CLIQUE NUMBER AND RANDOM GRAPHS

DYLAN SPARROW

Abstract. This paper investigates the behavior of the clique number of random graphs. It begins by presenting a famous result of the probabilistic method, whose proof utilizes independence number and random graphs. Then, the paper presents two proofs of the two-point concentration theorem, the latter of which employing The Janson Inequality.

Contents

1. Introduction to Clique Number

A clique with k vertices, denoted K_k, is a graph with the property that all of its vertices are adjacent to each other. The clique number of a graph G, denoted $\omega(G)$, is the number of vertices in the largest clique that is a sub-graph of G. The independence number of G, denoted $\alpha(G)$, is the number of vertices in the largest independent set (a set of vertices where no two vertices are adjacent) of G. The complement of a graph G, denoted $\overline{G}$, is a graph with the same vertices as G defined as follows: two vertices of $\overline{G}$ are adjacent if and only if they are nonadjacent in G. The clique number and independence number of a graph are closely related: $\alpha(G) = \omega(\overline{G})$. We define $\mathscr{G}(n,p)$ as the probability space produced by constructing a graph of n vertices by choosing to add an edge between each pair of vertices independently with probability p (this is the Erdős-Renyi model). We will conclude this paper by using two different proofs to show that, for $G \in \mathscr{G}(n, 1/2)$, the probability that $\omega(G)$ will be one of two values approaches 1 as n approaches infinity. This is known as the two-point concentration of the clique number.

A legal *vertex coloring* of a graph is an assignment of each vertex to a color such that no adjacent vertices are assigned the same color. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colors needed to legally color G.

A *trail* of length $j \geq 3$ is a sequence of vertices $v_1 v_2 v_3 \dots v_j$ of a graph G such that every member is distinct and every pair of adjacent members of the sequence are adjacent in G. A *cycle* of size j is a trail with the property that v_1 and v_j share an edge. The *girth* of a graph, denoted $\text{girth}(G)$, is the length of the smallest cycle in G. By convention, if G does not have any cycles, it has infinite girth.

Note that since a graph with large girth has no cycles of small length, graphs with large girth often have lower chromatic number as the graph is less "interconnected." An interesting question is thus whether we can find graphs with both large girth and large chromatic number.

We will prove that this is in fact possible in our first theorem, which utilizes the independence number of random graphs to show that there actually does exist a graph with any choice of arbitrarily large girth and chromatic number. Before we get to the first theorem, we must prove some facts about chromatic number and cycles.

Proposition 1.1. *For any graph G,*

$$\chi(G) \geq \frac{n}{\alpha(G)}.$$

Proof. Let $\chi(G) = k$. Let ϕ denote any k-coloring of G, and for each i, let S_i denote the set of vertices colored by the ith color. Every S_i must be independent; if two vertices shared an edge, it would not be a legal coloring. Therefore, $\alpha(G) \geq |S_i|$ for all i. Hence,

$$n = \left| \bigcup_{i=1}^{k} S_i \right| \leq \alpha(G)k \quad \Rightarrow \quad \frac{n}{\alpha(G)} \leq k.$$

□

Proposition 1.2. *The number of possible cycles with i vertices that are subgraphs of a graph G with n vertices is $\frac{(n)_i}{2i}$, where $(n)_i := n(n-1)\dots(n-i+1)$.*

Proof. Let A be the set of orderings of any i vertices of G from left to right in a straight line. Note that $|A| = \frac{(n)_i}{2i}$. For $a \in A$, let $f(a)$ be the cycle of length i formed by the edges connecting any two adjacent members a, and the edge connecting the right-most and left-most members of a. We claim that f is a $2i$-to-1 map.

Let $a' \in A$ be a translation of each member of a left or right some number of times. Notice that $f(a)$ is the same cycle as $f(a')$. For each a, there exist i such a''s (including a itself).

Let us denote a in more detail as $v_1v_2v_3v_4 \dots v_{\lceil \frac{i+1}{2} \rceil} \dots v_{i-2}v_{i-1}v_i$. Let $a'' \in A$ be the ordering $v_1v_iv_{i-1}v_{i-2} \dots v_{\lceil \frac{i+1}{2} \rceil} \dots v_4v_3v_2$. Notice that $f(a)$ and $f(a'')$ are the same cycle. Therefore, f is a $2i$-to-1 map. Hence, there are $\frac{(n)_i}{2i}$ possible cycles of length i.

