FROBENIUS ALGEBRAS AND COHOMOLOGY RINGS WITH POINCARÉ DUALITY

WENYUAN SHI

ABSTRACT. In this paper we discuss how cohomology rings with Poincaré duality have a Frobenius algebra structure. We first introduce the definitions of Frobenius algebras and cohomology rings with Poincaré duality before showing that the latter is in fact an example of the former.

CONTENTS

1. INTRODUCTION

The notion of Poincaré duality comes from the idea of "holes" on manifolds. Specifically, the number of k-dimensional holes and the number of $(n-k)$-dimensional holes on a closed (that is, compact and without boundary) orientable n-manifold are equal. This idea was further developed with the advent of homology and cohomology. The modern form of Poincaré duality states that for a closed (that is, compact and without boundary) orientable n-manifold M, its kth cohomology group is isomorphic to its $(n-k)$th homology group.

On the other hand, Frobenius algebras are a type of algebraic structure that has popped up in a variety of places from physics to computer science. In short, a Frobenius algebra is a finite-dimensional algebra equipped with an associative nondegenerate pairing. Frobenius algebras carry a deep connection to topology.

This paper shall show that the duality provided by Poincaré duality gives rise to an associative nondegenerate pairing on the cohomology ring, thus giving the cohomology ring the structure of a Frobenius algebra.

2. Frobenius algebras

We will review some algebraic preliminaries and establish the notation used in this paper. First, recall the universal property of tensor product: there exists a bilinear map $\tau : V \times W \to V \otimes W$ such that for every bilinear map $f : V \times W \to T$ there exists a unique linear map $\tilde{f} : V \otimes W \to T$ such that $f = \tilde{f} \circ \tau$.

$$\begin{tikzcd} V \times W \arrow[r, "\tau"] \arrow[dr, "f"'] & V \otimes W \arrow[d, dashed, "!\tilde{f}"] \\ & T \end{tikzcd}$$

Thus, having a linear map $V \otimes W \to T$ is equivalent to having a bilinear map $V \times W \to T$. This paper will generally use the notation of tensor products.

Definition 2.1. Given a vector space V over a field K, the **dual space** V^* is another vector space whose elements are the linear forms on V. Given a linear map $f : V \to W$, the **dual map** is $f^* : W^* \to V^*$ by $\psi \mapsto \psi \circ f$.

If V is finite dimensional with a basis $\{e_1, \cdots, e_n\}$, then the basis of V^* is $\{e^1, \cdots, e^n\}$ where $e^i(e_i) = \delta^i_j$. Thus, we see $\dim(V) = \dim(V^*)$ and $V \cong V^*$, however, there is no canonical isomorphism between a vector space and its dual.

Remark 2.2. The linear map $V \to V^{**}$ by $v \mapsto [\psi \mapsto \psi(v)]$ is injective, and it's an isomorphism if V is finite dimensional.

Lemma 2.3. If $f : V \to W$ is injective, then $f^* : W^* \to V^*$ is surjective; and if $f : V \to W$ is surjective, then $f^* : W^* \to V^*$ is injective.

Proof. Say f is injective. Then f has a left inverse g where $g \circ f = 1_V$. Then, we have $(g \circ f)^* = f^* \circ g^* = 1_{V^*}$. Since f^* has a right inverse, it is surjective. Now, say f is surjective. Then, if $f^*(\psi) = \psi \circ f = 0$, it must be that $\psi = 0$, so f^* is injective. □

Definition 2.4. A **pairing** of two vector spaces V and W is a linear map $\beta : V \otimes W \to K$, denoted $v \otimes w \mapsto \beta(v, w)$.

Remark 2.5. For a pairing $\beta : V \otimes W \to K$, we can define the linear maps $\beta_{\mathrm{L}} : W \to V^*$ by $w \mapsto [v \mapsto \beta(v, w)]$ and $\beta_{\mathrm{R}} : V \to W^*$ by $v \mapsto [w \mapsto \beta(v, w)]$. And if we have a map of the form $V \to W^*$ where V and W have finite dimension, then we have a paring $\beta : V \otimes W \to K$.

