GROUND STATES, CONCENTRATION COMPACTNESS, AND BLOW-UP DYNAMICS IN THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. This paper studies the focusing nonlinear Schrödinger (NLS) equation, where solutions may either exist globally or blow up in finite time. The analysis relies on the existence of ground state (or energy-minimizing) solutions, which we show are also related to the sharp constant in the Gagliardo-Nirenberg inequality. Using the concentration compactness principle, we establish the existence of such ground states by proving convergence of a minimizing sequence. We then show how these ground states determine a threshold for global existence versus finite-time blow-up in the mass-critical case, connecting variational methods with the dynamics of the NLS equation. We assume the reader is familiar with basic real and functional analysis, as well as Sobolev spaces and their properties.

Contents

Date: October 5, 2025.

1. Introduction

The nonlinear Schrödinger (NLS) equation is one of the most fundamental equations in the study of mathematical physics and partial differential equations. It is a nonlinear version of the classical linear Schrödinger equation, with applications in geometric optics, fluid dynamics, and Bose-Einstein condensates.

1.1. **Analyzing the Equation.** The focusing NLS equation is given by:

$$\begin{cases} iu_t + \Delta u = -|u|^{p-1}u \text{ on } \mathbb{R}^d_x \times \mathbb{R}_t, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^d) \end{cases} \tag{1.1}$$

where $p \in (1,\infty)$, and we assume that it is satisfied in the weak sense. Two exponents play a special role: the *mass-critical* exponent $p_m = 1 + \frac{4}{d}$, and (for $d \geq 3$) the *energy-critical* exponent $p_e = 1 + \frac{4}{d-2}$. These exponents arise from the scaling symmetry $u(t,x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$: the mass-critical exponent corresponds to invariance of the L^2 (mass) norm, and the energy-critical exponent (when $d \geq 3$) to invariance of the $\dot{H}^1$ (energy) norm.

The minus sign on the term on the right-hand side indicates that we are considering the *focusing* nonlinearity (as opposed to the *defocusing* nonlinearity, which would have a plus sign instead). We will analyze this difference further in Section 2, where we also discuss other key properties of the equation such as conservation of energy and mass, and the different regimes of criticality depending on the parameter p. However, we can already glean from the equation why the focusing case is more intricate and can yield a richer and more intricate theory: the Laplacian term Δu is dispersive, i.e. it tends to spread out and regularize the solution, while the focusing nonlinearity instead tends to concentrate the solution.

One way to gain intuition for this is to consider smooth initial data $u_0 = Af$ and let $A \to \infty$. Then $|\Delta u| \asymp A$ while $\left||u|^{p-1}u\right| \asymp A^p$, so the nonlinearity term will dominate. Heuristically, the solution should initially be close to that of $iu_t = -|u|^{p-1}u$, which is $u(t,x) = Ae^{itA^{p-1}|f(x)|^{p-1}} f(x)$. Integration by parts shows that for any domain Ω, we have

$$\frac{d}{dt}\int_\Omega |u|^2 = \int_{\partial\Omega} \vec{N} \cdot 2\,\mathrm{Re}(-\overline{u} i \Delta u),$$

where $\vec{N}$ is the inward-pointing normal vector to $\partial\Omega$ [4]. Plugging in our approximate solution for small t, this becomes

$$\frac{d}{dt}\int_\Omega |u|^2 \approx \int_{\partial\Omega} \vec{N} \cdot 2A^{p+1} t |f|^2 \nabla(|f|^{p-1}).$$

Thus, if f is a localized bump so that $\nabla(|f|^{p-1})$ points inward on $\partial\Omega$, then we expect the mass in Ω to increase for small t. This illustrates how the focusing nonlinearity is concentrating the solution – and if we used the defocusing nonlinearity, we would end up with an extra minus sign and the opposite effect. This also helps provide intuition for the terminology 'focusing' and 'defocusing.'

This tension between spreading from Δu and concentrating from the focusing nonlinearity $-|u|^{p-1}u$ naturally leads to the question of whether there exist special solutions that balance these two effects. It turns out that these indeed exist, and are referred to as *solitons*.

1.2. **Solitons, Ground States, and the Weinstein Functional.** Solitons will play a central role in our analysis of the NLS equation. We will be able to show that these are linked to maximizers of a variational quantity called the Weinstein functional, defined as follows:

$$W(u) = \frac{\int_{\mathbb{R}^d} |u|^{p+1}}{\left(\int_{\mathbb{R}^d} |u|^2\right)^{1-\frac{(d-2)(p-1)}{4}} \left(\int_{\mathbb{R}^d} |\nabla u|^2\right)^{\frac{d(p-1)}{4}}}. \tag{1.2}$$

The *Gagliardo-Nirenberg inequality* [14] [15] states exactly that W is bounded from above, i.e., there exists a constant $C_{d,p}$ such that

$$\|u\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} \leq C_{d,p} \left(\int_{\mathbb{R}^d} |u(x)|^2 \, \mathrm{d}x\right)^{1-\frac{(d-2)(p-1)}{4}} \left(\int_{\mathbb{R}^d} |\nabla u(x)|^2 \, \mathrm{d}x\right)^{\frac{d(p-1)}{4}}. \tag{1.3}$$

In fact, the best constant $C_{d,p}$ is exactly $W_{max} = \sup\{W(u) \ : \ 0 \neq u \in H^1(\mathbb{R}^d)\}$. In Section 2, we will prove that a maximizer Q of the Weinstein functional satisfies an equation of the form

$$\Delta Q + \alpha |Q|^{p-1} Q = \beta Q,$$

for some $\alpha, \beta > 0$, and that this is also exactly the form we need for Q to be a soliton solution to the NLS equation. Such maximizers are called *ground state* solutions; a key challenge is proving their existence.