□

Proposition 1.3. *Let $G \in \mathscr{G}(n,p)$. The expected number of cycles of length i in G is*

$$\frac{(n)_i}{2i} p^i.$$

Proof. Let S be the set of possible cycles of length i in G and assign it an arbitrary ordering. By Proposition 1.2, we have $|S| = \frac{(n)_i}{2i}$. Let X_k be a random variable whose value is 1 if the kth cycle in S is present in G, and 0 otherwise (called an "indicator variable"). The expected number of cycles of size i is therefore

$$\mathbb{E}\left[\sum_{k=1}^{|S|} X_k\right] = \sum_{k=1}^{|S|} \mathbb{E}[X_k] = \sum_{k=1}^{|S|} \mathbb{P}[\text{the } k\text{th cycle in } S \text{ is present in } G] = \frac{(n)_i}{2i} p^i.$$

□

Lemma 1.4 (Union Bound)**.** *For any finite probability space Ω, and events $\{A_i\}_{i=1}^n$ in Ω,*

$$\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} \mathbb{P}[A_i].$$

Now we are equipped with the tools needed to prove that there exists a graph with any choice of arbitrarily large girth and chromatic number. The proof famously demonstrates the utility of the probabilistic method—the technique of proving the existence of something by showing it has positive probability.

Theorem 1.5 (Erdős 1959)**.** *There exists a graph G such that $\chi(G) > j$ and* $\mathrm{girth}(G) > k$ *for any j, k.*

Proof. The general strategy for this proof will be to prove the existence of a graph with sufficiently low independence number and sufficiently few cycles of length k or less. We will then remove a vertex from each such cycle, ensuring that the girth of the modified graph is greater than k. Since we removed sufficiently few vertices and since the original graph had sufficiently low independence number, we can apply Proposition 1.1 to show that the chromatic number of the modified graph can be made arbitrarily large.

Let $\theta > 0$ be such that $\theta k < 1$ and let $G \in \mathscr{G}(n, n^{\theta-1})$. By Proposition 1.3 the expected number of cycles whose lengths are less than or equal to k is

$$\mathbb{E}[X] = \sum_{i=3}^{k} \frac{(n)_i}{2i} n^{\theta i - i} \leq \sum_{i=3}^{k} \frac{n^{\theta i}}{2i} \leq \sum_{i=3}^{k} \frac{n^{\theta k}}{2i} \leq k n^{\theta k}.$$

Because $\theta k < 1$, we have that $\mathbb{E}[X] = o(n)$. Therefore, we have that

$$\frac{n}{2} \cdot \mathbb{P}\left[X \geq \frac{n}{2}\right] \leq \mathbb{E}[X] = o(n)$$
$$\mathbb{P}\left[X \geq \frac{n}{2}\right] = o(1).$$

Let $x = \lceil 3n^{1-\theta} \ln n \rceil$. There are $\binom{n}{x}$ distinct sets of size x in G. Any such set is an independent set if and only if each of the $\binom{n}{x}$ pairs of vertices are nonadjacent. This has probability $(1 - n^{\theta-1})^{\binom{x}{2}}$. Therefore, by the Union Bound

$$\mathbb{P}[\alpha(G) \geq x] \leq \binom{n}{x} \left(1 - n^{\theta-1}\right)^{\binom{x}{2}}.$$

Since $1 - n^{\theta-1} \leq e^{-n^{\theta-1}}$, and $\binom{n}{x} \leq n^x$, we have

$$\mathbb{P}[\alpha(G) \geq x] \leq \left(n e^{-n^{\theta-1}(x-1)/2}\right)^x \sim \left(n^{-\frac{1}{2}} e^{\frac{n^{\theta-1}}{2}}\right)^{3n^{1-\theta} \ln n} = o(1).$$