Lemma 2.6. In the case that V and W are of finite dimension, β_{R} is the dual map of β_{L} (identifying $V \xrightarrow{\cong} V^{**}$) and β_{L} is the dual map of β_{R} (identifying $W \xrightarrow{\cong} W^{**}$).

Proof. By definition, we have $\beta_L^* : V^{**} \to W^*$ by $\omega \mapsto \omega \circ \beta_L$. Composing with the isomorphism $v \mapsto [\psi \mapsto \psi(v)]$ from Remark 2.2, we have a map $V \to V^{**} \to W^*$ by $v \mapsto [\psi \mapsto \psi(v)] \circ [w \mapsto [v \mapsto \beta(v, w)]]$. Observe that this resulting map is the one that sends w to $\beta(v, w)$, which is β_R.

Similarly, we can see β_{L} is the dual map of β_{R}. □

Definition 2.7. The pairing $\beta : V \otimes W \to K$ is nondegenerate in the variable V if V is finite-dimensional and the induced map β_{R} is injective. Similarly, β is nondegenerate in the variable W if W is finite-dimensional and β_{L} is injective. β is simply called **nondegenerate** if it is nondegenerate in both V and W.

Lemma 2.8. Given a pairing $\beta : V \otimes W \to K$ between finite dimensional vector spaces, the following are equivalent:

(i). β is nondegenerate,
(ii). The induced linear map $\beta_{\mathrm{R}} : V \to W^*$ is an isomorphism,
(iii). The induced linear map $\beta_{\mathrm{L}} : W \to V^*$ is an isomorphism.

If we know $\dim(V) = \dim(W)$, then nondegeneracy can also be characterised by:

(ii′). $\beta(v, w) = 0$ for all $v \in V$ implies $w = 0$,
(iii′). $\beta(v, w) = 0$ for all $w \in W$ implies $v = 0$.

Proof. If β is nondegenerate, then both β_L and β_R are injective. By the duality of β_L and β_R, they are also both surjective. Thus, β_L and β_R are isomorphisms. On the other hand, if either β_L or β_R is an isomorphism, the other will also be an an isomorphism by duality. Thus, β will be nondegenerate. □

Definition 2.9. Let K be a field. A ***K*-algebra** is a K-vector space A together with two K-linear maps multiplication $\mu : A \otimes A \to A$, written $x \otimes y \mapsto xy$, and unit $\eta : K \to A$, such that for all $x, y, z \in A$, $(xy)z = x(yz)$ and $Ix = x = xI$ where $I = \eta(1_K)$.

This definition implies that A is a ring. In fact, we can also define a K-algebra as a ring A equipped with a ring homomorphism $\eta : K \to A$.

Definition 2.10. A **right *A*-module** is a vector space M with a K-linear map $M \otimes A \to M$, written $x \otimes a \mapsto x \cdot a$, such that for all $x \in M$ and for all $a, b \in A$, $(x \cdot a) \cdot b = x \cdot (ab)$ and $x \cdot 1 = x$. A **left *A*-module** is a vector space N with a K-linear map $A \otimes N \to N$, written $a \otimes x \mapsto a \cdot x$, such that for all $x \in N$ and for all $a, b \in A$, $a \cdot (b \cdot x) = (ab) \cdot x$ and $1 \cdot x = x$.

Definition 2.11. A K-linear map $\phi : M \to P$ between two right A-modules M and P is called a **right *A*-homomorphism** if for all $x \in M$ and for all $a \in A$, $\phi(x \cdot a) = \phi(x) \cdot a$. A K-linear map $\psi : N \to Q$ between two left A-modules N and Q is called a **left *A*-homomorphism** if for all $x \in N$ and for all $a \in A$, $\psi(a \cdot x) = a \cdot \psi(x)$.

