direct summands of a polynomial $G/H \to G/L$, we see that every element of $\underline{\mathbf{A}}[G/H](G/L)$ is uniquely a sum of (isomorphism classes of) polynomials of the form

$$G/H \leftarrow \coprod_i G/H_i \to G/K \to G/L,$$

i.e. a polynomial $G/H \leftarrow A \to B \to G/L$ where B is transitive. We will call such a polynomial *irreducible.* It is clear that isomorphism classes of irreducible polynomials form a $\mathbf{Z}$-basis for $\mathcal{P}_G(G/H, G/L)$.

Recall the following definition:

Definition 3.3. A finite group G is said to be a *Dedekind group* if all subgroups are normal.

The only facts about Dedekind groups which will be relevant to us are the following.

(a) There exists a G-equivariant map $G/H \to G/K$ iff $H \leq K$. In this case, the equivariant map $G/H \to G/K$ sending H to gK is well-defined for all $g \in G$.
(b) For any two subgroups $H, K \leq G$, $HK = KH$ is also a subgroup.
(c) All double cosets $H\backslash G/K$ are the same as the one-sided cosets $G/HK = HK\backslash G$.

When G is a Dedekind group, any irreducible polynomial of the form

$$G/H \leftarrow \coprod_i G/H_i \to G/K \to G/L$$

must satisfy $H_i \leq K \leq L$ and $H_i \leq H$. Furthermore, by composing appropriate elements of G to G/K and each factor of G/H_i, we can arrange for the representative for the isomorphism class of the diagram above to be of the form

$$G/H \xleftarrow{\sqcup_i f_i} \coprod_i G/H_i \xrightarrow{\sqcup_i \mathrm{pr}} G/K \xrightarrow{\mathrm{pr}} G/L$$

where pr is generic notation for the natural projection $G/S \to G/T$ whenever $S \leq T$ are subgroups and $f : G/H \to G/K$ denotes the equivariant map sending H to fK. Thus, an element $\underline{\mathbf{A}}[G/H](G/L)$ can be represented by some tuple $((H_1, f_1), \ldots, (H_n, f_n))_K$, where f_i are elements of G/H which keep track of the image of H_i under the component $G/H_i \to G/H$. Note here that the order of the (H_i, f_i) does not matter, as shuffling the order gives an isomorphic polynomial.

Lemma 3.4. *Two tuples $((H_1, f_1), \ldots, (H_n, f_n))_K$ and $((H'_1, g_1), \ldots, (H'_m, g_n))_K$ yield isomorphic polynomials iff $K = K'$, $m = n$, we have $H'_i = H_i$ up to reordering, and the following condition is true: there exists some $\ell \in L/K$ and lifts $\ell_1, \ldots, \ell_n \in L$ such that*

$$\ell_i f_i = g_i$$

up to reordering.

Proof. The condition that $K = K'$ and $\{H_i\} = \{H'_i\}$ is obvious, since $G/K \cong G/K'$ as G-sets iff $K = K'$. Thus we want to determine when there exists an isomorphism of irreducible polynomials

$$\begin{tikzcd}
& \coprod_i G/H_i \arrow[r, "\sqcup_i \mathrm{pr}"] \arrow[dl, "\sqcup_i f_i"'] \arrow[dd, "\sqcup r_i"] & G/K \arrow[dr, "\mathrm{pr}"] \arrow[dd, "s"] & \\
G/H & & & G/L \\
& \coprod_i G/H_i \arrow[r, "\sqcup_i \mathrm{pr}"'] \arrow[ul, "\sqcup_i g_i"] & G/K \arrow[ur, "\mathrm{pr}"'] &
\end{tikzcd}$$

We see that in order for the diagram to commute, we must have $s \in L/K$, $r_i \in L/H_i$ must lift $s \in L/K$, and $f_i = r_i g_i$. $\square$

Remark 3.5. This equivalence relation is hard to describe cleanly and is a source of difficulty for tracking the combinatorics involved in analyzing the ring structure of $\underline{\mathbf{A}}[G/H]$. We will later isolate some special cases for which this task is easier.

3.2. **A Levelwise Grading for Polynomials.** The polynomial Tambara functors admit a levelwise $\mathbf{N}$-grading for arbitrary finite G; when G is a Dedekind group, an explicit computation offers a slight refinement.

Definition 3.6. For any finite group G, let $\mathcal{O}^w_G$ consist of the pairs

$$(n, K)$$

where $K \leq G$ and $n \in \mathbf{Z}_{\geq 0}$ is a nonnegative integer. $\mathcal{O}^w_G$ assembles into a commutative monoid under the operation

$$(n, K) + (m, L) = (n + m, K \cap L).$$

In other words, $\mathcal{O}^w_G \cong \mathbf{N} \times \mathcal{O}_G$ as a monoid, where $\mathcal{O}_G$ is the monoid of subsets of G under intersection.

Definition 3.7. Let G be a finite group and $b = [G/H \leftarrow A \rightarrow B \rightarrow G/L]$ be an irreducible polynomial, i.e. one where B is transitive. We define the *degree* of b to be the degree of $A \rightarrow B$ as a map of G-sets.

Definition 3.8. When G is Dedekind, define the degree of an irreducible polynomial

$$((H_1, f_1), \ldots, (H_n, f_n))_K$$

to be

$$\left(\sum_i \frac{|K|}{|H_i|}, K\right) \in \mathcal{O}_L^w \subseteq \mathcal{O}_G^w.$$

Here, the integer $\sum_i |K|/|H_i|$ is again the degree of the map

$$\coprod_i G/H_i \rightarrow G/K.$$

For the rest of this section, we will assume by default that G is Dedekind; however, most of the results are analogous in the general case and can be obtained by simply dropping the non-numerical component of $\mathcal{O}_L^w$.

It follows immediately from definition that

$$\underline{\mathbf{A}}[G/H](G/L) \cong \bigoplus_{d \in \mathcal{O}_L^w} S_d,$$

as groups, where S_d is the group generated by the irreducible polynomials of degree $d \in \mathcal{O}_L^w$. This also gives a description of $\underline{\mathbf{A}}[G/H]/(G/L)$ as a graded ring.

Lemma 3.9. *$((H_1, f_1), \ldots, (H_n, f_n))_K \cdot ((H_1', g_1), \ldots, (H_m', g_m))_{K'}$ is a homogeneous element of degree*

$$\begin{aligned} &\deg((H_1, f_1), \ldots, (H_n, f_n))_K + \deg((H_1', g_1), \ldots, (H_m', g_m))_{K'} \\ &= \left(\sum_i \frac{|K|}{|H_i|} + \sum_j \frac{|K'|}{|H_j'|}, K \cap K'\right) \end{aligned}$$

Proof. Recall that multiplication on polynomials is given by

$$[X \leftarrow A \rightarrow B \rightarrow Y] \cdot [X \leftarrow A' \rightarrow B' \rightarrow Y] = [X \leftarrow (A \times_Y B') \sqcup (A' \times_Y B) \rightarrow B \times_Y B' \rightarrow Y].$$

If $A \rightarrow B$ is of degree n, then so is $A \times_Y B' \rightarrow B \times_Y B'$, as pullback preserves degrees; the same holds for $A' \rightarrow B'$. Hence $(A \times_Y B') \sqcup (A' \times_Y B) \rightarrow B \times_Y B'$ is of degree $n + m$, and the fact that the associated subgroup is $K \cap K'$ follows from $G/K \times_{G/L} G/K' \cong \coprod G/(K \cap K')$. $\square$

Remark 3.10. Note that the elements of degree $(0, -)$ form a subring which we temporarily denote by $\underline{\mathbf{A}}_0[G/H](G/L)$. This ring is generated by irreducible polynomials of the form

$$G/H \leftarrow \varnothing \rightarrow G/K \rightarrow G/L,$$

so we can identify the $\mathbf{Z}$-basis with the poset $\mathcal{O}_L$ of subgroups $K \leq L$.

In particular, $\underline{\mathbf{A}}_0[G/H](G/L)$ has finite $\mathbf{Z}$-rank. Note moreover that since polynomials of the form

$$G/H \leftarrow \varnothing \rightarrow X \rightarrow G/L$$

where $X \rightarrow G/L$ is an equivariant map of finite G-sets are in bijection with polynomials $\varnothing \rightarrow G/L$, we have an isomorphism between $\underline{\mathbf{A}}_0[G/H](G/L)$ and $\underline{\mathbf{A}}(G/L)$, where $\underline{\mathbf{A}}$ is the Burnside ring $\underline{\mathbf{A}} = \underline{\mathbf{A}}[\varnothing] = \mathcal{P}(\varnothing, -)$. In other words, the sub-Tambara functor $\mathcal{P}(\varnothing, -) \hookrightarrow \mathcal{P}(G/H, -)$ has as levelwise image the elements of degree $(0, -)$.

Remark 3.11. The elements of degree $(-, L)$ also form a subring of $\underline{\mathbf{A}}[G/H](G/L)$, which we denote by $\underline{\mathbf{A}}^0[G/H](G/L)$. This ring is generated by irreducible polynomials of the form

$$G/H \leftarrow \coprod_i G/H_i \to G/L \to G/L.$$

In particular, the elements of degree $(-, L)$ forms an $\mathbf{N}$-graded subring, since

$$(n, L) + (m, L) = (n + m, L)$$

in $\mathcal{O}_L^w$. It is easy to describe the multiplication of such irreducible polynomials: we have

$$((H_i, f_i)_{i \in I})_L \cdot ((H'_j, g_j)_{j \in J})_L = ((H_i, f_i)_{i \in I}, (H'_j, g_j)_{j \in J})_L.$$

In particular, we see that $\underline{\mathbf{A}}^0(G/L)$ is a finite type $\mathbf{Z}$-algebra. The equivalence relation in Lemma 3.4 is also simple; the set of representatives for the equivalence is given by orbits of collections $((H_1, f_1), \ldots, (H_n, f_n))_L$ under the L^n-action

$$((H_1, f_1), \ldots, (H_n, f_n))_L \mapsto ((H_1, \ell_1 f_1), \ldots, (H_n, \ell_n f_n)),$$

so each of the f_i is a well-defined element of G/HL. While $\underline{\mathbf{A}}^0$ is preserved by restriction and norm maps, it is not preserved by transfer, so it is not a sub-Tambara functor.

Example 3.12. For $G = C_p$, we have $\underline{\mathbf{A}}_0[C_p/C_p](C_p/C_p) \cong \mathbf{Z}[t]/(t^2 - pt)$, where here t is the element corresponding to the polynomial

$$C_p/C_p \leftarrow \varnothing \to C_p/e \to C_p/C_p$$

while $\underline{\mathbf{A}}_0[C_p/C_p](C_p/e) \cong \mathbf{Z}$. On the other hand, we have $\underline{\mathbf{A}}^0[C_p/C_p](C_p/e) = \mathbf{Z}[x]$, where here x is the element corresponding to the polynomial

$$C_p/C_p \leftarrow C_p/e \to C_p/e \to C_p/e.$$

$\underline{\mathbf{A}}^0[C_p/C_p](C_p/C_p) = \mathbf{Z}[x]$, where x is the element corresponding to the polynomial

$$C_p/C_p \leftarrow C_p/C_p \to C_p/C_p \to C_p/C_p$$

Lemma 3.13. *The Weyl group action on $\underline{\mathbf{A}}[G/H](G/L)$ preserves degrees.*

Proof. Obvious. □

Lemma 3.14. *Let $L \leq L'$. $\mathrm{Tr} : \underline{\mathbf{A}}[G/H](G/L) \to \underline{\mathbf{A}}[G/H](G/L')$ and $\mathrm{Res} : \underline{\mathbf{A}}[G/H](G/L') \to \underline{\mathbf{A}}[G/H](G/L)$ are graded homomorphisms, in the sense that there are monoid homomorphisms $\phi : \mathcal{O}_L^w \to \mathcal{O}_{L'}^w$ and $\psi : \mathcal{O}_{L'}^w \to \mathcal{O}_L^w$ such that*

$$\mathrm{Tr}\left(\underline{\mathbf{A}}[G/H](G/L)_d\right) \subseteq \underline{\mathbf{A}}[G/H](G/L)_{\phi(d)}, \quad \mathrm{Res}\left(\underline{\mathbf{A}}[G/H](G/L')_{d'}\right) \subseteq \underline{\mathbf{A}}[G/H](G/L)_{\psi(d')}$$

for any $d \in \mathcal{O}_L^w$, $d' \in \mathcal{O}_{L'}^w$. Explicitly,

$$\phi(n, K) = (n, K), \quad \psi(n, K) = (n, K \cap L)\,.$$

Proof. Observe that ϕ, ψ indeed define monoid homomorphisms. The fact that Res and Tr are graded in this way more or less follows immediately from the explicit description of transfer and restriction in Lemma 3.2, the fact that pullback preserves degrees, and the fact that $G/K \times_{G/L'} G/L = \coprod G/(K \cap L)$. □

We summarize these observations in the following, more general lemma.