Let A be the event that $X > \frac{n}{2}$ and B be the event that $\alpha(G) \geq x$. Let n_0 be chosen arbitrarily among the integers large enough such that $\mathbb{P}[A] < \frac{1}{2}$ and $\mathbb{P}[B] < \frac{1}{2}$. By the Union Bound $\mathbb{P}[A \cup B] < 1$, which means that $\mathbb{P}[\overline{A} \cap \overline{B}] > 0$ (where $\overline{A}$ and $\overline{B}$ are the complements of A and B respectively). Since $\overline{A} \cap \overline{B}$ has positive probability, there must exist a graph G with n_0 vertices where $X < \frac{n_0}{2}$ and $\alpha(G) < x$. Remove a vertex from each cycle of length at most k to produce the graph G'. Notice that $\text{girth}(G') > k$. By Proposition 1.1,

$$\chi(G') \geq \frac{|V(G')|}{\alpha(G')}.$$

Two vertices in $V(G')$ are adjacent in G' if and only if they are adjacent in G. Therefore, every independent set of G' is an independent set of G. Hence, $\alpha(G) \geq \alpha(G')$. We removed fewer than $\frac{n_0}{2}$ vertices; hence, $|V(G')| > \frac{n_0}{2}$. Thus,

$$\chi(G') \geq \frac{n_0}{2\alpha(G)} = \frac{n_0^{\theta}}{6 \ln n_0}$$

The right-hand side tends to infinity as $n_0 \to \infty$. Since we chose n_0 arbitrarily, we can choose n_0 large enough so that $\chi(G') > j$. □

2. An Important Statistical Theorem

Before progressing further with cliques, we must take a detour and prove Theorem 2.7, a statistical theorem vital to the proofs to come. The results presented after this section will investigate events whose probabilities change as n, the number of vertices in the random graph in question, tends to infinity. If, for an event A where $\mathbb{P}[A]$ depends on n, we have $\lim_{n\to\infty} \mathbb{P}[A] = 1$, then we say it occurs *with high probability* with respect to n. Theorem 2.7 will be valuable because it provides us with conditions on a random variable X—which, in these results, counts the number of occurrences of something important in a random graph—that, if are fulfilled, yields $X > 0$ with high probability. In other words, that important thing exists in the random graph with high probability.

Definition 2.1. The *variance* of a random variable X, denoted $\text{Var}[X]$, is defined as

$$\text{Var}[X] := \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Definition 2.2. The *covariance* of two random variables X and Y, denoted $\text{Cov}[X, Y]$, is defined as

$$\text{Cov}[X, Y] := \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Definition 2.3. Let X be a nonnegative random variable. Let $X = X_1 + \cdots + X_n$, where X_i is the indicator variable for the event A_i, and $\{A_i\}_{i=1}^n$ are symmetric events. For $i \neq k$, we say that $i \sim k$ if A_i and A_k are not independent. Define $S(X)$ as

$$S(X) := \sum_{i \sim 1} \mathbb{P}[A_i | A_1].$$

Proposition 2.4. *Let $X = X_1 + \cdots + X_n$, as in Definition 2.3. Then*

$$\text{Var}[X] \leq \mathbb{E}[X] + \sum_{i \neq k} \text{Cov}[X_i, X_k].$$

Proof. We have

$$\begin{aligned}\mathrm{Var}[X] &= \sum_i \mathbb{E}[X_i^2] + \sum_{i\neq k}\mathbb{E}[X_iX_k] - \sum_i \mathbb{E}[X_i]^2 - \sum_{i\neq k}\mathbb{E}[X_i]\mathbb{E}[X_k] \\ &= \sum_i \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 + \sum_{i\neq k}\mathbb{E}[X_iX_k] - \mathbb{E}[X_i]\mathbb{E}[X_k] \\ &= \sum_i \mathrm{Var}[X_i] + \sum_{i\neq k}\mathrm{Cov}[X_i, X_k].\end{aligned}$$

Notice that $\mathrm{Var}[X_i] = \mathbb{P}[A_i] - \mathbb{P}[A_i]^2 \leq \mathbb{P}[A_i]$. Hence

$$\mathrm{Var}[X] \leq \mathbb{E}[X] + \sum_{i\neq k}\mathrm{Cov}[X_i, X_k].$$

□

Proposition 2.5. $\mathrm{Var}[X] \leq \mathbb{E}[X] + \mathbb{E}[X]S(X)$.