Definition 2.12. For a right A-module M and a left A-module N, a pairing $\beta : M \otimes N \to K$ is called **associative** if for all $x \in M$, for all $y \in N$, and for all $a \in A$, $\beta(x \cdot a, y) = \beta(x, a \cdot y)$.

Definition 2.13. A **Frobenius algebra** is a K-algebra A of finite dimension, with any of the following equivalent structures:

(1). a linear form $\varepsilon : A \to K$ whose nullspace contains no nontrivial left ideals, called the **Frobenius form**,
(2). an associative nondegenerate pairing $\beta : A \otimes A \to K$, called the **Frobenius pairing**,
(3). a left A-isomorphism to its dual,
(4). a right A-isomorphism to its dual.

3. Cohomology and Poincaré Duality

3.1. **Singular Homology and Cohomology.** Poincaré duality is a relation between the homology and cohomology groups of manifolds. First, we will need to define homology and cohomology.

Definition 3.1.1. A **singular n-simplex** in a topological space X is a continuous map $\sigma : \Delta^n \to X$, where Δ^n is the standard n-simplex:

$$\Delta^n = \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} | \Sigma_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}.$$

The map σ need not be a nice embedding and it can have "*singular*ities".

We may designate σ by its vertices by writing $[v_0, \cdots, v_n] = [\sigma(e_0), \cdots, \sigma(e_n)]$ where $[e_0, \cdots, e_n]$ are the vertices of the standard n-simplex. The faces of an n-simplex are the $(n-1)$-simplices $\sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_n]}$ where $\hat{v}_i$ denotes that the vertex v_i is removed. The **boundary map** ∂_n sends an n-simplex to a sum of its faces:

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_n]}.$$

The signs are inserted so the faces have a coherent orientation.

Let $C_n(X)$ be the free abelian group with a basis of the set of all possible n-simplices in X. Elements of $C_n(X)$ are called **singular n-chains**. The boundary map ∂ may be extended to n-chains, in fact, $\partial_n : C_n(X) \to C_{n-1}(X)$ is a homomorphism. Thus, we have a **singular chain complex**:

$$\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots$$

Elements in the kernal of ∂ are called the **cycles**. Elements in the image of ∂ are called the **boundaries**.

Definition 3.1.2. The **n^{th} singular homology group** is the quotient group $H_n(X) = \ker(\partial_n)/\text{im}(\partial_{n+1})$.

In the previous definition, the n-chains are in the form $\sum_i n_i\sigma_i$ where each n_i is an integer and each σ_i is a singular n-simplex. We can extend the definition by allowing the coefficients n_i to come from some other fixed abelian group G rather than $\mathbb{Z}$. The groups of these n-chains are denoted $C_n(X;G)$. The boundary maps ∂ still work for arbitrary G, thus, we can also form chain complexes with $C_n(X;G)$. The resulting homology groups $H_n(X;G)$ are called **homology groups with coefficients in G**.

Given a space X with a subspace $A \subset X$, we denote the quotient group $C_n(X)/C_n(A)$ as $C_n(X,A)$. The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ induces a quotient boundary map $\partial'_n : C_n(X,A) \to C_{n-1}(X,A)$. Thus, we have a chain complex

$$\cdots \to C_{n+1}(X,A) \xrightarrow{\partial'_{n+1}} C_n(X,A) \xrightarrow{\partial'_n} C_{n-1}(X,A) \to \cdots$$

from which we get the **relative homology group** $H_n(X,A) := \ker(\partial'_n)/\text{im}(\partial'_{n+1})$.

We can also form relative homology groups with coefficients in G, denoted $H_n(X,A;G)$, by forming a complex with $C_n(X,A;G) = C_n(X;G)/C_n(A;G)$.

Now, we shall move on to cohomology. Previously we defined dual vector spaces in Definition 2.1. Similarly, we can define the dual of a group G with respect to some other group H as $\text{Hom}(G,H)$. Dual groups are key to the definition of singular cohomolgy.