Lemma 3.15. *Let $\mathcal{F}$ be a collection of subgroups of L and let $\mathcal{O}^w_{L,\mathcal{F}}$ be the subset of $\mathcal{O}^w_L$ consisting of all pairs of the form (n, K) where $K \in \mathcal{F}$.*

(a) If $L \in \mathcal{F}$ and $\mathcal{F}$ is closed under intersections, then the elements of degree in $\mathcal{O}^w_{L,\mathcal{F}}$ form a subring of $\underline{\mathbf{A}}[G/H](G/L)$.
(b) If for every $K \in \mathcal{F}$ and $K' \leq L$, $K \cap K' \in \mathcal{F}$, then the elements of degree in $\mathcal{O}^w_{L,\mathcal{F}}$ form an ideal of $\underline{\mathbf{A}}[G/H](G/L)$. In particular, any family of subgroups $\mathcal{F}$ which is "downwards closed" in the sense that $K' \in \mathcal{F}$ whenever $K' \leq K$ and $K \in \mathcal{F}$ defines an ideal of $\underline{\mathbf{A}}[G/H](G/L)$.
(c) For $L \leq L'$, the image of transfer $\mathrm{Tr} : \underline{\mathbf{A}}[G/H](G/L) \to \underline{\mathbf{A}}[G/H](G/L')$ is the ideal corresponding to the family $\mathcal{F} = \{K \leq L\}$.

Proof. Obvious. □

Remark 3.16. The subring $\underline{\mathbf{A}}^0[G/H](G/L)$ can be identified with the quotient of $\underline{\mathbf{A}}[G/H](G/L)$ by the ideal corresponding to the family $\mathcal{F} = \{K \lneq L\}$.

Remark 3.17. $\mathrm{Nm}_L^{L'} : \underline{\mathbf{A}}[G/H](G/L) \to \underline{\mathbf{A}}[G/H](G/L')$ is not generally graded.

Remark 3.18. None of the results on the grading of $\underline{\mathbf{A}}[G/H](G/L)$ depend on the transitivity of $X = G/H$. In particular, for any finite G-set X, the ring $\underline{\mathbf{A}}[X](G/L)$ is graded over $\mathcal{O}^w_L$ in the way described: an irreducible polynomial

$$X \leftarrow \coprod_i G/H_i \to G/K \to G/L$$

has degree $(\sum_i |K|/|H_i|, K)$, which turns $\underline{\mathbf{A}}[X](G/L)$ into a graded ring for each $L \leq G$. Restriction and transfer maps for $\underline{\mathbf{A}}[X]$ are also graded in the same way. For now, we call this the *naive grading* on $\underline{\mathbf{A}}[X]$. In §6, we will show that the gradings on $\underline{\mathbf{A}}[G/H]$ for transitive G/H induce a grading on $\underline{\mathbf{A}}[X]$ for arbitrary G-sets X, which coincides with the naive grading.

4. Finite Generation for Dedekind Groups

4.1. **Finiteness Results for General G.** The main goal of this section is to prove that the polynomial Tambara functors $\underline{\mathbf{A}}[G/H]$ are levelwise finitely generated for Dedekind G and relatively finite-dimensional for general G. First, some definitions.

Definition 4.1. Let T be a G-Tambara functor and let R be a T-algebra, i.e. R is a G-Tambara functor equipped with a morphism $f : T \to R$. We say that R is *levelwise finitely generated over* T if each $R(G/H)$ is a finite type $T(G/H)$-algebra. When $T = \underline{\mathbf{A}} = \underline{\mathbf{A}}[\varnothing]$ is the Burnside Tambara functor, we simply say that R is *levelwise finitely generated.*

Remark 4.2. Since $\underline{\mathbf{A}}$ is levelwise a finite $\mathbf{Z}$-module, R is levelwise finitely generated iff each $R(G/H)$ is a finite type $\mathbf{Z}$-algebra.

Definition 4.3 ([CW25, Definition 3.30])**.** A Green or Tambara functor R is *relatively finite-dimensional* if for all $H \leq K \leq G$, the restriction map $\mathrm{Res}^K_H : R(G/K) \to R(G/H)$ is a finite ring map. Equivalently, $\mathrm{Res}^G_H : R(G/G) \to R(G/H)$ is a finite ring map for all $H \leq G$.

Relative finite-dimensionality is a well-behaved finiteness condition on Green and Tambara functors which imposes many niceties on their category of modules (see [CW25, §3] for more details); we will use it extensively in §6. Next, we make some remarks on finite generation for general finite groups G.

Theorem 4.4 ([Bru05, Theorem A]). *Let G be a finite group, X a finite G-set. Then $\underline{\mathbf{A}}[X](G/e)$ is a free polynomial ring over $\mathbf{Z}$ on $|X|$ generators.*

In particular, $\underline{\mathbf{A}}[X](G/e)$ is a finite type $\mathbf{Z}$-algebra. We also cite a fact about Tambara functors we will use repeatedly.

Lemma 4.5 ([SSW25], Lemma 3.3). *Let G be a finite group and T a G-Tambara functor. For all $H \leq K$, restriction $\mathrm{Res}_H^K : T(G/K) \to T(G/H)$ is an integral ring map.*

Definition 4.6. Let G be a finite group, $H, L \leq G$. Let $\underline{\mathbf{A}}^0[G/H](G/L)$ denote the subring of $\underline{\mathbf{A}}[G/H](G/L)$ generated by those irreducible polynomials

$$G/H \leftarrow A \to B \to G/L$$

such that $B \to G/L$ is an isomorphism.

This is indeed a subring, since if $B \to G/L$ and $B' \to G/L$ are isomorphisms, then so is $B \times_{G/L} B' \to G/L$. Note that for Dedekind G, this coincides with our previous definition of $\underline{\mathbf{A}}^0$.

Lemma 4.7. *$\underline{\mathbf{A}}^0[G/H](G/L)$ is a finitely generated $\mathbf{Z}$-algebra.*

Proof. An irreducible polynomial in $\underline{\mathbf{A}}^0[G/H](G/L)$ is of the form

$$G/H \leftarrow \coprod_i G/H_i \to G/g^{-1}Lg \to G/L.$$

By replacing the G/H_i with isomorphic G-sets, we may assume that all $H_i \leq g^{-1}Lg$, so that it has a representative of the form

$$G/H \xleftarrow{\sqcup_i f_i} \coprod_i G/H_i \to G/g^{-1}Lg \xrightarrow{g^{-1}} G/L,$$

where $f_i : G/H_i \to G/H$ is the unique G-map sending $H_i \in G/H_i$ to $f_iH \in G/H$ (here $H_i \leq f_iHf_i^{-1}$) and each $G/H_i \to G/g^{-1}Lg$ is the natural projection. Multiplication of two such polynomials gives

$$\begin{aligned}
&[G/H \xleftarrow{\sqcup_i f_i} \coprod_i G/H_i \to G/g^{-1}Lg \xrightarrow{g^{-1}} G/L] \\
&\quad \cdot [G/H \xleftarrow{\sqcup_j g_j} \coprod_j G/H'_j \to G/g^{-1}Lg \xrightarrow{g^{-1}} G/L] \\
&= [G/H \xleftarrow{\sqcup_i f_i \sqcup_j g_j} \coprod_{i,j} G/H_i \sqcup G/H'_j \to G/g^{-1}Lg \xrightarrow{g^{-1}} G/L],
\end{aligned}$$

where we are using the fact that $G/g^{-1}Lg \times_{G/L} G/g^{-1}Lg = G/g^{-1}Lg$ and $G/H_i \times_{G/L} G/g^{-1}Lg = G/H_i \times_{G/g^{-1}Lg} G/g^{-1}Lg = G/H_i$. Hence, we can take as a set of generators all irreducible polynomials of the form

$$G/H \leftarrow G/H' \to G/g^{-1}Lg \to G/L$$

for varying $g \in G$ and $H' \leq g^{-1}Lg$. □

Lemma 4.8. *$\underline{\mathbf{A}}[G/H]$ is relatively finite-dimensional.*

Proof. Write $T = \underline{\mathbf{A}}[G/H]$ and $T^0(G/L) = \underline{\mathbf{A}}^0[G/H](G/L)$ for brevity. We prove that Res^G_L is a finite ring map by induction on the size of L. Since $T(G/e)$ is a finite type $\mathbf{Z}$-algebra by Theorem 4.4, Res^G_e is a finite type ring map, hence a finite ring map since all restrictions maps are integral (Lemma 4.5).

Now suppose $L \leq G$. It is clear from definition that every irreducible polynomial in $T(G/L)$ is either in $T^0(G/L)$ or is of the form $\mathrm{Tr}^L_K(x)$ for some $x \in T(G/K)$, $K \lneq L$. By Frobenius reciprocity, $\mathrm{Res}^G_L(y)$ where $y \in T(G/G)$ acts by multiplication on $\mathrm{Tr}^L_K(x)$ via

$$\mathrm{Res}^G_L(y)\,\mathrm{Tr}^L_K(x) = \mathrm{Tr}^L_K(\mathrm{Res}^G_K(y)x),$$

so it follows that as a $T(G/G)$-module, $T(G/L)$ is generated by $T^0(G/L)$ and modules isomorphic to quotients of $T(G/K)$ for $K \lneq L$. By assumption, the $T(G/K)$ are all finite $T(G/G)$-modules, so it suffices to show that the $T(G/G)$-module generated by $T^0(G/L)$ in $T(G/L)$ is contained in some finite $T(G/G)$-module as well. But $T^0(G/L)$ is a finite type $\mathbf{Z}$-algebra, so it generates a finite type subalgebra over $T(G/G)$ in $T(G/L)$. Since this subalgebra is integral over $T(G/G)$ by Lemma 4.5 again, it must be finite over $T(G/G)$, completing the proof. □

Corollary 4.9. *$\underline{\mathbf{A}}[G/H]$ is levelwise finitely generated over $\mathbf{Z}$ iff $\underline{\mathbf{A}}[G/H](G/G)$ is a finite type $\mathbf{Z}$-algebra.*

Proof. The only if direction is obvious. The converse follows from the fact that any algebra which is module-finite over a finite type $\mathbf{Z}$-algebra is also a finite type $\mathbf{Z}$-algebra. □

4.2. **Dedekind G, Transitive Case.** We introduce some auxiliary terms on levelwise finite generation.

Definition 4.10. Let G be a finite group. We say that G is *Tambara finite* if $\underline{\mathbf{A}}[X]$ is levelwise finitely generated for all finite G-sets X, and *transitively Tambara finite* or just *transitively finite* if $\underline{\mathbf{A}}[X]$ is levelwise finitely generated for all transitive finite G-sets X.

We will show in §6 that transitively Tambara finite groups are Tambara finite. For the rest of this section, assume that G is a finite Dedekind group; our goal now is to show (Theorem 4.14) that G is transitively finite.

Lemma 4.11. *Let $H \leq H'$. The morphism of Tambara functors $\underline{\mathbf{A}}[G/H] \to \underline{\mathbf{A}}[G/H']$ induced by restriction along the projection morphism $G/H \to G/H'$ contains in its levelwise image the set of irreducible polynomials $((H_i, f_i)_{i\in I})_K$ where $H_i \leq H$.*

Proof. This follows because the composition of the polynomials

$$G/H' \xleftarrow{\mathrm{pr}} G/H \to G/H \to G/H, \quad G/H \xleftarrow{\sqcup f_i} \coprod_i G/H_i \to G/K \to G/L$$

is the polynomial

$$G/H' \xleftarrow{\mathrm{pr}\circ\sqcup f_i} \coprod_i G/H_i \to G/K \to G/L.$$
□

The punchline is that for G Dedekind, the H_i must all collectively factor through $H \cap K$—thus, for the finite generation of $\underline{\mathbf{A}}[G/H](G/L)$, we only need to consider (by induction on the size of H) those irreducible polynomials with $H \leq K \leq L$ (so in particular, $H \leq L$).

Lemma 4.12. *When $H \leq K \leq L$, the isomorphism class of the polynomial $((H_i, f_i)_{i\in I})_K$ in $\underline{\mathbf{A}}[G/H](G/L)$ is determined by the reduction of the f_i mod K/H.*

Proof. Apply Lemma 3.4 with $s = 1$. □

Thus, we can view our polynomials $((H_i, f_i)_{i\in I})_K$ as having $f_i \in G/K$.

Lemma 4.13. *Let $H \leq K \leq K' \leq L$. Then*

$$((H_i, f_i)_{i\in I})_K \cdot ((H'_j, g_j)_{j\in J})_{K'} = \sum_{s\in L/K'} \left((H_i, f_i)_{i\in I} \sqcup \coprod_{\substack{j\in J \\ r\in L/K \\ r\mapsto s}} (H'_j, rg_j) \right)_K$$

in $\underline{\mathbf{A}}[G/H](G/L)$, where here $((H_1, f_1) \sqcup (H_2, f_2))$ just means the pair $((H_1, f_1), (H_2, f_2))$.