Proof. We will show that $\sum_{i\neq k}\mathrm{Cov}[X_i, X_k] \leq \mathbb{E}[X]S(X)$ and then apply Proposition 2.4. We have that

$$\mathbb{E}[X]S(x) = \sum_{i=1}^{n}\mathbb{P}[A_i]\cdot\sum_{k\sim 1}\mathbb{P}[A_k|A_1] = \sum_{i=1}^{n}\mathbb{P}[A_i]\sum_{k\sim 1}\mathbb{P}[A_k|A_1].$$

Since all A_i are symmetric, $\sum_{k\sim 1}\mathbb{P}[A_k|A_1] = \sum_{k\sim i}\mathbb{P}[A_k|A_i]$ for all fixed i. Thus,

$$\begin{aligned}\mathbb{E}[X]S(x) &= \sum_{i=1}^{n}\mathbb{P}[A_i]\sum_{k\sim i}\mathbb{P}[A_k|A_i] \\ &= \sum_{i=1}^{n}\sum_{k\sim i}\mathbb{P}[A_i]\mathbb{P}[A_k|A_i] \\ &= \sum_{i=1}^{n}\sum_{k\sim i}\mathbb{P}[A_k\cap A_i] \\ &= \sum_{k\sim i}\mathbb{P}[A_k\cap A_i] \\ &= \sum_{k\sim i}\mathbb{E}[X_kX_i]\end{aligned}$$

where the sum ranges over all possible i and k (neither are fixed). Now let's analyze $\sum_{i\neq k}\mathrm{Cov}[X_i, X_k]$. If X_i and X_k are independent, then $\mathrm{Cov}[X_i, X_k] = 0$. Therefore,

$$\sum_{i\neq k}\mathrm{Cov}[X_i, X_k] = \sum_{i\sim k}\mathrm{Cov}[X_i, X_k] = \sum_{i\sim k}\mathbb{E}[X_iX_k] - \mathbb{E}[X_i]\mathbb{E}[X_k] \leq \sum_{k\sim i}\mathbb{E}[X_kX_i].$$

Applying Proposition 2.4 gives

$$\mathrm{Var}[X] \leq \mathbb{E}[X] + \mathbb{E}[X]S(X).$$

□

The last tool we need to prove Theorem 2.7 is Chebyshev's Inequality. We omit the proof here, but a proof can be found in section 7.10 of [3].

Lemma 2.6 (Chebyshev's Inequality)**.** *For any $a > 0$,*

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq a] \leq \frac{\mathrm{Var}[X]}{a^2}.$$

Theorem 2.7. *If all X_i are symmetric, $\lim_{n\to\infty} \mathbb{E}[X] = \infty$, and $S(X) = o(\mathbb{E}[X])$, then $X > 0$ with high probability with respect to n.*

Proof. Dividing Proposition 2.5 through by $\mathbb{E}[X]^2$, we have that

$$\frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \leq \frac{1}{\mathbb{E}[X]} + \frac{S(X)}{\mathbb{E}[X]}. \tag{2.8}$$

By Chebyshev's Inequality,

$$\mathbb{P}[|X - \mathbb{E}[X]| \leq \mathbb{E}[X]] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2}. \tag{2.9}$$

Notice that $\mathbb{P}[X = 0] \leq \mathbb{P}[|X - \mathbb{E}[X]| \geq \mathbb{E}[X]]$. Combining (2.6) and (2.7),

$$\mathbb{P}[X = 0] \leq \frac{1}{\mathbb{E}[X]} + \frac{S(X)}{\mathbb{E}[X]}.$$

As $n \to \infty$, $\mathbb{P}[X = 0] \to 0$. Hence, $\mathbb{P}[X > 0] \to 1$. □

3. First Proof of Two-Point Concentration of Clique Number

With Theorem 2.7 in our repertoire, we return to the clique number of random graphs and provide our first proof of the two-point concentration. That proof will rely on knowledge of the expected number of k-cliques in a random graph of n vertices.

Proposition 3.1. *Let $G \in \mathscr{G}(n, p)$. The expected number of k-cliques in G is*

$$\binom{n}{k} p^{\binom{k}{2}}.$$

Proof. Let S be the set of k-sets of the vertices of G. Notice that $|S| = \binom{n}{k}$. For each k-set, the probability that their induced graph is a clique is $p^{\binom{k}{2}}$, as K_k has $\binom{k}{2}$ edges and each one is selected with probability p. Using an indicator variable argument similar to that in the proof of Proposition 1.3, the expected number of k-cliques in G is

$$\binom{n}{k} p^{\binom{k}{2}}.$$

□

The following theorem investigates the clique number of a random graph $G \in \mathscr{G}(n, 1/2)$ when the expected number of k_n-cliques tends to infinity as n tends to infinity, where $k_n \sim 2\log_2 n$. It is instrumental in our first proof of the two-point concentration because it gives the clique number of G a lower bound: $\omega(G) \geq k_n$ with high probability.