Fixing an abelian group G, we define the group $C^n(X;G)$ of **singular n-cochains with coefficients in G** as $\text{Hom}(C_n(X),G)$, the dual group of $C_n(X)$. We define the

coboundary map $d_n : C^n(X; G) \to C^{n+1}(X; G)$ as the dual map of ∂_n. Thus, we have a **singular cochain complex**:

$$\cdots \leftarrow C^{n+1} \xleftarrow{d_n} C^n \xleftarrow{d_{n-1}} C^{n-1} \leftarrow \cdots$$

Elements in the kernal of d are called the **cocycles**. Elements in the image of d are called the **coboundaries**.

Definition 3.1.3. The $\boldsymbol{n^{\text{th}}}$ **singular cohomology group with coefficients in $\boldsymbol{G}$** is the quotient group $H^n(X; G) = \ker(d_n)/\operatorname{im}(d_{n-1})$.

Example 3.1.4. Consider a space $A = \{a\}$ with a single point. Clearly, there exists only one singular n-simplex for any n — namely the map $\sigma_n : \Delta^n \to A$ by $x \mapsto a$. Observe that $\partial_n(\sigma_n) = \sum_i (-1)^i \sigma_{n-1}$ is the zero map when n is odd and σ_{n-1} when n is even. Thus, we have the following chain complex:

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$

And the homology groups are

$$\begin{cases} H_n(A) = 0 & \text{if } n > 0 \\ H_0(A) = \mathbb{Z} & \text{if } n = 0. \end{cases}$$

Now consider the cohomology. Let's fix $\mathbb{Z}$ as our abelian group for example, then, $H^n(A; \mathbb{Z}) = \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$; the dual maps of the isomophisms are still isomorphisms and the dual maps of the zero maps are still the zero maps. Thus, we have the following cochain complex:

$$\cdots \xleftarrow{0} \mathbb{Z} \xleftarrow{\cong} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\cong} \cdots \xleftarrow{0} \mathbb{Z} \xleftarrow{\cong} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{0} 0$$

And the cohomology groups are

$$\begin{cases} H^n(A; \mathbb{Z}) = 0 & \text{if } n > 0 \\ H^0(A; \mathbb{Z}) = \mathbb{Z} & \text{if } n = 0. \end{cases}$$

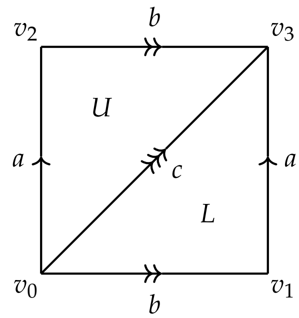
For another example, let's fix $\mathbb{Z}/2\mathbb{Z}$ as our abelian group, then, $H^n(A; \mathbb{Z}/2\mathbb{Z}) = \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$; once again the dual maps of the isomophisms are isomorphisms and the dual maps of the zero maps are zero maps. Thus, we have the following cochain complex:

$$\cdots \xleftarrow{0} \mathbb{Z}/2\mathbb{Z} \xleftarrow{\cong} \mathbb{Z}/2\mathbb{Z} \xleftarrow{0} \mathbb{Z}/2\mathbb{Z} \xleftarrow{\cong} \cdots \xleftarrow{0} \mathbb{Z}/2\mathbb{Z} \xleftarrow{0} 0$$

And the cohomology groups are

$$\begin{cases} H^n(A; \mathbb{Z}/2) = 0 & \text{if } n > 0 \\ H^0(A; \mathbb{Z}/2) = \mathbb{Z}/2\mathbb{Z} & \text{if } n = 0. \end{cases}$$

Example 3.1.5. We shall compute the homology and cohomology groups of a torus T. However, while singular homology and cohomology are useful for theoretical proofs, they are unwieldy for doing computations. It turns out that there is a related theory to singular homology and cohomology — simplicial homology and cohomology — which is isomorphic to the former (see [3]). Consider the following Δ-complex:



Note that v_0, v_1, v_2, v_3 are identified with the same 0-simplex which we shall denote v, they are ordered so as to make the Δ-complex well defined. We have:

$$\begin{aligned}
C_0(T;\mathbb{Z}) &= \langle v \rangle \\
C_1(T;\mathbb{Z}) &= \langle a, b, c \rangle \\
C_2(T;\mathbb{Z}) &= \langle U, L \rangle
\end{aligned}$$

Thus,

$$\begin{aligned}
\ker(\partial_1) &= \langle a, b, c \rangle & \ker(\partial_2) &= \langle U - L \rangle \\
\operatorname{im}(\partial_1) &= 0 & \operatorname{im}(\partial_2) &= \langle a + b - c \rangle
\end{aligned}$$

Thus,

$$\begin{aligned}
H_0(T) &= \langle v \rangle \cong \mathbb{Z} \\
H_1(T) &= \langle a, b, c \rangle / \langle a + b - c \rangle \cong \mathbb{Z}^2 \\
H_2(T) &= \langle U - L \rangle \cong \mathbb{Z}
\end{aligned}$$

Now we consider the cohomology. We use the $*$ to denote the duals, e.g. a^* is the map that sends a to 1 and b and c to 0. We have:

$$\begin{aligned}
C^0(T;\mathbb{Z}) &= \langle v^* \rangle \\
C^1(T;\mathbb{Z}) &= \langle a^*, b^*, c^* \rangle \\
C^2(T;\mathbb{Z}) &= \langle U^*, L^* \rangle
\end{aligned}$$

Thus,

$$\begin{aligned}
\ker(d_0) &= \langle v^* \rangle & \ker(d_1) &= \langle a^* + c^*, b^* + c^* \rangle \\
\operatorname{im}(d_0) &= 0 & \operatorname{im}(d_1) &= \langle U^* - L^* \rangle
\end{aligned}$$

Thus,

$$\begin{aligned}
H^0(T;\mathbb{Z}) &= \langle v^* \rangle \cong \mathbb{Z} \\
H^1(T;\mathbb{Z}) &= \langle a^* + c^*, b^* + c^* \rangle \cong \mathbb{Z}^2 \\
H^2(T;\mathbb{Z}) &= \langle U^*, L^* \rangle / \langle U^* - L^* \rangle = \langle U^* \rangle \cong \mathbb{Z}
\end{aligned}$$

3.2. **Cohomology Rings.** Singular cohomology is defined using singular homology. In this section we shall see that singular cohomology in fact has some additional structure than singular homology if the group of coefficients is a ring.

Definition 3.2.1. For a space X and a ring R, for the cochains $\phi \in C^p(X; R)$ and $\psi \in C^q(X; R)$, the **cup product** $\phi \smile \psi \in C^{p+q}(X; R)$ is defined as follows:

$$(\phi \smile \psi)(\sigma) = \phi(\sigma|_{[v_0,\cdots,v_p]})\psi(\sigma|_{[v_p,\cdots,v_{p+q}]}).$$

The coboundary of a cup product is given by:

$$d(\phi \smile \psi) = d\phi \smile \psi + (-1)^p \phi \smile d\psi.$$

We can see that the cup product of two cocycles is another cocycle, and the cup product of a cocycle and a coboundary is a coboundary. By construction, the cup product is bilinear. Thus, this induces a tensor product on cohomology groups:

$$\smile : H^p(X; R) \otimes H^q(X; R) \to H^{p+q}(X; R).$$

Definition 3.2.2. The direct sum $H^*(X; R) = \bigoplus_i H^i(X; R)$ together with the cup product is a graded ring, called the **cohomology ring** of X.

Example 3.2.3. The cohomology ring of a space A with one point from Example 3.1.4 is $\mathbb{Z}$ concentrated in the zeroth degree. More specifically, it's the free group generated by the cochain $\phi : C_0(A) \to \mathbb{Z}$ where $\sigma_0 \mapsto 1$. For the cup product, we have $\phi \smile \phi = \phi$.