Proof. We will unravel the multiplication rule

$$[X \leftarrow A \to B \to Y] \cdot [X \leftarrow A' \to B' \to Y] = [X \leftarrow (A \times_Y B') \sqcup (A' \times_Y B) \to B \times_Y B' \to Y]$$

in this case. Here

$$B \times_Y B' = G/K \times_{G/L} G/K' \cong \coprod_{s\in L/K'} G/K,$$

where the identification is given by noting that $G/K \times_{G/L} G/K'$ is the union of orbits of the form

$$G/K \times_{G/L} G/K' = \coprod_{s\in L/KK'} G \cdot (K, sK') = \coprod_{s\in L/K'} \frac{G}{K} \cdot (K, sK').$$

We also have

$$\begin{aligned} A \times_Y B' = \left(\coprod_i G/H_i\right) \times_{G/L} G/K' &= \coprod_i G/H_i \times_{G/L} G/K' \\ &= \coprod_i \coprod_{r\in L/K'} \frac{G}{H_i} \cdot (H_i, rK'), \end{aligned}$$

$$\begin{aligned} A' \times_Y B = \left(\coprod_j G/H'_j\right) \times_{G/L} G/K &= \coprod_j G/H'_j \times_{G/L} \times G/K \\ &= \coprod_j \coprod_{r\in L/K} \frac{G}{H'_j} \cdot (rH'_j, K). \end{aligned}$$

Which components of $A \times_Y B'$ and $A' \times_Y B$ lie over the component G/K corresponding to a given $s \in L/K'$? These are the components of $A \times_Y B'$ with

$$(H_i, rK') \mapsto (K, sK')$$

and the components of $A' \times_Y B$ with

$$(rH'_j, K) \mapsto (K, sK').$$

In the first case we need $r \in L/K'$ to map to $s \in L/K'$ and in the second case we need $r \in L/K$ to map to $s \in L/K'$.

Under this identification, we then see that the product is a sum of irreducible polynomials, indexed over $s \in L/K'$, of the form

$$G/H \leftarrow \left(\coprod_{i\in I} G/H_i\right) \sqcup \left(\coprod_{\substack{i\in I\\ r\in L/K\\ r\mapsto s}} G/H_j'\right) \to G/K \to G/L.$$

Now we analyze each component $G/H_i \to G/H$ and $G/H_j' \to G/H$. For a given $i \in I$ and $s \in L/K'$, the element in the component G/H_i of $A \times_Y B'$ lying above the s piece G/K corresponding to $H_i \in G/H_i$ is (H_i, rK'). The map to G/H is then projection onto the first coordinate followed by f_i, so we see that $G/H_i \ni H_i \mapsto f_iH \in G/H$. Similarly, for a given $j \in J$ and $r \in L/K$ mapping to $s \in L/K'$, the element in $A' \times_Y B$ corresponding to $H_j' \in G/H_j'$ is (rH_j', K). The map to G/H is projection onto the first coordinate followed by g_i, so $G/H_j' \ni H_j' \mapsto rg_iH \in G/H$. This completes the proof. □

Theorem 4.14. *Let G be a finite Dedekind group. Then G is transitively finite, i.e. $\underline{\mathbf{A}}[G/H]$ is levelwise finitely generated for any $H \leq G$.*

Proof. By Corollary 4.9, it suffices to show that $\underline{\mathbf{A}}[G/H](G/G)$ is finitely generated. Now fix an N large enough so that for any polynomial $((H_i, f_i)_{i\in I})_K$, where $H \leq K$ and $\deg((H_i, f_i)_{i\in I})_K = (n, K)$ with $n > N$, there exists some $H_0 \in \{H_i\}_{i\in I}$ which appears more than $1+2+\cdots+|G/K|$ times in the tuple above. We take as generators the set of all polynomials with degree (n, K) such that $n \leq N$, as well as the all the polynomials of the form $((H_i, f_i)_{i\in I})_{K'}$ where $K \leq K'$ and $|I| \leq |G/K|$. (Because these polynomials have degree (n, K') with n bounded, we can forget about this extra class of generators when N is sufficiently large.) We show that any polynomial with degree (n, K) with $n > N$ can be written as a combination of the generators above and polynomials of degree (n', K) with $n' < n$, which suffices by induction on n (and our remarks above on the reduction to the case $H \leq K$).

Let $b = ((H_i, f_i)_{i\in I})_K$ be any such polynomial and let H_0 be as above. Writing (H_0, f_1), $\ldots$, (H_0, f_m) for all the pairs in our collection associated to H_0, we have $m > |G/K|$ by assumption. Fix this H_0 for the rest of the proof.

Now let's fix some notation. Given any polynomial b and $f \in G/K$, let $m_f(b)$ denote the number of times that (H_0, f) appears in b. Let $\vec{v}(b) \in \mathbf{N}^{G/K}$ (here $\mathbf{N}^{G/K}$ is the set of $\mathbf{N}$-valued vectors indexed in elements of G/K) denote the vector whose f coordinate has value $m_f(b)$. Similarly, given any vector $\vec{v} \in \mathbf{N}^{G/K}$, let $m_f(\vec{v})$ denote the value of $\vec{v}$ at the f coordinate. Furthermore, there is an G/K action on $\mathbf{N}^{G/K}$ given by $m_f(g \cdot \vec{v}) = m_{g^{-1}f}(\vec{v})$.

If $\vec{w} \in \mathbf{N}^{G/K}$ with $\vec{w} \leq \vec{v}(b)$ ($m_f(\vec{w}) \leq m_f(b)$ for all $f \in G/K$), write $b/\vec{w}$ for the polynomial obtained by deleting $m_f(\vec{w})$ copies of (H_0, f) from b for each $f \in G/K$. If $\vec{w} \in \mathbf{N}^{G/K}$, we let $b_{K'}(\vec{w})$ denote the irreducible polynomial

$$b_{K'}(\vec{w}) = \left(\coprod_{f\in G/K} \coprod_{m_f(\vec{w})} (H_0, f)\right)_{K'}$$

i.e. (H_0, f) appears in $b_{K'}(\vec{w})$ exactly $m_f(\vec{w})$ times.

Our proof strategy is a generalization of the corresponding proof for $G = C_p$ in [4DS]. We adopt analogous notation: for any $j \geq 0$ let

$$S_j(\vec{v}) = \{f \in G/K : m_f(\vec{v}) = j\}$$

and define $S_j(b) = S_j(\vec{v}(b))$. Similarly define

$$T_j(\vec{v}) = \bigcup_{i \geq j} S_i(\vec{v}), \quad T_j(b) = T_j(\vec{v}(b)).$$

Given any subset $A \subseteq G/K$, let $\vec{e}(A) \in \mathbf{N}^{G/K}$ be the vector with

$$m_f(\vec{e}(A)) = \begin{cases} 1 & f \in A, \\ 0 & \text{otherwise.} \end{cases}$$

We use a nested induction argument. First, we induct on $|S_0(b)|$. The base case $|S_0(b)| = 0$ follows from the fact that

$$b = (b/\vec{e}(G/K)) \cdot b_G(\vec{e}(\{1\}))$$

by the formula given in Lemma 4.13. We then have the inductive hypothesis

(0) Suppose there is an $m_0 > 0$ such that b is a combination of lower-degree terms and generators whenever $|S_0(b)| < m_0$.

We prove the inductive step for $|S_0(b)| = m_0$ by induction on $|S_1(b)|$. At the base case $|S_1(b)| = 0$, we have $G/K = T_1(b) \sqcup S_0(b) = T_2(b) \sqcup S_0(b)$. Let K'/K be the stabilizer of $T_2(b)$ under the action of G/K, so that $T_2(b)$ is the union of cosets

$$T_2(b) = (g_1K'/K) \sqcup \cdots \sqcup (g_tK'/K).$$

Note that by assumption $S_0 \neq \varnothing$ so $K'/K \neq G/K$. Then we have

$$\begin{aligned} b &= (b/\vec{e}(T_2(b))) \cdot b_{K'}(\vec{e}(\{g_1, \ldots, g_t\})) \\ &\quad - \sum_{0 \neq s \in G/K'} b\left(\vec{v}(b) - \vec{e}(T_2(b)) + \sum_{\substack{r \in G/K \\ r \mapsto s}} r \cdot \vec{e}(\{g_1, \ldots, g_t\})\right) \\ &= (b/\vec{e}(T_2(b))) \cdot b_{K'}(\vec{e}(\{g_1, \ldots, g_t\})) \\ &\quad - \sum_{0 \neq s \in G/K'} b\left(\vec{v}(b) - \vec{e}(T_2(b)) + r \cdot \vec{e}(T_2(b))\right) \end{aligned}$$

again from the formula in Lemma 4.13, and where on the last line we simply choose for each $s \in G/K'$ an element $r \in G/K$ mapping to s. Since no nonzero element of $s \in G/K'$ stabilizes $T_2(b)$ and $T_2(b) \cap S_0 = \varnothing$, any $r \in G/K$ mapping to nonzero $s \in G/K'$ has $(r \cdot T_2(b)) \cap S_0(b) \neq \varnothing$, so we must have

$$|S_0\left(\vec{v}(b) - \vec{e}(T_2(b)) + r \cdot \vec{e}(T_2(b))\right)| < |S_0(b)| = m_0$$

for each such r. Thus by the inductive hypothesis (0) we have established the base case for $|S_1(b)|$. We now have the inductive hypothesis

(1) Suppose there is an $m_1 > 0$ such that b is a combination of lower degree terms and generators whenever $|S_1(b)| < m_1$.

We prove the inductive step for $|S_1(b)| = m_1$ by induction on $|S_2(b)|$. At the base case $|S_2(b)| = 0$, we have $G/K = T_2(b) \sqcup S_1(b) \sqcup S_0(b) = T_3(b) \sqcup S_1(b) \sqcup S_0(b)$. Again let K'/K be the stabilizer of $T_3(b)$ under the G/K action, so that $T_3(b)$ is the union

$$T_3(b) = (g_1K'/K) \sqcup \cdots \sqcup (g_tK'/K).$$

By assumption $S_1, S_0 \neq \varnothing$, so $K'/K \neq G/K$ and

$$\begin{aligned} b = (b/\vec{e}(T_3(b))) \cdot b_{K'}(\vec{e}(\{g_1, \dots, g_t\})) \\ - \sum_{0 \neq s \in G/K'} b\left(\vec{v}(b) - \vec{e}(T_3(b)) + r \cdot \vec{e}(T_3(b))\right) \end{aligned}$$

Since no nonzero element of $s \in G/K'$ stabilizes $T_3(b)$, any $r \in G/K$ mapping to a nonzero element in G/K' has $(r \cdot T_3(b)) \cap (S_1 \sqcup S_0) \neq \varnothing$, so the vector

$$|\vec{v}(b) - \vec{e}(T_3(b)) + r \cdot \vec{e}(T_3(b))|$$

either has $|S_1| < m_1$ or $|S_0| < m_0$. We continue with the inductive hypothesis

(2) Suppose there is an $m_2 > 0$ such that b is a combination of lower degree terms and generators whenever $|S_2(b)| < m_2$.

Continue in this way; the base case for the inductive hypothesis (i) is implied by the inductive hypotheses for (j) with $j < i$. Proving the inductive step for any (i) establishes the inductive step for (0) and thus completes the proof.

Eventually, we arrive at the inductive hypothesis for (M), where $M = (\sum_{f \in G/K} m_f(b)) + 1$. The base case is proved by the inductive hypotheses for (j), $j < M$, so we prove the inductive step: if $m_M > 0$ and $|S_M(b)| = m_M$, then b is a combination of lower degree terms and generators. But $m_M > 0$ and $|S_M(b)| = m_M$ cannot both be true at the same time, so the inductive step is vacuous; hence only the base case matters and the theorem is proved. $\square$

We remark that the proof of this theorem actually gives us a little bit more:

Lemma 4.15. *Let G be a finite group, and let $H \leq K \leq G$ be normal subgroups of G. There exist a finite set S of polynomials in $R = \underline{\mathbf{A}}[G/H](G/G)$ such that every irreducible polynomial of the form*

$$G/H \leftarrow \coprod G/H_i \to G/K \to G/G$$

has an algebraic expression by elements of S.

Most of the notation from the proof carries over; in particular, we can again represent all irreducible bispans of the above form via

$$G/H \xleftarrow{\sqcup f_i} \coprod_i G/H_i \to G/K \to G/G,$$

where the f_i assmeble into some vector $\vec{v} \in \mathbf{N}^{G/K}$. The main point in the proof where some care needs to be taken is in the decomposition of $T_n(b)$ into *right cosets*

$$T_n(b) = (K'/K)g_1 \sqcup \cdots \sqcup (K'/K)g_t,$$

since the stabilizer K'/K of $T_n(b)$ under the G/K-action need not be a normal subgroup of G/K, and the observation that the multiplication formula in Lemma 4.13

$$b_K(\vec{v}) \cdot b_{K'}(\vec{w}) = \sum_{s \in G/K'} b\left(\vec{v} + \sum_{\substack{r \in G/K \\ r \mapsto s}} r \cdot \vec{w}\right)$$

has the effect of adding all elements of the form $s(K'/K)g_j$ to the tuple given by $\vec{v}$, where here the elements $g_j \in G/K$ are specified by the vector $\vec{w} \in \mathbf{N}^{G/K}$. We will use this more general lemma in §5, when we prove finite generation in special cases.