Theorem 3.2. *Let $G \in \mathscr{G}(n, 1/2)$. If k_n is a sequence with $\lim_{n\to\infty} \frac{\binom{n}{k_n}}{2^{\binom{k_n}{2}}} = \infty$ and $k_n \sim 2\log_2 n$, then $\omega(G) \geq k_n$ with high probability with respect to n.*

Proof. We will use k to represent k_n for efficiency. Let X be the random variable whose value is the number k-cliques in G. Let A be the set of k-sets of the vertices of G. Assign an ordering to S, and let S_i denote the ith element of S. Let A_i be the event that the k-clique whose vertices are S_i is present in G, and let X_i be its indicator variable. Then,

$$X = \sum_{i=1}^{\binom{n}{k}} X_i.$$

Let us compute $S(X)$. For $i \sim 1$, it must be that A_i and A_1 share j vertices, where $2 \leq j \leq n-1$. If they shared fewer than 2 vertices, A_i and A_1 would be independent, and if they shared more than $n-1$ vertices, they would be identical. If S_i and S_1 share j vertices, $\mathbb{P}[A_i|A_1] = \frac{1}{2^{\binom{k}{2}-\binom{j}{2}}}$. This is because, since A_1 is given, out of the $2^{\binom{k}{2}}$ edges needed for the k-clique on S_i, $\binom{j}{2}$ are already chosen. Now, let's count the number of possible S_i which share j vertices with S_1. There are $\binom{k}{j}$ vertices they could share, and $\binom{n-k}{k-j}$ ways to choose the remaining $k-j$ vertices of S_i (since they cannot be any of the k vertices in S_1). Hence,

$$S(X) = \sum_{j=2}^{k-1} \binom{k}{j}\binom{n-k}{k-j}\frac{1}{2^{\binom{k}{2}-\binom{j}{2}}}.$$

We intend to show that $S(X) = o(\mathbb{E}[X])$. Let us, then, analyze $\frac{S(X)}{\mathbb{E}[X]}$. We have

$$\frac{S(X)}{\mathbb{E}[X]} = \sum_{j=2}^{k-1} \frac{\binom{k}{j}\binom{n-k}{k-j}}{\binom{n}{k}} 2^{\binom{j}{2}}.$$

This sum is dominated by the terms produced by $j = 2$ and $j = k-1$. When $j = 2$,

$$\frac{S(X)}{\mathbb{E}[X]} = \sum_{j=2}^{k-1} 2\frac{\binom{k}{2}\binom{n-k}{k-2}}{\binom{n}{k}} \sim \frac{\frac{k^2 n^{k-2}}{(k-2)!}}{\frac{n^k}{k!}} \sim \frac{k^4}{n^2} \sim \frac{16(\log_2 n)^4}{n^2} = o(1)$$

because $k \sim 2\log_2 n$. When $j = k-1$,

$$\frac{S(X)}{\mathbb{E}[X]} = \frac{2k(n-k)2^{-k}}{\binom{n}{k}\frac{1}{2^{\binom{k}{2}}}} \sim \frac{2kn2^{-k}}{\mathbb{E}[X]} \sim \frac{4\log_2 n}{n} \cdot \frac{1}{\mathbb{E}[X]} = o(1)$$

since $\lim_{n\to\infty} \mathbb{E}[X] = \infty$. Therefore, $S(X) = o(\mathbb{E}[X])$. We can use Theorem 2.7 because all A_i are symmetric. Hence, $X > 0$ with high probability with respect to n. Thus, there exists a k-clique with high probability. Therefore, $\omega(G) \geq k$ with high probability.

$\square$

Now we are ready for the first proof of the two-point concentration.