Example 3.2.4. Consider the cohomology groups of the torus from 3.1.5. Observe that if ϕ is any cochain, then $v^* \smile \phi = \phi \smile v^* = \phi$.

Now, we can calculate the cup product of two elements in $H^1(T, \mathbb{Z})$:

$$((a^* + c^*) \smile (b^* + c^*))(U) = (a^* + c^*)(a)(b^* + c^*)(b) = 1$$
$$((a^* + c^*) \smile (b^* + c^*))(L) = (a^* + c^*)(b)(b^* + c^*)(a) = 0$$

Thus, we have $(a^* + c^*) \smile (b^* + c^*) = U^*$.

3.3. **Poincaré Duality.** Under certain conditions, which we will now explain, the structure of the homology and cohomology groups of orientable closed manifolds present an interesting symmetry.

Definition 3.3.1. An ***n*-dimensional manifold without boundary**, or n-manifold, is a Hausdorff space M which is locally homeomorphic to $\mathbb{R}^n$. A compact manifold without boundary is called **closed**.

For any point p in a manifold M, there is a neighborhood U of p and a homeomorphism $\phi : U \to \phi(U) \in \mathbb{R}^n$ called a **coordinate chart**. We denote the chart by (U, ϕ). A family of charts $\{(U_\alpha, \phi_\alpha)\}$ which covers M is called an **atlas**. For two charts (U, ϕ) and (V, ψ) that overlap, there is a **transition function** $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ which is a homeomorphism.

Definition 3.3.2. A **local orientation** of an n-manifold M at a point p is a choice of generator for the group $H_n(M, M \setminus \{p\}) \cong \mathbb{Z}$.

A transition function $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is said to be orientation preserving if for each p in $U \cap V$, it fixes the generators of $H_N(U \cap V, U \cap V \setminus \{p\})$ which is isomorphic to $H_n(M, M \setminus \{p\})$ by excision.

Definition 3.3.3. A manifold M is called **orientable** if it admits an atlas for which all the transition functions are orientation preserving. An **orientation** for M is a choice of atlas which satisfies the previous condition.

More generally, we can consider $H_n(M, M \setminus \{p\}; R) \cong R$ for any commutative ring R with identity. Then, an ***R*-orientation** is a choice of atlas for M where all the transition functions fix the generators of $H_n(M, M \setminus \{p\}; R)$, and M is called ***R*-orientable** if it admits such an atlas. It can be shown that M is R-orientable if and only if for all $p \in M$, the map $H_n(M; R) \to H_n(M, M \setminus \{p\}; R) \cong R$ is an isomorphism. An element of $H_n(M; R)$ whose image in $H_n(M, M \setminus \{p\}; R)$ is a generator for all p is called a **fundamental class** of M.

Definition 3.3.4. For a space X and a ring R, for the chain $\sigma \in C_p(X; R)$ and cochain $\phi \in C^q(X; R)$, the **cap product** $\sigma \frown \phi \in C_{p-q}(X; R)$ is defined as follows:

$$\sigma \frown \phi = \phi(\sigma|_{[v_0, \cdots, v_q]})\sigma|_{[v_q, \cdots, v_p]}.$$

The boundary of a cap product is given by:

$$\partial(\sigma \frown \phi) = (-1)^q(\partial\sigma \frown \phi - \sigma \frown d\phi).$$

We can see that the cap product of a cycle and a cocycle is a cycle, the cap product of a cycle and a coboundary is a boundary, and the cap product of a boundary and a cocycle is a boundary. Thus, this induces a product:

$$\frown : H_p(X; R) \otimes H^q(X; R) \to H_{p-q}(X; R).$$

Theorem 3.3.5. (Poincaré duality). If M is an R-orientable closed n-manifold, with a fundemental class $[M] \in H_n(M; R)$, the map

$$\begin{aligned} D : H^p(M; R) &\to H_{n-p}(M; R) \\ \phi &\mapsto [M] \frown \phi \end{aligned} \tag{1}$$

is an isomorphism for all p.