5. Finite Generation in Special Cases

5.1. **A Strong Constraint on G.** Consider the following condition on a finite group G:

($*$) Every proper, nontrivial subgroup of G is maximal, and every subgroup is either normal or satisfies $N_G(H) = H$.

In particular, any two proper subgroups have trivial intersection. In this section, we prove directly that G is transitively Tambara finite when G satisfies ($*$) or when $G \cong D_8$, the dihedral group of order 8. The following lemma gives a class of groups which satisfy ($*$) (in particular, these include the dihedral groups D_{2p} of order $2p$ for $p > 2$ a prime).

Proposition 5.1. *Let $p > q$ be primes and suppose $\phi : C_q \to \mathrm{Aut}(C_p)$ is a faithful C_q-action on C_p. Then $G = C_p \rtimes_\phi C_q$ satisfies ($*$).*

Proof. Write $\phi_y = \phi(y)$ for any $y \in C_q$. It is clear that every proper nontrivial subgroup is maximal for order reasons. For $e \neq y \in C_q$ and $x \in C_p$, we have

$$x^k y x^{-k} = x^k \phi_y(x^{-k}) y = x^{k(1-t)} y,$$

where $\phi_y(x) = x^t$, $t \not\equiv 1 \bmod p$. In particular, taking $k = n/(1-t) \bmod p$, we see that the elements $x^n y$ and y are conjugate. From this, we find that the only proper nontrivial subgroups of G are C_p (which is normal) and those generated by $\langle x^n y\rangle$, $e \neq y \in C_q$, which are all non-normal of order q. Furthermore, the computation above shows that $N_G(\langle y\rangle) = \langle y\rangle$ and thus $N_G(\langle x^n y\rangle) = \langle x^n y\rangle$, since the two subgroups are conjugate. Hence G satisfies ($*$). □

First, we remark that the finite generation of $\underline{\mathbf{A}}[X]$ for a top-level generator X is known in general:

Theorem 5.2 ([SSW25, Proposition 3.12])**.** *Let G be a finite group. $\underline{\mathbf{A}}[G/G](G/H)$ is a finitely generated ring for all $H \leq G$.*

Proposition 5.3. *Let G be a finite group satisfying ($*$). Then G is transitively Tambara finite.*

Proof. We wish to show that $\underline{\mathbf{A}}[G/H]$ is levelwise finitely generated for all $H \leq G$. Recall by Corollary 4.9 that it suffices to show that $R = \underline{\mathbf{A}}[G/H](G/G)$ is a finite type $\mathbf{Z}$-algebra. In general, we will say that an irreducible polynomial of the form

$$G/H \leftarrow A \to G/K \to G/G = *$$

is "of type K." We will say that type K polynomials are *finitely generated* if there exists a finite set S of polynomials in R such that every irreducible polynomial of type K can be expressed as some combination of polynomials in S.

Suppose first that $H = e$. The type K polynomials for K normal are all finitely generated by Lemma 4.15. If K is not normal, then we compute

$$\begin{aligned}
&[G/e \xleftarrow{\sqcup f_i} \coprod G/e \to G/K \to *] \cdot [G/e \xleftarrow{\sqcup g_j} \coprod G/e \to G/K \to *] \\
&= [G/e \xleftarrow{\sqcup f_i \sqcup g_j} \coprod G/e \to G/K \to *] + \text{type } e,
\end{aligned}$$

where here we use the fact that

$$G/K \times G/K \cong G/K \sqcup \coprod_{e \neq g \in K\backslash G/K} G/(K \cap gKg^{-1}) \cong G/K \sqcup \coprod G/e,$$

since $K \cap gKg^{-1} = e$ for any $g \notin K$. Thus, we need only take as additional type K generators (when K is not normal) those polynomials of the form

$$[G/e \xleftarrow{f} G/e \to G/K \to *].$$

Now suppose $H \neq e$ is a proper subgroup. If H is normal, then a type K polynomial

$$[G/H \xleftarrow{\sqcup f_i} \coprod G/H_i \to G/K \to *]$$

must satisfy $H_i \leq K \cap H$ for all i. If $H \not\leq K$, then this implies all the $H_i = e$ and hence all such polynomials lie in the image of the map $\underline{\mathbf{A}}[G/e] \to \underline{\mathbf{A}}[G/H]$ induced by restriction, from which we are done by the $H = e$ case. If $H \leq K$, then K is normal and we are done by Lemma 4.15.

If H is not normal, then we again have two cases. A type K polynomial

$$[G/H \xleftarrow{\sqcup f_i} \coprod G/H_i \to G/K \to *]$$

with K normal must have $H_i \leq K$ and $H_i \leq gHg^{-1}$ for some $g \in G$. This can only occur if $H_i = e$ or $K = G$; in the first case, these lie in the image of $\underline{\mathbf{A}}[G/e] \to \underline{\mathbf{A}}[G/H]$ and we are done. In the second case, these are elements of $\underline{\mathbf{A}}^0[G/H](G/G)$, which we know is finitely generated by Lemma 4.7.

Thus, suppose K is non-normal. If K is not conjugate to H, then for all i, we have $H_i \leq K \cap gHg^{-1} = e$ for some $g \in G$, in which case these lie in the image of $\underline{\mathbf{A}}[G/e] \to \underline{\mathbf{A}}[G/H]$ and we are done. Thus, we can assume K is conjugate to H, from which we may assume $K = H$, since type K polynomials are isomorphic to type H polynomials. In this case we must have all the $H_i = e$, or $H_i = H$ with the corresponding map $G/H_i \to G/H$ equal to the identity. Thus all type H polynomials look like

$$G/H \xleftarrow{\sqcup f_i \sqcup \mathrm{pr}_j} \coprod G/e \sqcup \coprod G/H \to G/H \to *,$$

in which case we have an analogous computation

$$\begin{aligned}
&[G/H \xleftarrow{\sqcup f_i \sqcup \mathrm{pr}_j} \coprod_i G/e \sqcup \coprod_j G/H \to G/H \to *] \\
&\cdot G/H \xleftarrow{\sqcup g_k \sqcup \mathrm{pr}_l} \coprod_k G/e \sqcup \coprod_l G/H \to G/H \to * \\
&= [G/H \xleftarrow{\sqcup f_i \sqcup g_k \sqcup \mathrm{pr}_j \sqcup \mathrm{pr}_l} \coprod_{i,k} G/e \sqcup \coprod_{j,l} G/H \to G/H \to *] + \text{type } e,
\end{aligned}$$

where we again have used the fact that $G/H \times G/H \cong G/H \sqcup \coprod G/e$. From this we see that we can take as additional type H generators those of the form

$$[G/H \xleftarrow{f} G/e \to G/H \to *], \quad [G/H \leftarrow G/H \to G/H \to *]$$

Lastly, we note that if $H = G$, then we are done by Theorem 5.2. □

5.2. $G \cong D_8$**, Transitive Case.** Now we give a proof of transitive finiteness for D_8. Neither of the conditions imposed in $(*)$ are true for D_8.

We use the presentation

$$D_8 \cong \langle a, x \mid a^4 = x^2 = e, ax = xa^{-1} \rangle.$$

There are three subgroups of D_8 of order 4, $\langle a^2, x\rangle$, $\langle a^2, ax\rangle$, and $\langle a\rangle$. All three are normal; the third is characteristic. There are five subgroups of order 2: $\langle x\rangle$, $\langle a^2x\rangle$ are conjugate, $\langle ax\rangle$, $\langle a^3x\rangle$ are conjugate, and $\langle a^2\rangle$ is the center of the group. Note in particular that $\langle x\rangle$ has normalizer $N_{D_8}(\langle x\rangle) = \langle a^2, x\rangle \lneq D_8$.

Proposition 5.4. *$\underline{\mathbf{A}}[D_8/e]$ is levelwise finitely generated.*

Proof. By Lemma 4.15, the only types of irreducible polynomials which may cause issues are

(a)
$$D_8/e \xleftarrow{\sqcup f_i} D_8/e \to \coprod D_8/\langle x\rangle \to *$$

(b)
$$D_8/e \xleftarrow{\sqcup f_i} \coprod D_8/e \to D_8/\langle ax\rangle \to *$$

Call all other types of irreducible polynomials "type (c)." We will only do the computation for type (a) since the argument for type (b) polynomials is identical.

Note that in the irreducible polynomials of type (a), the f_i are well-defined up to a coset representative $\langle x\rangle\backslash D_8$. Thus, we can assume that the f_i are of the form a^k for some $0 \le k \le 3$. Represent the f_i via some vector $\vec{v} \in \mathbf{N}^4$, where (n_0, n_1, n_2, n_3) means a^k appears n_k times. Note that $C_4 = \langle a\rangle$ acts on these vectors via $a \cdot (n_0, n_1, n_2, n_3) = (n_3, n_0, n_1, n_2)$. We compute

$$\begin{aligned}
&[D_8/e \xleftarrow{\sqcup f_i} \coprod D_8/e \to D_8/\langle x\rangle \to *] \cdot [D_8/e \xleftarrow{g} D_8/e \to D_8/\langle x\rangle \to *] \\
&= [D_8/e \xleftarrow{\sqcup f_i \sqcup g} \coprod D_8/e \to D_8/\langle x\rangle \to *] \\
&\quad + [D_8/e \xleftarrow{\sqcup f_i \sqcup a^2 g} \coprod D_8/e \to D_8/\langle x\rangle \to *] + \text{type (c)},
\end{aligned}$$

$$\begin{aligned}
&[D_8/e \xleftarrow{\sqcup f_i} \coprod D_8/e \to D_8/\langle x\rangle \to *] \cdot [D_8/e \xleftarrow{g} D_8/e \to D_8/\langle a^2, x\rangle \to *] \\
&= [D_8/e \xleftarrow{\sqcup f_i \sqcup g \sqcup a^2 g} \coprod D_8/e \to D_8/\langle x\rangle \to *] \\
&\quad + [D_8/e \xleftarrow{\sqcup f_i \sqcup ag \sqcup a^3 g} \coprod D_8/e \to D_8/\langle x\rangle \to *] + \text{type (c)},
\end{aligned}$$

$$\begin{aligned}
&[D_8/e \xleftarrow{\sqcup f_i} \coprod D_8/e \to D_8/\langle x\rangle \to *] \cdot [D_8/e \leftarrow D_8/e \to * \to *] \\
&= [D_8/e \xleftarrow{\sqcup f_i \sqcup e \sqcup a \sqcup a^2 \sqcup a^3} \coprod D_8/e \to D_8/\langle x\rangle \to *]
\end{aligned}$$

From this and a simple induction argument (reminiscent of Theorem 4.14), we need only take as additional generators those corresponding to vectors $\vec{v} \in \mathbf{N}^4$ with $n_0 + n_1 + n_2 + n_3 \le 4$. $\square$

Proposition 5.5. *$\underline{\mathbf{A}}[D_8/\langle x\rangle]$ is levelwise finitely generated.*

Proof. Again by Lemma 4.15, the only types of irreducible polynomials which may cause issues are

(a)
$$D_8/\langle x\rangle \xleftarrow{\sqcup f_i \sqcup g_j} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \to D_8/\langle x\rangle \to *$$

(b)
$$D_8/\langle x\rangle \xleftarrow{\sqcup f_i \sqcup g_j \sqcup h_k} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \sqcup \coprod D_8/\langle a^2 x\rangle \to D_8/\langle a^2, x\rangle \to *$$

Call all other "type (c)." In the type (a) polynomials, we can view the f_i as taking values in $D_8/\langle x\rangle = \{e\langle x\rangle, a\langle x\rangle, a^2\langle x\rangle, a^3\langle x\rangle\}$, while the g_j take values in $N_{D_8}(\langle x\rangle)/\langle x\rangle = \langle a^2, x\rangle/\langle x\rangle$, so we can take them to be e or a^2. We compute

$$\begin{aligned}
&[D_8/\langle x\rangle \xleftarrow{\sqcup f_i \sqcup g_j} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \to D_8/\langle x\rangle \to *] \\
&\quad \cdot [D_8/\langle x\rangle \leftarrow D_8/e \to * \to *] \\
&= [D_8/\langle x\rangle \xleftarrow{(\sqcup f_i \sqcup e \sqcup a \sqcup a^2 \sqcup a^3)\sqcup g_j} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \to D_8/\langle x\rangle \to *],
\end{aligned}$$

$$\begin{aligned}
&[D_8/\langle x\rangle \xleftarrow{\sqcup f_i \sqcup g_j} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \to D_8/\langle x\rangle \to *] \\
&\quad \cdot [D_8/\langle x\rangle \xleftarrow{f} D_8/e \to D_8/\langle x\rangle \to *] \\
&= [D_8/\langle x\rangle \xleftarrow{(\sqcup f_i \sqcup f)\sqcup g_j} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \to D_8/\langle x\rangle \to *] \\
&\quad + [D_8/\langle x\rangle \xleftarrow{(\sqcup f_i \sqcup a^2 f)\sqcup g_j} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \to D_8/\langle x\rangle \to *] \\
&\quad + \text{type (c)}
\end{aligned}$$

$$\begin{aligned}
&[D_8/\langle x\rangle \xleftarrow{\sqcup f_i \sqcup g_j} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \to D_8/\langle x\rangle \to *] \\
&\quad \cdot [D_8/\langle x\rangle \xleftarrow{f} D_8/e \to D_8/\langle a^2, x\rangle \to *] \\
&= [D_8/\langle x\rangle \xleftarrow{(\sqcup f_i \sqcup f \sqcup a^2 f)\sqcup g_j} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \to D_8/\langle x\rangle \to *] \\
&\quad + [D_8/\langle x\rangle \xleftarrow{(\sqcup f_i \sqcup af \sqcup a^3 f)\sqcup g_j} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \to D_8/\langle x\rangle \to *]
\end{aligned}$$