Theorem 3.3 (Two-Point Concentration)**.** *There exists a sequence k_n such that $\omega(G) = k_n$ or $\omega(G) = k_n + 1$ with high probability with respect to n.*

Proof. For any $n \geq 5$, the expected number of 3-cliques is greater than 1, and the expected number of n-cliques is less than 1. Thus, for any $n \geq 5$, there exists a unique j_n such that

$$\frac{\binom{n}{j_n}}{2^{\binom{j_n}{2}}} \geq 1 > \frac{\binom{n}{j_n+1}}{2^{\binom{j_n+1}{2}}}.$$

For convenience, we will refer to j_n as j. By Stirling's Formula, which states

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

(proven in [5]), we have that

$$j_n \sim 2\log_2 n.$$

Therefore, the ratio of the expected number $j+1$-cliques to the expected number of j-cliques is

$$\frac{n-j}{j+1}2^{-j} \sim \frac{n-2\log_2 n}{2\log_2 n+1} \cdot \frac{1}{n^2} = \frac{1}{n^{1+o(1)}}. \tag{3.4}$$

We claim that

$$k_n = \begin{cases} j-1 & \text{if} \quad \dfrac{\binom{n}{j+1}}{2^{\binom{j+1}{2}}} \leq \dfrac{1}{\sqrt{n}} \\ \\ j & \text{if} \quad \dfrac{\binom{n}{j+1}}{2^{\binom{j+1}{2}}} > \dfrac{1}{\sqrt{n}}. \end{cases}$$

Let n be sufficiently large so that $n^{\frac{3}{4}} \leq n^{1+o(1)}$ from (3.4). Consider if $\dfrac{\binom{n}{j+1}}{2^{\binom{j+1}{2}}} \leq \dfrac{1}{\sqrt{n}}$. Then,

$$\frac{1}{\sqrt{n}} \geq \mathbb{P}[\text{There exists a clique of size } j+1] = \mathbb{P}[\omega(G) \geq j+1].$$

Hence $\mathbb{P}[\omega(G) \geq j+1] \to 0$. By the definition of j, we have $\dfrac{\binom{n}{j}}{2^{\binom{j}{2}}} \geq 1$. Hence,

$$\frac{\binom{n}{j-1}}{2^{\binom{j-1}{2}}} \geq n^{\frac{3}{4}}.$$

Therefore, $\dfrac{\binom{n}{j-1}}{2^{\binom{j-1}{2}}} \to \infty$. Thus, Theorem 3.2 tells us that $\omega(G) \geq j-1$ with high probability with respect to n. Hence, $\omega(G) = j-1$ or $\omega(G) = j$ with high probability.

Now consider the case $\dfrac{\binom{n}{j+1}}{2^{\binom{j+1}{2}}} > \dfrac{1}{\sqrt{n}}$. In this case:

$$\frac{\binom{n}{j}}{2^{\binom{j}{2}}} > n^{\frac{1}{4}},$$

which means $\frac{\binom{n}{j}}{2^{\binom{j}{2}}} \to \infty$. Therefore, Theorem 3.2 tells us that $\omega(G) \geq j$ with high probability. By the definition of j, we have $1 \geq \frac{\binom{n}{j+1}}{2^{\binom{j+1}{2}}}$. Hence,

$$\frac{1}{n^{\frac{3}{4}}} \geq \frac{\binom{n}{j+2}}{2^{\binom{j+2}{2}}},$$

which means $\mathbb{P}[\omega(G) \geq j+2] \to 0$. Thus, $\omega(G) = j$ or $\omega(G) = j+1$ with high probability.

□

4. Second Proof of Two-Point Concentration of Clique Number

This second proof of the two-point concentration relies on The Janson Inequality. We omit the proof here, but a proof can be found in chapter 8 of [1].

Lemma 4.1 (The Janson Inequality)**.** *Let S be set. Let $A = \{A_1, \dots, A_n\}$ be a collection of subsets of S. Let R be a random subset of S, with p_s being the probability that an arbitrary $s \in S$ is chosen to be in R. For any $A_i \subseteq S$, let B_i be the event that $A_i \not\subseteq R$. Let the random variable X be number of elements of A in R. Then*

$$\prod_{i=1}^{n} \mathbb{P}[B_i] \leq \mathbb{P}\left[\bigcap_{i=1}^{n} B_i\right] \leq \exp\left(-\mathbb{E}[X] + \frac{\mathbb{E}[X]S(X)}{2}\right).$$

Furthermore, if $\mathbb{P}[A_i \subset R] \leq \epsilon = o(1)$ for all $A_i \in A$,

$$\prod_{i=1}^{n} \mathbb{P}[B_i] \sim e^{-\mathbb{E}[X]}.$$

Finally, now equipped with The Janson Inequality, we provide the second proof of the two-point concentration theorem. This proof offers more insight into the two values on which the clique number concentrates.