The universal coefficient theorem (see [3]) provides a stronger duality if R is a field, in that case, we have $(H_n(X; R))^* = \mathrm{Hom}(H_n(X; R), R) \cong H^n(X; R)$.

4. Frobenius Algebra structure of cohomology rings

Finally, we shall show that cohomology rings with Poincaré duality are examples of Frobenius algebras. We will take K to be a field. First, we shall define some useful isomorphisms.

The **Kronecker index** is the map

$$\langle -, - \rangle : H_*(M; K) \otimes H^*(M; K) \to K$$

where $\langle \sigma, \phi \rangle := \phi(\sigma)$ if $\sigma \in H_p(M; K)$ and $\phi \in H^q(M; K)$ where $p = q$, and $\langle \sigma, \phi \rangle := 0$ otherwise. By Remark 2.2, We have the isomorphism

$$\begin{aligned} ** : H_{n-p}(M; K) &\to (H_{n-p}(M; K))^{**} \\ \sigma &\mapsto \langle \sigma, - \rangle. \end{aligned} \tag{2}$$

We also note that the cap product, cup product, and evaluation pairing are related by the identity

$$\langle \phi \frown \sigma, \psi \rangle = \langle \sigma, \phi \smile \psi \rangle$$

where σ is a chain and ϕ and ψ are cochains, i.e. the cap and cup products are adjoint. Thus, we can define an isomorphism

$$\begin{aligned} ad : (H_{n-p}(M; K))^{**} &\to (H^{n-p}(M; K))^* \\ \langle \phi \frown [M], - \rangle &\mapsto \langle [M], \phi \smile - \rangle \end{aligned} \tag{3}$$

Finally, we can prove the main theorem:

Theorem 4.1. Cohomology rings with Poincaré duality are Frobenius algebras.

Proof. We can compose the isomorphisms (1), (2), and (3) to get an isomorphism

$$\begin{aligned} ad \circ ** \circ D : H^p(M;K) &\to (H^{n-p}(M;K))^* \\ \phi &\mapsto \langle [M], \phi \smile - \rangle. \end{aligned}$$

By Remark 2.5, we have a pairing

$$\begin{aligned} \beta : H^p(M;K) \otimes H^{n-p}(M;K) &\to K \\ \phi \otimes \psi &\mapsto \langle [M], \phi \smile \psi \rangle. \end{aligned}$$

Thus, we have a pairing on the cohomology ring

$$\begin{aligned} \beta : H^*(M;K) \otimes H^*(M;K) &\to K \\ \phi \otimes \psi &\mapsto \langle [M], \phi \smile \psi \rangle. \end{aligned}$$

Bilinearity and associativity of β follow from the fact that the cup product is bilinear and associative:

$$\beta(\phi \smile \psi, \omega) = \langle [M], \phi \smile \psi \smile \omega \rangle = \beta(\phi, \psi \smile \omega).$$

Since $ad \circ ** \circ D = \beta_R$ is an isomorphism, β is nondegenerate by Lemma 2.8. Thus, cohomology rings with Poincaré duality are Frobenius algebras. □

Acknowledgements

I would like to thank Professor Carmen Rovi, Yuqin Kewang, and Professor Peter May for their guidence though this REU. I would especially like to thank Professor Rovi for originally introducing the idea of Frobenius algebras to me in her talks, and for her support in putting the project together. I would also like to thank Professor May in organising the REU. Finally, I would like to thank all my mentors for putting up with me in a rather difficult time in my life.

References

[1] Joachim Kock. *Frobenius Algebras and 2D Topological Quantum Field Theories.* Cambridge University Press. 2003.

[2] J. P. May. *A Concise Course in Algebraic Topology.*

[3] Allen Hatcher. *Algebraic Topology.* Cambridge University Press. 2002.