In addition, we have

$$\begin{aligned}
&[D_8/\langle x\rangle \xleftarrow{\sqcup f_i \sqcup g_j} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \to D_8/\langle x\rangle \to *] \\
&\quad \cdot [D_8/\langle x\rangle \leftarrow D_8/\langle x\rangle \to D_8/\langle x\rangle \to *] \\
&= [D_8/\langle x\rangle \xleftarrow{\sqcup f_i (\sqcup g_j \sqcup e)} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \to D_8/\langle x\rangle \to *] \\
&\quad + [D_8/\langle x\rangle \xleftarrow{\sqcup f_i (\sqcup g_j \sqcup a^2)} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \to D_8/\langle x\rangle \to *] \\
&\quad + \text{type (c)}.
\end{aligned}$$

$$\begin{aligned}
&[D_8/\langle x\rangle \xleftarrow{\sqcup f_i \sqcup g_j} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \to D_8/\langle x\rangle \to *] \\
&\quad \cdot [D_8/\langle x\rangle \xleftarrow{g} D_8/\langle x\rangle \to D_8/\langle a^2, x\rangle \to *] \\
&= [D_8/\langle x\rangle \xleftarrow{\sqcup f_i (\sqcup g_j \sqcup e \sqcup a^2)} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \to D_8/\langle x\rangle \to *] \\
&\quad + [D_8/\langle x\rangle \xleftarrow{(\sqcup f_i \sqcup ag)\sqcup g_j} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \to D_8/\langle x\rangle \to *].
\end{aligned}$$

Using the last two identities, we can arrange for any polynomial of type (a) to be a combination of type (a) polynomials where the number of g_j is ≤ 2 and type (c) polynomials. Then using the first three identities and an inductive argument as in Theorem 4.14, we can arrange for these polynomials to be combinations of type (c) polynomials and type (a) polynomials where the number of f_i is ≤ 4 and where the number of g_j is ≤ 2. Hence the type (a) polynomials are finitely generated.

For the type (b) polynomials, we note that in the polynomial

$$D_8/\langle x\rangle \xleftarrow{\sqcup f_i \sqcup g_j \sqcup h_k} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \sqcup \coprod D_8/\langle a^2x\rangle \to D_8/\langle a^2,x\rangle \to *$$

the f_i, g_j, h_k are all well-defined as cosets $D_8/\langle a^2,x\rangle = \{e\langle a^2,x\rangle, a\langle a^2,x\rangle\}$. Observe that

$$\begin{aligned}
&[D_8/\langle x\rangle \xleftarrow{\sqcup f_i \sqcup g_j \sqcup h_k} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \sqcup \coprod D_8/\langle a^2x\rangle \to D_8/\langle a^2,x\rangle \to *] \\
&\quad \cdot [D_8/\langle x\rangle \leftarrow D_8/e \to * \to *] \\
&= [D_8/\langle x\rangle \xleftarrow{(\sqcup f_i \sqcup e \sqcup a) \sqcup g_j \sqcup h_k} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \sqcup \coprod D_8/\langle a^2x\rangle \to D_8/\langle a^2,x\rangle \to *],
\end{aligned}$$

and similarly with multiplication by $[D_8/\langle x\rangle \leftarrow D_8/\langle x\rangle \to * \to *]$ and $[D_8/\langle x\rangle \leftarrow D_8/\langle a^2x\rangle \to * \to *]$, with contributions of e and a to the g_j and h_k instead in those cases. In addition,

$$\begin{aligned}
&[D_8/\langle x\rangle \xleftarrow{\sqcup f_i \sqcup g_j \sqcup h_k} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \sqcup \coprod D_8/\langle a^2x\rangle \to D_8/\langle a^2,x\rangle \to *] \\
&\quad \cdot [D_8/\langle x\rangle \leftarrow D_8/e \to D_8/\langle a^2,x\rangle \to *] \\
&= [D_8/\langle x\rangle \xleftarrow{(\sqcup f_i \sqcup e) \sqcup g_j \sqcup h_k} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \sqcup \coprod D_8/\langle a^2x\rangle \to D_8/\langle a^2,x\rangle \to *] \\
&\quad + [D_8/\langle x\rangle \xleftarrow{(\sqcup f_i \sqcup a) \sqcup g_j \sqcup h_k} \coprod D_8/e \sqcup \coprod D_8/\langle x\rangle \sqcup \coprod D_8/\langle a^2x\rangle \to D_8/\langle a^2,x\rangle \to *],
\end{aligned}$$

again with similar expressions when $[D_8/\langle x\rangle \leftarrow D_8/e \to D_8/\langle a^2,x\rangle \to *]$ is replaced by $[D_8/\langle x\rangle \leftarrow D_8/\langle x\rangle \to D_8/\langle a^2,x\rangle \to *]$ and $[D_8/\langle x\rangle \leftarrow D_8/\langle a^2x\rangle \to D_8/\langle a^2,x\rangle \to *]$. From this we see that we can take as additional type (b) generators those in which there are ≤ 2 of the f_i, g_j, h_k. $\square$

Proposition 5.6. $\underline{\mathbf{A}}[D_8/\langle a^2\rangle]$ *is levelwise finitely generated.*

Proof. Since all irreducible polynomials

$$D_8/\langle a^2\rangle \leftarrow \coprod D_8/H_i \to D_8/H \to *$$

must have $H_i \leq \langle a^2\rangle \cap H$, these lie in the image of previously-computed cases unless $\langle a^2\rangle \leq H$, in which case they are covered by Lemma 4.15, since all subgroups containing $\langle a^2\rangle$ are normal. $\square$

Proposition 5.7. $\underline{\mathbf{A}}[D_8/H]$ *is levelwise finitely generated, where H is any normal subgroup of order* 4*.*

Proof. All irreducible polynomials

$$D_8/H \leftarrow \coprod D_8/H_i \to D_8/K \to *$$

must again have $H_i \leq H \cap K$, in which case they either lie in the image of previously-computed cases or $H \leq K$, in which case they are covered by Lemma 4.15 since all such H are maximal. $\square$

Proposition 5.8. $\underline{\mathbf{A}}[D_8/D_8]$ *is levelwise finitely generated.*

Proof. Follows from Theorem 5.2. $\square$

Remark 5.9. Note that the difference in the computation for D_8 (as opposed to the $(*)$ case or the Dedekind case) arose from the fact that there were terms arising from the double coset formula

$$D_8/\langle x\rangle \times D_8/\langle x\rangle \cong D_8/\langle x\rangle \sqcup D_8/e \sqcup D_8/\langle x\rangle$$

(corresponding to the double cosets $\langle x\rangle e\langle x\rangle$, $\langle x\rangle a\langle x\rangle$, and $\langle x\rangle a^2\langle x\rangle$) which either (1) did not immediately reduce to lower-level computations or (2) created complications in the $\coprod_i G/H_i$ piece of the polynomial

$$G/H \leftarrow \coprod_i G/H_i \to G/K \to G/G,$$

causing the various H_i to change under multiplication.

We thus conclude:

Theorem 5.10. *Let G be a finite group satisfying $(*)$ or $G \cong D_8$. Then G is transitively Tambara finite.*

6. Extension to Non-Transitive Sets

6.1. **A General Formula for the Box Product.** Our goal for this section is to show that the levelwise grading and finite generation results can be extended to $\underline{\mathbf{A}}[X]$ when X is not transitive—in particular, that transitively finite groups are Tambara finite. We review the construction of the box product and its relation with Dress pairings. Let G be a finite group. The category of finite G-sets is symmetric monoidal with respect to the product $A \times B$ of two G-sets—this induces a symmetric monoidal product on the categories $\mathcal{P}_G$ and $\mathcal{P}_G^+$. The *box product* of two Mackey functors M, N is defined to be the Day convolution of M and N with respect to this monoidal structure, i.e. $M \boxtimes N$ is the left Kan extension

$$\begin{tikzcd} \mathcal{P}_G^+ \times \mathcal{P}_G^+ \arrow[rr, "M\times N"] \arrow[dr, "\otimes"'] & \arrow[d, phantom, "\Downarrow"] & \mathbf{Set} \\ & \mathcal{P}_G^+ \arrow[ur, "M\boxtimes N"'] & \end{tikzcd}$$

More concretely, there exist for any finite G-sets X, Y maps

$$M(X) \times N(Y) \to (M \boxtimes N)(X \times Y)$$

natural in X and Y, and moreover $M \boxtimes N$ is initial with respect to this property. The universal property of $M \boxtimes N$ can also be described using Dress pairings:

Lemma 6.1 ([Lew81]). *A morphism $M \boxtimes N \to P$ is equivalent to the following data: for each subgroup $H \le G$, a bilinear map $f_H : M(G/H) \otimes N(G/H) \to P(G/H)$ such that the following are satisfied:*

(a) $f_H \circ (\mathrm{Res}_H^K \otimes \mathrm{Res}_H^K) = \mathrm{Res}_H^K \circ f_K$ whenever $H \le K$.
(b) $f_H \circ (c_g \otimes c_g) = c_g \circ f_H$ for all $g \in G$ and $H \le G$.
(c) For any $H \le K$,

$$\mathrm{Tr}_H^K \circ f_H \circ (\mathrm{Res}_H^K \otimes \mathrm{id}) = f_K \circ (\mathrm{id} \otimes \mathrm{Tr}_H^K), \quad \mathrm{Tr}_H^K \circ f_H \circ (\mathrm{id} \otimes \mathrm{Res}_H^K) = f_K \circ (\mathrm{Tr}_H^K \otimes \mathrm{id}).$$

The box product of two Tambara functors is again Tambara functor in a canonical way:

Lemma 6.2 ([Str12]). *If T, R are Tambara functors, there is a unique Tambara functor structure for the Mackey functor $T \boxtimes R$ such that the Dress pairings*

$$f_H : T(G/H) \otimes R(G/H) \to (T \boxtimes R)(G/H)$$

are ring maps and satisfy $f_K(\mathrm{Nm}_H^K(x) \otimes \mathrm{Nm}_H^K(y)) = \mathrm{Nm}_H^K(f_H(x \otimes y))$ whenever $x \otimes y \in T(G/H) \otimes R(G/H)$ and $H \le K$. Such defined, $\boxtimes$ is the coproduct in the category of Tambara functors.

A slight generalization of the Dress pairing conditions can be formulated for an n-fold box product.

Lemma 6.3. *A morphism $M_1 \boxtimes \cdots \boxtimes M_N \to P$ is equivalent to the following data: for each subgroup $H \le G$, a multilinear map $f_H : M_1(G/H) \otimes \cdots \otimes M_N(G/H) \to P(G/H)$ such that the following are satisfied:*

(a) $f_H \circ (\mathrm{Res}_H^K \otimes \cdots \otimes \mathrm{Res}_H^K) = \mathrm{Res}_H^K \circ f_K$ whenever $H \le K$.
(b) $f_H \circ (c_g \otimes \cdots \otimes c_g) = c_g \circ f_H$ for all $g \in G$ and $H \le G$.
(c) For any $i = 1, \ldots, N$ and $H \le K$,

$$\begin{aligned}\mathrm{Tr}_H^K \circ f_H \circ (\mathrm{Res}_H^K \otimes \cdots \otimes \mathrm{Res}_H^K \otimes \mathrm{id}_i \otimes \mathrm{Res}_H^K \otimes \cdots \otimes \mathrm{Res}_H^K) \\ = f_K \circ (\mathrm{id}_1 \otimes \cdots \otimes \mathrm{id}_{i-1} \otimes \mathrm{Tr}_H^K \otimes \mathrm{id}_{i+1} \otimes \cdots \otimes \mathrm{id}_N),\end{aligned}$$

where id_i means that on the i^{th} factor, id is being applied.

Proof Sketch. Given natural morphisms

$$M_1(X_1) \times \cdots \times M_N(X_N) \to P(X_1 \times \cdots \times X_N),$$

the multilinear map f_H is defined via the composition

$$M_1(G/H) \times \cdots \times M_N(G/H) \to P(G/H \times \cdots \times G/H) \xrightarrow{R_\delta} P(G/H),$$

where R_δ is restriction along the diagonal map $\delta : G/H \to G/H \times \cdots \times G/H$.