Proof. Let m'_k be the real number such that

$$\frac{\binom{m'_k}{k}}{2^{\binom{k}{2}}} = 1.$$

Let m_k be the least integer such that

$$\frac{\binom{m_k}{k}}{2^{\binom{k}{2}}} \geq 1.$$

Notice that $m_k \geq m'_k \geq m_k - 1$, which means $m'_k \sim m_k$. Hence

$$\frac{\binom{m_k}{k}}{2^{\binom{k}{2}}} \sim 1,$$

so

$$\frac{\binom{m_k}{k}}{2^{\binom{k}{2}}} = 1 + o(1).$$

Let

$$n_k = m_k \left(1 + \frac{\lambda + o(1)}{k}\right)$$

where λ is some real number. We have that

$$\frac{\binom{n_k}{k}}{2^{\binom{k}{2}}} \sim \frac{m_k^k \left(1 + \frac{\lambda+o(1)}{k}\right)^k}{k! 2^{\binom{k}{2}}} \sim \frac{\binom{m_k}{k}}{2^{\binom{k}{2}}} \left(1 + \frac{\lambda + o(1)}{k}\right)^k \sim \left(1 + \frac{\lambda + o(1)}{k}\right)^k$$

because $\dfrac{\binom{m_k}{k}}{2^{\binom{m}{2}}} = 1 + o(1)$. Therefore,

$$\begin{aligned}
\frac{\binom{n_k}{k}}{2^{\binom{k}{2}}} &= (1 + o(1)) \left(1 + \frac{\lambda + o(1)}{k}\right)^k \\
&= \left[(1 + o(1))^{\frac{1}{k}} \left(1 + \frac{\lambda + o(1)}{k}\right)\right]^k \\
&= \left[(1 + o(1)) \left(1 + \frac{\lambda + o(1)}{k}\right)\right]^k \\
&= \left(1 + \frac{\lambda + o(1)}{k}\right)^k \\
&= e^{\lambda} + o(1)
\end{aligned}$$

Therefore, as k approaches infinity, the expected number of k-cliques in a random graph G with n_k vertices and edge probability $1/2$ approaches e^{λ}. Let us apply The Janson Inequality. Let C be the set of all k-sets of $V(G)$ and assign C an arbitrary ordering, where C_i denotes the ith member of C. In this situation, S from The Janson Inequality is the set of all possible edges of G, R is the set of edges chosen with probability $1/2$, and a particular A_i is the set of all possible edges produced by the vertices in C_i; that is, the edges of the k-clique whose vertices are C_i. Additionally, X is the number of k-cliques in the random graph. Hence, $\mathbb{E}[X] = \frac{\binom{n_k}{k}}{2^{\binom{k}{2}}} \to e^{\lambda}$ which means that $k \sim 2 \log_2 n$. Therefore, we see from the proof of Theorem 3.2 that $S(X) = o(\mathbb{E}[X]) = o(e^{\lambda} + o(1))$. Hence, $S(X) = o(1)$. Since

$$\mathbb{P}[A_i] = \frac{1}{2^{\binom{k}{2}}} = o(1)$$

for all i, The Janson Inequality tells us that

$$\lim_{k \to \infty} \prod_{i=1}^{\binom{n_k}{k}} \mathbb{P}(B_i) = e^{-e^{\lambda}}.$$

Thus, The Janson Inequality further tells us that

$$e^{-e^{\lambda}} + o(1) \le \mathbb{P}\left[\bigcap_{i=1}^{\binom{n_k}{k}} B_i\right] \le \exp\left(\mathbb{E}[X] + \frac{\mathbb{E}[X]S(X)}{2}\right).$$

We know from the proof of Theorem 3.2 that $S(X) = o(1)$. Since $\mathbb{E}[X]$ approaches a constant,

$$e^{-e^{\lambda}} + o(1) \le \mathbb{P}\left[\bigcap_{i=1}^{\binom{n_k}{k}} B_i\right] \le e^{-e^{\lambda}+o(1)}.$$