Conversely, given multilinear maps $f_H : M_1(G/H) \otimes \cdots \otimes M_N(G/H) \to P(G/H)$, we extend these to multilinear maps $f_X : M_1(X) \times \cdots \times M_1(X) \to P(X)$ for a nontransitive G-set $X = G/H_1 \sqcup \cdots \sqcup G/H_m$, using the fact that

$$M_j(\sqcup_i G/H_i) \cong \prod M_j(G/H_i)$$

and setting

$$f_X(x_1, \ldots, x_N) = \begin{cases} f_{H_j}(x_1, \ldots, x_N) & \text{if } x_i \in M_i(G/H_j) \text{ for all } i, \\ 0 & \text{otherwise.}\end{cases}$$

Then the maps $M_1(X_1) \times \cdots \times M_N(X_N) \to P(X_1 \times \cdots \times X_N)$ are given by

$$M_1(X_1) \times \cdots \times M_N(X_N) \xrightarrow{R_{\mathrm{pr}_i}} \prod_{i=1}^N M_i\left(\prod_{j=1}^j X_j\right) \xrightarrow{f_{\prod_{j=1}^N X_j}} P\left(\prod_{j=1}^N X_j\right). \qquad \square$$

We describe a formula for the n-fold box product of two Mackey functors over an arbitrary finite group G. First, a lemma.

Lemma 6.4. *Let G be a group and let H, K, L, M be subgroups, with $H \le K \le L$ and $M \le L$. Then the double coset representatives of $H \backslash L / M$ are in bijection with the elements $\{y_x \cdot x\}$, where the x range over the double coset representatives of $K \backslash L / M$ and the y_x, for each fixed x, range over the double coset representatives of $H \backslash K / (xMx^{-1} \cap K)$.*

Proof. Suppose $y_x x$ and $y_{x'} x'$ represent the same double coset $H\backslash L/M$. Then $y_{x'}x' = hy_x xm$ for $h \in H$ and $m \in M$, which implies x and x' represent the same $K\backslash L/M$ coset, so $x = x'$. Now if $y'_x x = hy_x xm$ for $h \in H$ and $m \in M$, then $y'_x = hy_x(xmx^{-1})$; since $y_x, y'_x \in K$, $xmx^{-1} \in xMx^{-1} \cap K$ and thus y'_x, y_x represent the same $H\backslash K/(xMx^{-1} \cap K)$ coset. This shows injectivity of the map $\{y_x x\} \to H\backslash L/M$.

For surjectivity, note that for fixed x, changing the representative of y_x does not affect the double coset $Hy_x xM$. Indeed, if $y'_x = hy_x\kappa$, then $\kappa = xmx^{-1}$ and $Hy'_x xM = Hy_x xM$.

Thus, what we have shown is this: the set of double cosets $\{Hy_x xM\}$ where x varies over $K\backslash L/M$ and y varies over $H\backslash K/M$ is equal to the set of double cosets $\{HyxM\}$ where y varies over K and x varies over $K\backslash L/M$. This is just the set of double cosets $\{HyxM\}$ where x, y vary arbitrarily in L, K, so we obtain surjectivity. $\square$

Theorem 6.5. *Let G be a finite group, $M_1, \ldots, M_N$ be a collection of G-Mackey functors and write $M = M_1 \boxtimes \cdots \boxtimes M_N$. Fix a subgroup $L \leq G$. For any subgroup $H \leq L$, define*

$$S_H^L = M_1(G/H) \otimes \cdots \otimes M_N(G/H).$$

Then we have

$$M(G/L) = \left(\bigoplus_{H \leq L} S_H^L \right) /F,$$

where F is the submodule generated by the Frobenius relations

$$\begin{aligned} S_K^L \ni\, & x_1 \otimes \cdots \otimes x_{i-1} \otimes \mathrm{Tr}_H^K(x_i) \otimes x_{i+1} \otimes \cdots \otimes x_N \\ &= \mathrm{Res}_H^K(x_1) \otimes \cdots \otimes \mathrm{Res}_H^K(x_{i-1}) \otimes x_i \otimes \mathrm{Res}_H^K(x_{i+1}) \otimes \cdots \otimes \mathrm{Res}_H^K(x_N) \in S_H^L \end{aligned}$$

for all $H \leq K \leq L$, and the Weyl relations

$$S_H^L \ni x_1 \otimes \cdots \otimes x_N = c_\ell(x_1) \otimes \cdots \otimes c_\ell(x_N) \in S_{\ell H \ell^{-1}}^L$$

for all $\ell \in L$.

The isomorphisms $c_g : M(G/L) \to M(G/gLg^{-1})$ are induced on each S_H^L by the maps

$$S_H^L \ni x_1 \otimes \cdots \otimes x_N \mapsto c_g(x_1) \otimes \cdots \otimes c_g(x_N) \in S_{gHg^{-1}}^{gLg^{-1}}.$$

The transfer maps $\mathrm{Tr}_L^{L'} : M(G/L) \to M(G/L')$ are induced by the obvious isomorphisms $S_H^L \to S_H^{L'}$. The restriction maps $\mathrm{Res}_L^{L'} : M(G/L') \to M(G/L)$ are defined on each component $S_H^{L'}$ by the formula

$$\mathrm{Res}_L^{L'}(x_1 \otimes \cdots \otimes x_N) = \sum_{g \in L\backslash L'/H} c_g\left(\mathrm{Res}_{H \cap g^{-1}Lg}^H(x_1)\right) \otimes \cdots \otimes c_g\left(\mathrm{Res}_{H \cap g^{-1}Lg}^H(x_N)\right),$$

where each $\mathrm{Res}_{H\cap g^{-1}Lg}^H(x_1) \otimes \cdots \otimes \mathrm{Res}_{H \cap g^{-1}Lg}^H(x_N)$ lies in $S_{gHg^{-1}\cap L}^L$. The Dress pairing $f_L : M_1(G/L) \otimes \cdots \otimes M_N(G/L) \to M(G/L)$ is induced by the obvious map

$$M_1(G/L) \otimes \cdots \otimes M_N(G/L) \to S_L^L.$$

Proof. For notational simplicity, we give the proof when $N = 2$, but the proof for arbitrary N is exactly the same. There is a long chain of straightforward but necessary verifications which need to be performed.

(1) Tr, Res, and c_g are well-defined.

This is obvious for $\mathrm{Tr}_L^{L'}$ and straightforward for c_g. The only nontrivial part is proving that $\mathrm{Res}_L^{L'}$ preserves Frobenius relations. Let $H \le K \le L'$. By definition,

$$\mathrm{Res}_L^{L'}(x \otimes \mathrm{Tr}_H^K(y)) = \sum_{g \in L\backslash L'/K} c_g \left(\mathrm{Res}^K_{K\cap g^{-1}Lg}(x)\right) \otimes c_g \left(\mathrm{Res}^K_{K\cap g^{-1}Lg} \mathrm{Tr}_H^K(y)\right).$$

On the other hand,

$$\mathrm{Res}_L^{L'}(\mathrm{Res}_H^K(x) \otimes y) = \sum_{g \in L\backslash L'/H} c_g \left(\mathrm{Res}^K_{H\cap g^{-1}Lg}(x)\right) \otimes c_g \left(\mathrm{Res}^H_{H\cap g^{-1}Lg}(y)\right).$$

By Lemma 6.4, this is equal to

$$\begin{aligned}
&\sum_{s\in L\backslash L'/K}\ \sum_{t\in(s^{-1}Ls\cap K)\backslash K/H} c_s c_t \left(\mathrm{Res}^K_{H\cap(st)^{-1}L(st)}(x)\right) \otimes c_s c_t \left(\mathrm{Res}^H_{H\cap(st)^{-1}L(st)}(y)\right)\\
&= \sum_{s\in L\backslash L'/K}\ \sum_{t\in(s^{-1}Ls\cap K)\backslash K/H} c_s \left(\mathrm{Res}^K_{tHt^{-1}\cap s^{-1}Ls}(x)\right) \otimes c_s \left(c_t \mathrm{Res}^H_{H\cap(st)^{-1}L(st)}(y)\right)\\
&= \sum_{s\in L\backslash L'/K}\ \sum_{t\in(s^{-1}Ls\cap K)\backslash K/H} c_s \left(\mathrm{Res}^K_{K\cap s^{-1}Ls}(x)\right) \otimes c_s \left(\mathrm{Tr}^{K\cap s^{-1}Ls}_{tHt^{-1}\cap sLs^{-1}} c_t \mathrm{Res}^H_{H\cap(st)^{-1}L(st)}(y)\right)\\
&= \sum_{s\in L\backslash L'/K} c_s \left(\mathrm{Res}^K_{K\cap s^{-1}Ls}(x)\right) \otimes c_s \left(\mathrm{Res}^K_{K\cap s^{-1}Ls} \mathrm{Tr}^K_H(y)\right)
\end{aligned}$$

which by the above is equal to $\mathrm{Res}_L^{L'}(x \otimes \mathrm{Tr}_H^K(y))$. Here we have used Frobenius relations on the second-to-last line and the fact that

$$\mathrm{Res}^K_{K\cap g^{-1}Lg} \mathrm{Tr}^K_H = \sum_{t\in(g^{-1}Lg\cap K)\backslash K/H} \mathrm{Tr}^{K\cap s^{-1}Ls}_{tHt^{-1}\cap sLs^{-1}} c_t \mathrm{Res}^H_{H\cap(gt)^{-1}L(gt)}$$

by the double coset formula.

(2) Tr, Res, c_g are functorial.

This is again more or less obvious by definition for Tr and c_g. For Res, let $L \le L' \le L''$, $H \le L''$, and suppose $x_1 \otimes \cdots \otimes x_N \in S_H^{L''}$. By definition,

$$\begin{aligned}
&\mathrm{Res}_L^{L'} \mathrm{Res}_{L'}^{L''}(x \otimes y)\\
&= \mathrm{Res}_L^{L'}\left(\sum_{g\in L'\backslash L''/H} c_g \left(\mathrm{Res}^H_{H\cap g^{-1}L'g}(x)\right) \otimes c_g \left(\mathrm{Res}^H_{H\cap g^{-1}L'g}(y)\right)\right)\\
&= \sum_{g\in L'\backslash L''/H}\ \sum_{t\in L\backslash L'/(gHg^{-1}\cap L')} c_t\Big(\mathrm{Res}^{L'\cap gHg^{-1}}_{t^{-1}Lt\cap gHg^{-1}} c_g \mathrm{Res}^H_{H\cap g^{-1}L'g}(x)\\
&\qquad\qquad \otimes \mathrm{Res}^{L'\cap gHg^{-1}}_{t^{-1}Lt\cap gHg^{-1}} c_g \mathrm{Res}^H_{H\cap g^{-1}L'g}(y)\Big)\\
&= \sum_{g\in L'\backslash L''/H}\ \sum_{t\in L\backslash L'/(gHg^{-1}\cap L')} c_{tg} \left(\mathrm{Res}^H_{H\cap(tg)^{-1}L(tg)}(x) \otimes \mathrm{Res}^H_{H\cap(tg)^{-1}L(tg)}(y)\right)\\
&= \sum_{g\in L\backslash L''/H} c_g \left(\mathrm{Res}^H_{H\cap g^{-1}Lg}(x) \otimes \mathrm{Res}^H_{H\cap g^{-1}Lg}(y)\right) = \mathrm{Res}_L^{L''}(x \otimes y),
\end{aligned}$$

where we have used Lemma 6.4 in the last line.

(3) M assembles into a Mackey functor via $\mathrm{Tr}, \mathrm{Res}, c_g$.

It is clear that c_g commutes with Tr. furthermore, $c_\ell = \mathrm{id}$ on $M(G/L)$ by definition. For commutation with Res, let $x_1 \otimes \cdots \otimes x_N \in S_H^{L'}$; we have

$$c_g \operatorname{Res}_L^{L'}(x_1 \otimes \cdots \otimes x_N) = \sum_{t \in L\backslash L'/H} c_g c_t \left(\operatorname{Res}^H_{H\cap t^{-1}Lt}(x_1) \otimes \cdots \otimes \operatorname{Res}^H_{H\cap t^{-1}Lt}(x_N)\right),$$

while on the other hand

$$\begin{aligned}
&\operatorname{Res}^{gL'g^{-1}}_{gLg^{-1}}(c_g(x_1) \otimes \cdots \otimes c_g(x_N)) \\
&= \sum_{t \in L\backslash L'/H} c_{gtg^{-1}} \left(\operatorname{Res}^{gHg^{-1}}_{gHg^{-1}\cap gt^{-1}Ltg^{-1}}(c_g(x_1)) \otimes \cdots \otimes \operatorname{Res}^{gHg^{-1}}_{gHg^{-1}\cap gt^{-1}Ltg^{-1}}(c_g(x_N))\right) \\
&= \sum_{t \in L\backslash L'/H} c_g c_t \left(\operatorname{Res}^H_{H\cap t^{-1}Lt}(x_1) \otimes \cdots \otimes \operatorname{Res}^H_{H\cap t^{-1}Lt}(x_N)\right)
\end{aligned}$$

as desired. Thus, it remains to verify the double coset formula. Let $L, L' \leq L''$ and $x_1 \otimes \cdots \otimes x_N \in S_H^{L'}$.

$$\operatorname{Res}_L^{L''} \operatorname{Tr}_{L'}^{L''}(x_1 \otimes \cdots \otimes x_N) = \sum_{g \in L\backslash L''/L'} c_g \left(\operatorname{Res}^H_{H\cap g^{-1}Lg}(x_1) \otimes \cdots \otimes \operatorname{Res}^H_{H\cap g^{-1}Lg}(x_N)\right),$$

while on the other hand

$$\begin{aligned}
&\sum_{g \in L\backslash L''/L'} \operatorname{Tr}^L_{L\cap gL'g^{-1}} c_g \operatorname{Res}^{L'}_{g^{-1}Lg\cap L'}(x_1 \otimes \cdots \otimes x_N) \\
&= \sum_{g \in L\backslash L''/L'} \sum_{t \in (g^{-1}Lg\cap L')\backslash L'/H} c_g c_t \left(\operatorname{Res}^H_{(gt)^{-1}L(gt)\cap H}(x_1) \otimes \cdots \otimes \operatorname{Res}^H_{(gt)^{-1}L(gt)\cap H}(x_N)\right) \\
&= \sum_{g \in L\backslash L''/H} c_g \left(\operatorname{Res}^H_{g^{-1}Lg\cap H}(x_1) \otimes \cdots \otimes \operatorname{Res}^H_{g^{-1}Lg\cap H}(x_N)\right),
\end{aligned}$$

using Lemma 6.4 again. Thus, M assembles into a Mackey functor.

(4) $M \cong M_1 \boxtimes \cdots \boxtimes M_N$.

We verify that M has the universal property of $M_1 \boxtimes \cdots \boxtimes M_N$ with respect to the proposed pairing $f_L : M_1(G/L) \otimes \cdots \otimes M_N(G/L) \to M(G/L)$. The fact that the f_L do indeed assemble into a Dress pairing is enforced by the Frobenius and Weyl relations. Given any Dress pairing $g_L : M_1(G/L) \otimes \cdots \otimes M_N(G/L) \to P(G/L)$, we define $h_L : M(G/L) \to P(G/L)$ by setting

$$h_L(x_1 \otimes \cdots \otimes x_N) = \operatorname{Tr}_H^L(g_H(x_1 \otimes \cdots \otimes x_N))$$

whenever $x_1 \otimes \cdots \otimes x_N \in S_H^L$. Conversely, given a morphism of Mackey functors $h : M \to P$, the Dress pairing $g_L : M_1(G/L) \otimes \cdots \otimes M_N(G/L) \to P(G/L)$ is recovered via the composition $g_L = h_L \circ f_L$, where $h_L : M(G/L) \to P(G/L)$ is the L-level of h. $\square$

The formula simplifies slightly in the abelian case:

Corollary 6.6. *Let G be a finite abelian group, $M_1, \ldots, M_N$ be a collection of G-Mackey functors and write $M = M_1 \boxtimes \cdots \boxtimes M_N$. Fix a subgroup $L \leq G$. For any subgroup $H \leq L$, define*

$$S_H^L = (M_1(G/H) \otimes \cdots \otimes M_N(G/H))/(L/H).$$

Then we have

$$M(G/L) = \left(\bigoplus_{H \leq L} S_H^L \right) / F,$$

where F is the submodule generated by the Frobenius relations

$$\begin{aligned} S_K^L \ni x_1 \otimes \cdots \otimes x_{i-1} \otimes \mathrm{Tr}_H^K(x_i) \otimes x_{i+1} \otimes \cdots \otimes x_N \\ = \mathrm{Res}_H^K(x_1) \otimes \cdots \otimes \mathrm{Res}_H^K(x_{i-1}) \otimes x_i \otimes \mathrm{Res}_H^K(x_{i+1}) \otimes \cdots \otimes \mathrm{Res}_H^K(x_N) \in S_H^L \end{aligned}$$

For all $H \leq K \leq L$. The transfer, restriction, and conjugation homomorphisms are defined analogously.

Remark 6.7. The formula of Theorem 6.5 generalizes the inductive description of $T \boxtimes R$ for two C_{p^n}-Tambara functors given in [Maz13] to all finite G. Intuitively, each S_H^L should be thought of as the set of formal expressions $\mathrm{Tr}_H^L(x_1 \otimes \cdots \otimes x_N)$ where $H \leq L$ and each $x_i \in M_i(G/H)$.

Remark 6.8. When $T_1, \ldots, T_N$ are Tambara functors, the universal morphisms $T_i \to T = T_1 \boxtimes \cdots \boxtimes T_N$ exhibiting T as a coproduct of the T_i are given by the compositions

$$T_i(G/L) \xrightarrow{g_i} T_1(G/L) \otimes \cdots \otimes T_N(G/L) \xrightarrow{f_L} T(G/L),$$

where $g_i : T_i(G/L) \to T_1(G/L) \otimes \cdots \otimes T_N(G/L)$ is the map sending x to $1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$, i.e. inclusion into the i^{th} factor. In the special case that $T_i = \underline{\mathbf{A}}[G/H_i]$ and $T = \boxtimes_{i=1}^N \underline{\mathbf{A}}[G/H_i] = \underline{\mathbf{A}}[X]$, the universal morphisms $T_i \to T$ are induced by $R_{g_i} \in \mathcal{P}_G(X, G/H_i)$, where $g_i : G/H_i \to X$ is the canonical inclusion (follows from an application of the Yoneda lemma).

Remark 6.9. Let T be a Tambara functor and let R, S be T-algebras. The box product $R \boxtimes_T S$ is given by modifying the definition of each S_H^L to be tensor products over $T(G/H)$.

6.2. Extending Finiteness Criteria to Non-Transitive Sets.

Proposition 6.10. *Let G be a finite group. If T, R are levelwise finitely generated G-Tambara functors, then so is $T \boxtimes R$.*

Proof. Fix an $L \leq G$. We claim that the image of each S_H^L in $(T \boxtimes R)(G/L)$ is a finitely generated S_L^L-module. Since $S_L^L \cong T(G/L) \otimes R(G/L)$ as a ring, which is finitely generated, this will imply $(T \boxtimes R)(G/L)$ is finitely generated. Indeed, the elements of S_H^L are all of the form $\mathrm{Tr}_H^L(x \otimes y)$ where $x \otimes y \in S_H^H$. S_L^L acts by

$$(x' \otimes y') \cdot \mathrm{Tr}_H^L(x \otimes y) = \mathrm{Tr}_H^L(\mathrm{Res}_H^L(x' \otimes y') \cdot (x \otimes y)) = \mathrm{Tr}_H^L(\mathrm{Res}_H^L(x')x \otimes \mathrm{Res}_H^L(y')y).$$

In other words, the image of S_H^L is a quotient of the S_L^L-module $T(G/H) \otimes R(G/H)$, with S_L^L-action given by the restriction map

$$\mathrm{Res}_H^L \otimes \mathrm{Res}_H^L : T(G/L) \otimes T(G/L) \to T(G/H) \otimes R(G/H).$$

Since T, R are levelwise finitely generated and the maps $\mathrm{Res}_H^L : T(G/L) \to T(G/H)$ are integral (Lemma 4.5), they are finite ring maps, whence $T(G/L) \otimes T(G/L) \to T(G/H) \otimes R(G/H)$ is a finite ring map and thus $T(G/H) \otimes R(G/H)$ is a finite S_L^L-module, from which the result follows. □

Remark 6.11. The same proof can be extended to show that if S is a G-Tambara functor and T, R are S-algebras, levelwise finitely generated over S, then so is $T \boxtimes_S R$.

Corollary 6.12. *Let G be a transitively Tambara finite group. Then G is also Tambara finite.*

Proof. This follows from Proposition 6.10 and the isomorphism $\underline{\mathbf{A}}[X \sqcup Y] \cong \underline{\mathbf{A}}[X] \boxtimes \underline{\mathbf{A}}[Y]$. $\square$

Example 6.13. We remark that Proposition 6.10 is not immediate from the formula in Theorem 6.5, since it is is false for Green functors. Let R be the C_p-Green functor given by the Lewis diagram

$$\begin{tikzcd} \mathbf{F}_p \arrow[d, shift right, "\mathrm{Res}_e^{C_p}"'] \\ \mathbf{F}_p[x] \arrow[u, shift right, "\mathrm{Tr}_e^{C_p}"'] \arrow[loop below, "\mathrm{id}"] \end{tikzcd}$$

where $\mathrm{Res}_e^{C_p} : \mathbf{F}_p \to \mathbf{F}_p[x]$ is the standard inclusion and $\mathrm{Tr}_e^{C_p} = 0$. Clearly R is a levelwise finite type $\mathbf{Z}$-algebra. $R \boxtimes R$ is given by

$$\begin{tikzcd} (\mathbf{F}_p \oplus \mathbf{F}_p[x,y])/F \arrow[d, shift right, "\mathrm{Res}_e^{C_p}"'] \\ \mathbf{F}_p[x,y] \arrow[u, shift right, "\mathrm{Tr}_e^{C_p}"'] \arrow[loop below, "\mathrm{id}"] \end{tikzcd}$$

The Frobenius relations enforce $0 = c \otimes \mathrm{Tr}(y^k) = \mathrm{Tr}(c \otimes y^k)$ and $0 = \mathrm{Tr}(x^k) \otimes c = \mathrm{Tr}(x^k \otimes c)$ for $k \geq 0$, and multiplication

$$\mathrm{Tr}(x^{n_1} \otimes y^{m_1})\,\mathrm{Tr}(x^{n_2} \otimes y^{m_2}) = \mathrm{Tr}\left((x^{n_1} \otimes y^{m_1})\,\mathrm{Res}\,\mathrm{Tr}(x^{n_2} \otimes y^{m_2})\right) = 0$$

Hence the top level $(R \boxtimes R)(C_p/C_p)$ is the ring consisting of elements

$$(n, p(x,y))$$

where $n \in \mathbf{F}_p$ and $p(x,y)$ is a polynomial with $\mathbf{F}_p$-coefficients having no terms of the form cx^k, cy^k, $k \geq 0$. Multiplication is given by $(n, p(x,y))(m, q(x,y)) = (nm, nq(x,y) + mp(x,y))$. In particular, the top level $(R \boxtimes R)(C_p/C_p)$ is not a finitely generated ring.

While levelwise finite generation is too much to ask for a box product of levelwise finitely generated Green functors (see Example 6.13), relative finite-dimensionality is not:

Lemma 6.14. *Let T, R be relatively finite dimensional Green functors. Then $T \boxtimes R$ is also relatively finite-dimensional.*

Proof. Let $L \leq L'$. As in the proof of Proposition 6.10, each S_H^L is a finite module over $S_L^L = T(G/L) \otimes R(G/L)$, since the S_L^L-module structure on $S_H^L = T(G/H) \otimes R(G/H)$ is induced by $\mathrm{Res}_H^L \otimes \mathrm{Res}_H^L : T(G/L) \otimes R(G/L) \to T(G/H) \otimes R(G/H)$ which is a finite ring map by assumption. Thus, $(T \boxtimes R)(G/L)$ is a finite $T(G/L) \otimes R(G/L)$-module. Since $\mathrm{Res}_L^{L'} : T(G/L') \otimes R(G/L') \to T(G/L) \otimes R(G/L)$ is a finite ring map, we therefore see that $\mathrm{Res}_L^{L'} : T(G/L') \otimes R(G/L') \to (T \boxtimes R)(G/L)$ is a finite ring map, and hence $\mathrm{Res}_L^{L'} : (T \boxtimes R)(G/L') \to (T \boxtimes R)(G/L)$ is as well. $\square$

We also remark that with relative finite-dimensionality hypotheses, "levelwise finite generation" is stable under base change:

Lemma 6.15. *Let T be a relatively finite-dimensional Tambara functor, and suppose R is levelwise finitely generated (over $\underline{\mathbf{A}}$). Then $T \boxtimes R$ is levelwise finitely generated over T.*

Proof. Let $L \leq G$. For each $H \leq L$, we have that the ring map $\mathrm{Res}^L_H \otimes \mathrm{Res}^L_H : T(G/L) \otimes R(G/L) \to T(G/H) \otimes R(G/H)$ is finite, from which we deduce (as in the proof of Proposition 6.10) that the ring map $T(G/L) \otimes R(G/L) \to (T \boxtimes R)(G/L)$ is finite. Since $R(G/L)$ is a finite type $\mathbf{Z}$-algebra, $T(G/L) \otimes R(G/L)$ is a finite type $T(G/L)$-algebra, hence $(T \boxtimes R)(G/L)$ is a finite type $T(G/L)$-algebra. □

Theorem 6.16. *Let G be a finite group and let X be a finite G-set. Then $\underline{\mathbf{A}}[X]$ is relatively finite-dimensional.*

Proof. Follows from Lemma 6.14, Lemma 4.8, and the fact that $\underline{\mathbf{A}}[X \sqcup Y] \cong \underline{\mathbf{A}}[X] \boxtimes \underline{\mathbf{A}}[Y]$. □

Consider the following condition on a finite group G.

(†) G satisfies at least one of the following conditions:
- (a) G is a Dedekind group.
- (b) Every proper, nontrivial subgroup of G is maximal, and every subgroup is either normal or satisfies $N_G(H) = H$.
- (c) $G \cong D_8$, the dihedral group of order 8.