Thus, $\mathbb{P}\left[\bigcap_{i=1}^{\binom{n_k}{k}} B_i\right] = e^{-e^{\lambda}} + o(1)$. Therefore,

$$\mathbb{P}\left[\omega\left(G\left(n_k, 1/2\right)\right) < k\right] = e^{-e^{\lambda}} + o(1). \tag{4.2}$$

Fix $K > 0$ arbitrarily. Let

$$I_k = \left[m_k\left(1 - \frac{K}{k}\right), m_k\left(1 + \frac{K}{k}\right)\right].$$

Since

$$\frac{\binom{m_k}{k}}{2^{\binom{k}{2}}} \sim 1,$$

Stirling's Formula tells us that

$$m_k^k \sqrt{2}^{k-k^2} \sim \left(\frac{k}{e}\right)^k \sqrt{2\pi k}$$

$$m_k \sim \frac{k}{e\sqrt{2}} \sqrt{2}^k.$$

Hence, $\frac{m_{k+1}}{m_k} \sim \sqrt{2}$. Thus, for any fixed K, there exists a j_K such that for all $k \ge j_K$, $\frac{m_{k+1}}{m_k}$ is sufficiently large and $\frac{K}{k}$ is sufficiently small so that the intervals $\{I_i\}_{i=k-2}^{\infty}$ are disjoint. Consider any $n \ge m_{j_k}$. Let k' be the smallest k such that $m_k \ge n$. Then $m_{k'} \ge m_{j_K}$, which means $k' \ge j_K$ since the sequence m_k is monotonically increasing. Therefore, the intervals $I_{k'-2}, I_{k'-1}, I_{k'}, I_{k'+1}$ are disjoint. Since $m_{k'} \ge n \ge m_{k'-1}$, either $n \in I_{k'}$, $n \in I_{k'-1}$, or n lies between $I_{k'}$ and $I_{k'-1}$.

If $n \in I_{k'}$, then

$$m_{k'-1}\left(1 + \frac{K}{k'-1}\right) \le n \le m_{k'+1}\left(1 - \frac{K}{k'+1}\right).$$

By (4.2), we have that

$$\mathbb{P}\left[\omega\left(G\left(m_{k'-1}\left(1 + \frac{K}{k'-1}\right), \frac{1}{2}\right)\right) < k' - 1\right] = e^{-e^{K}} + o(1).$$

Because $m_{k'-1}\left(1 + \frac{K}{k'-1}\right) \le n$, we have

$$\mathbb{P}\left[\omega\left(G\left(n, 1/2\right)\right) < k' - 1\right] \le e^{-e^{K}} + o(1).$$

Since $k' \to \infty$ as $n \to \infty$, we see that the $o(1)$ term goes to 0 as $n \to \infty$. Moreover, K can be made arbitrarily large, making e^{-e^K} arbitrarily small. Hence, $\omega((G(n, 1/2)) \geq k' - 1$ with high probability with respect to n.

Using similar logic, we have

$$\mathbb{P}\left[\omega\left(G\left(n, 1/2\right)\right) < k' + 1\right] \geq e^{-e^{-K}} + o(1)$$

which implies that $\omega((G(n, 1/2)) < k' + 1$ with high probability with respect to n. Therefore, $\omega((G(n, 1/2)) = k' - 1$ or $\omega((G(n, 1/2)) = k'$ with high probability.

Analogously, if $n \in I_{k'-1}$, we have that $\omega((G(n, 1/2)) = k' - 2$ or $\omega((G(n, 1/2)) = k' - 1$ with high probability.

Now consider if n lies between $I_{k'-1}$ and $I_{k'}$. Then,

$$m_{k'-1}\left(1 + \frac{K}{k'-1}\right) \leq n \leq m_{k'}\left(1 - \frac{K}{k'}\right).$$

Hence,

$$\mathbb{P}\left[\omega\left(G\left(n, 1/2\right)\right) < k' - 1\right] \leq e^{-e^{K}} + o(1)$$

and

$$\mathbb{P}\left[\omega\left(G\left(n, 1/2\right)\right) < k'\right] \geq e^{-e^{-K}} + o(1)$$

which means that $\omega\left(G\left(n, 1/2\right)\right) = k' - 1$ with high probability with respect to n. $\square$

Remark 4.3. It turns out that n falls between $I_{k'-1}$ and $I_{k'}$ more often than it falls in $I_{k'-1}$ or $I_{k'}$. Thus, $\omega(G)$ is concentrated on a single value for most n.

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