We showed that all finite groups satisfying (†) are transitively finite in §4 and §5. We now put these facts to use.

Theorem 6.17. *Let G be a finite group satisfying (†). Then G is Tambara finite. In particular, since $\underline{\mathbf{A}}[X]$ is a levelwise finitely generated Tambara functor for any G-set X, it is also levelwise Noetherian and relatively finite-dimensional.*

Proof. Follows from Corollary 6.12. □

Theorem 6.18. *Let G be a Tambara finite and let X be a finite G-set. Then $\underline{\mathbf{A}}[X]$ is module-Noetherian, in the sense that every submodule of a finitely generated module over $\underline{\mathbf{A}}[X]$ is finitely generated. In particular, this holds for all finite groups satisfying (†).*

This is a consequence of Theorem 6.17 and a more general fact about relatively finite-dimensional Green functors:

Lemma 6.19 ([CW25, Corollary 3.35])**.** *Let R be a relatively finite-dimensional Green functor. Then R is module-Noetherian iff $R(G/G)$ is Noetherian iff $R(G/H)$ is Noetherian for all $H \leq G$.*

Tambara finite groups also satisfy a weak Hilbert basis theorem:

Theorem 6.20 (Weak Hilbert Basis Theorem)**.** *Let G be a Tambara finite group, and suppose T is levelwise Noetherian and relatively finite-dimensional. Then $T[X]$ is also levelwise Noetherian and relatively finite-dimensional for all finite G-sets X. If T is levelwise finitely generated, then so is $T[X]$.*

Proof. $T[X]$ is relatively finite-dimensional by Lemma 6.14. For the levelwise Noetherian statement, observe as in the proof of Proposition 6.10 that each ring map $T(G/L) \otimes \underline{\mathbf{A}}[X](G/L) \to T[X](G/L)$ is finite; since $T(G/L) \otimes \underline{\mathbf{A}}[X](G/L)$ is a finite type $T(G/L)$-algebra by Theorem 6.17, so is $T[X](G/L)$, whence $T[X](G/L)$ is Noetherian. The levelwise finitely generated part of the statement follows from Theorem 6.17 and Proposition 6.10. □

6.3. **Stability of the Grading.** To conclude this section, we provide proofs to some earlier remarks about the stability of the grading we have introduced on polynomials under box products.

Corollary 6.21. *Let G be a finite group. The gradings on $\underline{\mathbf{A}}[G/H]$ for $H \le G$ induce a grading on $\underline{\mathbf{A}}[X]$ for arbitrary finite G-sets X via the box product; for now, we call this grading the box grading.*

Proof. We explain this for the Dedekind case; the nonabelian case is obtained by dropping the nonnumerical component of the degree. For each i, let $T_i = \underline{\mathbf{A}}[G/H_i]$ and $T = \underline{\mathbf{A}}[X] = \boxtimes_{i=1}^N T_i$. We adopt the same notation for the S_H^L in Theorem 6.5. Note that each S_H^L has a grading over $\mathcal{O}_H^w$ induced from the grading on each $T_i(G/H)$; by identifying $\mathcal{O}_H^w$ as a subset of $\mathcal{O}_L^w$, this induces a grading of S_H^L over $\mathcal{O}_L^w$. We claim that this endows $T(G/L)$ with the structure of a graded ring over $\mathcal{O}_L^w$, for which it suffices to show that the Frobenius relations are homogeneous.

Indeed, suppose $H \le K$ and x_i, $i = 1, \dots, \hat{j}, \dots N$ are elements of degree (n_i, J_i) in $T_i(G/K)$, while x_j is an element of degree (n_j, J_j) in $T_j(G/H)$. Then $\mathrm{Tr}_H^K(x_j)$ is also an element of degree (n_j, J_j) in $T_j(G/K)$ by Lemma 3.14, so $x_1 \otimes \cdots \otimes \mathrm{Tr}_H^K(x_j) \otimes \cdots \otimes x_N$ is of degree $(\sum n_i, \bigcap J_i)$ in S_K^L. On the other hand, $\mathrm{Res}_H^K(x_1) \otimes \cdots \otimes x_j \otimes \cdots \otimes \mathrm{Res}_H^K(x_N)$ is of degree $(\sum n_i, \bigcap J_i \cap H)$ in S_K^L. Since $J_j \le H$ by assumption, these two degrees coincide and $\underline{\mathbf{A}}[X]$ is graded. $\square$

Remark 6.22. We note that the natural isomorphism $\underline{\mathbf{A}}[X] \to \underline{\mathbf{A}}[X \sqcup \varnothing] \cong \underline{\mathbf{A}}[X] \otimes \underline{\mathbf{A}}[\varnothing] \cong \underline{\mathbf{A}}[X]$ is graded by inspection (when $\underline{\mathbf{A}}[X \sqcup \varnothing]$ is given the box grading), since all elements of $\underline{\mathbf{A}}[\varnothing](G/L)$ have degree $(0, L)$. Hence the box grading is well-defined.

Lemma 6.23. *The box grading on $\underline{\mathbf{A}}[X]$ coincides with the naive grading of Remark 3.18.*

Proof. Again, we explain this for the Dedekind case; the nonabelian case is obtained by dropping the nonnumerical component of the degree. Write $X = \sqcup_i G/H_i$, $T = \underline{\mathbf{A}}[X]$, and $T_i = \underline{\mathbf{A}}[G/H_i]$. We want to show that the grading coincides on $\underline{\mathbf{A}}[X](G/L)$. Since the transfer homomorphisms $\mathrm{Tr}_L^{L'}$ are graded in the same way for both gradings, it suffices by induction to show that the grading coincides on elements of $\underline{\mathbf{A}}[X](G/L)$ which do not lie in the image of transfer from lower levels, i.e. it suffices to show this for elements in the image of the Dress pairing $f_L : T_1(G/L) \otimes \cdots \otimes T_N(G/L) \to T(G/L)$. This is a graded ring homomorphism for the box grading (essentially by definition), so it suffices to show that the elements $f_L(1 \otimes \cdots \otimes 1 \otimes x_i \otimes 1 \otimes \cdots \otimes 1)$ have the same degree with respect to both the box and naive gradings. Now $f_L(1 \otimes \cdots \otimes 1 \otimes x_i \otimes 1 \otimes \cdots \otimes 1)$ is the image of $x_i \in T_i(G/L)$ under the natural map $T_i \to T$, which is given by

$$[G/H_i \xleftarrow{f} A \to B \to G/L] \mapsto [X \xleftarrow{j \circ f} A \to B \to G/L],$$

where $j : G/H_i \to X$ is the natural inclusion. Thus, if x_i has degree (n, K) with respect to the naive grading, then it also has degree (n, K) with respect to the box grading. $\square$

Corollary 6.24. *Let $f : X \to Y$ be a map of finite G-sets. The induced map $R_f^* : \underline{\mathbf{A}}[X] \to \underline{\mathbf{A}}[Y]$ is levelwise graded with respect to the identity homomorphism $\mathcal{O}_L^w \to \mathcal{O}_L^w$. If moreover f has a well-defined degree, e.g. if Y is transitive, then $N_f^* : \underline{\mathbf{A}}[Y] \to \underline{\mathbf{A}}[X]$ is levelwise graded with respect to the monoid homomorphism $\phi : \mathcal{O}_L^w \to \mathcal{O}_L^w$ given by*

$$\phi(n, K) = (\deg(f)n, K).$$

Proof. R_f^* is given by

$$[X \xleftarrow{p} A \to B \to G/L] \mapsto [Y \xleftarrow{f \circ p} A \to B \to G/L],$$

which evidently preserves the degree of $A \to B$. N_f^* is given by

$$[Y \leftarrow A \to B \to G/L] \mapsto [X \leftarrow A \times_Y X \to B \to G/L],$$

which changes the degree by a multiplicative factor of $\deg(f)$. □

Remark 6.25. The corresponding result for general G is obtained by dropping the non-numerical component.

Lemma 6.26. *Let X be a finite G-set and T a Tambara functor. The free T-algebra $T[X] \cong T \boxtimes \underline{\mathbf{A}}[X]$ inherits a levelwise grading from $\underline{\mathbf{A}}[X]$ over $\mathbf{N}$.*

Proof. We again adopt the notation of Theorem 6.5. Each $T(G/L)$ is trivially graded over $\mathbf{N}$ by giving every element degree 0. Hence, each S_H^L is graded over $\mathbf{N}$. It remains to verify that the Frobenius relations are homogeneous. But this is obvious by inspection, since Res_H^K and Tr_H^K on $\underline{\mathbf{A}}[X]$ do not alter the numerical component of the degree. □

7. Applications to Norm Functors

Let G be a finite group and suppose $H \leq G$. The functor Ind_H^G sending an H-set X to its induced G-set $G \times_H X$ is left adjoint to the functor Res_H^G which sends a G-set to its underlying H-set. This induces an adjunction on the polynomial categories $\mathcal{P}_H$ and $\mathcal{P}_G$, which further induces an adjunction on the categories $H\mathbf{Tamb}$ and $G\mathbf{Tamb}$. More precisely, defining for any Tambara functor $\mathrm{Res}_H^G T = T \circ \mathrm{Ind}_H^G$ and $\mathrm{Coind}_H^G T = T \circ \mathrm{Res}_H^G$, we have

$$\mathrm{Hom}_{H\mathbf{Tamb}}(\mathrm{Res}_H^G T, R) \cong \mathrm{Hom}_{G\mathbf{Tamb}}(T, \mathrm{Coind}_H^G R).$$

The functor Res_H^G has a left adjoint, denoted $n_H^G : H\mathbf{Tamb} \to G\mathbf{Tamb}$, which is defined pointwise by the left Kan extension

$$\begin{tikzcd} \mathcal{P}_H \arrow[rr, "T"] \arrow[dr, "\mathrm{Ind}_H^G"'] & \arrow[d, phantom, "\Downarrow"] & \mathbf{Set} \\ & \mathcal{P}_G \arrow[ur, "n_H^G T"'] & \end{tikzcd}$$

and thus given by the formula

$$n_H^G T(Y) = \int^{X \in \mathcal{P}_H} \mathcal{P}_G(G \times_H X, Y) \times T(Y).$$

In particular, we see from this formula that n_H^G preserves quotients (levelwise surjections) $T \to R$. See [Hoy14] or [HM19] for more details on the functor n_H^G.

If X is an H-set, we compute

$$\begin{aligned} \mathrm{Hom}_{G\mathbf{Tamb}}(n_H^G \underline{\mathbf{A}}[X], T) &\cong \mathrm{Hom}_{H\mathbf{Tamb}}(\underline{\mathbf{A}}[X], \mathrm{Res}_H^G T) \\ &\cong (\mathrm{Res}_H^G T)(X) \\ &\cong T(G \times_H X), \end{aligned}$$

so there is a natural isomorphism $n_H^G \underline{\mathbf{A}}[X] \cong \underline{\mathbf{A}}[G \times_H X]$.

Lemma 7.1. *Let G be a finite group. The following are equivalent:*

(a) The norm functors n_H^K preserve levelwise finite generation over $\mathbf{Z}$ for all finite groups $H \leq K \leq G$.
(b) Every subgroup $H \leq G$ is Tambara finite.
(c) Every subgroup $H \leq G$ is transitively Tambara finite.

Proof. The equivalence of (b) and (c) follows from our work in §6. Assume (a); in light of the isomorphism $n_H^K \underline{\mathbf{A}}[H/H] \cong \underline{\mathbf{A}}[K/H]$ and the levelwise finite generation of $\underline{\mathbf{A}}[H/H]$ established in [SSW25], $\underline{\mathbf{A}}[K/H]$ is finitely generated and thus (a) implies (c). Now assume (b) and let T be a levelwise finitely generated H-Tambara functor. Then T receives a (levelwise) surjection from $\underline{\mathbf{A}}[X]$ for some finite H-set X: the (finite) collection of generators x_i in the $T(H/H_i)$ for $H_i \leq H$ induce a morphism $\boxtimes_i \underline{\mathbf{A}}[H/H_i] \to T$ which includes all the x_i in its image, so $\boxtimes_i \underline{\mathbf{A}}[H/H_i] \to T$ is a surjection. In our situation, the surjection $\underline{\mathbf{A}}[X] \to T$ induces a surjection $\underline{\mathbf{A}}[K \times_H X] \cong n_H^K \underline{\mathbf{A}}[X] \to n_H^K T$, so $n_H^K T$ is levelwise finitely generated. □

Corollary 7.2. *The norm functors n_H^G preserve levelwise finite generation over* $\mathbf{Z}$ *for all finite $H \leq G$ iff all finite groups G are Tambara finite.*

Adopting the notation of §6, we again consider the following hypothesis on a finite group G:

(†) G satisfies at least one of the following conditions:
- (a) G is a Dedekind group.
- (b) Every proper, nontrivial subgroup of G is maximal, and every subgroup is either normal or satisfies $N_G(H) = H$.
- (c) $G \cong D_8$, the dihedral group of order 8.

Theorem 7.3. *Let $H \leq G$, where G is a finite group satisfying* (†). *Then n_H^G preserves levelwise finite generation.*

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