1.3. **Concentration Compactness and Existence of Maximizers.** In variational problems such as proving the existence of a maximizer to the Weinstein functional, a common strategy is to take a maximizing sequence and attempt to prove that it has a subsequence converging to a maximizer. When we have compactness, this is much more straightforward – for example, when our sequence is bounded in a Sobolev norm and we are working on a bounded domain, we can use the Rellich-Kondrachov compactness theorem to get a convergent subsequence that has an admissible limit. However, in problems lacking compactness (such as ours), the concentration compactness principle provides a framework to analyze the limiting behavior of our sequence.

To gain intuition, let's begin by considering a sequence of functions g_n in $L^2(\mathbb{R}^d)$ such that $g_n \rightharpoonup g$ weakly, but not strongly, such that $\|g_n - g\|_{L^2(\mathbb{R}^d)} = 1$ for all n. Equivalently, we can consider functions $f_n = g_n - g$ which weakly converge to 0 but have a constant (unit) $L^2(\mathbb{R}^d)$ norm. In Figure 1 we illustrate four key ways this can happen: vanishing (the function spreads out everywhere), concentration (the function concentrates at a point, like a Dirac δ function), translation (the function shifts off to infinity), and oscillations (the function oscillates more and more rapidly). Of course, via addition and composition of these four examples, we can create many more functions that weakly converge to zero but have constant norm.

In Section 3, we will see how a principle called Concentration Compactness tells us that in certain settings, a subsequence of our functions must exhibit one of the following behaviors: vanishing, concentration, or dichotomy (concentrating into multiple bubbles). Therefore, we can extract a strongly convergent subsequence provided we can rule out the behaviors that we don't want (vanishing and dichotomy). This will then allow us in Section 4 to prove the existence of maximizers

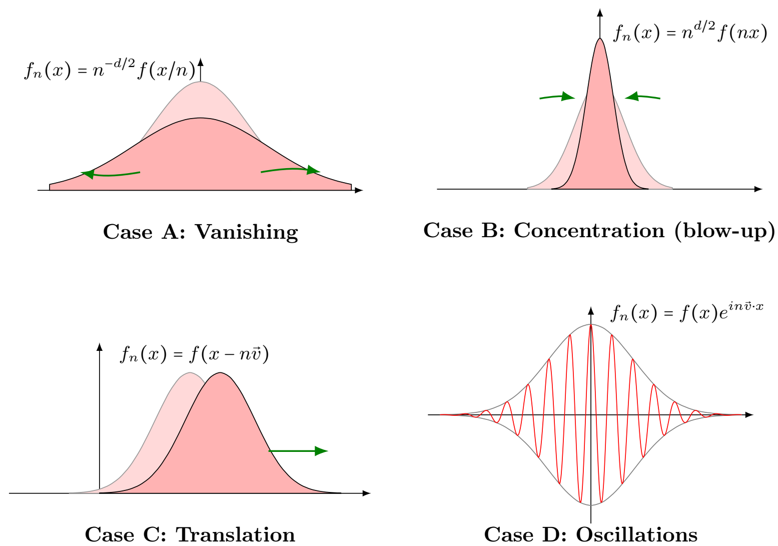

FIGURE 1. Examples of behaviors of function sequences such that $f_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d)$ while $\|f_n\|^2_{L^2(\mathbb{R}^d)} = \|f\|^2_{L^2(\mathbb{R}^d)}$ stays constant. (Adapted from [3].)

to the Weinstein functional, which give us the ground state solutions Q we will need in our further study of the NLS equation.

1.4. **Well-Posedness and Blow-up.** In Section 5, we will cover the theory of local well-posedness for the NLS equation, following the exposition in [12] using Strichartz estimates. In particular, this will immediately imply global well-posedness in the defocusing case. However, the focusing case is more intricate, and we will explore it in Section 6.

In particular, for the mass-critical exponent $p = p_m$, we will see how the mass of the ground state Q provides a threshold for global existence and blow-up in finite time. This will culminate in the following theorem, which partially characterizes when blow-up occurs:

Theorem 1.4. *Suppose $u_0 \in H^1(\mathbb{R}^d)$ and let $u(t,x)$ be the corresponding solution to the mass-critical focusing NLS equation with maximal lifespan $I \subseteq \mathbb{R}$ and initial data $u(0,x) = u_0(x)$. We denote by $M(u)$ and $E(u)$ the mass and energy functionals for NLS, which will be defined later in (2.1) and (2.2) respectively. Let Q be a ground state solution, i.e., a maximizer of the Weinstein functional (1.2). Then:*

(1) If the initial energy

$$E(u_0) = \frac{1}{2}\|\nabla u_0\|^2_{L^2} - \frac{1}{p+1}\|u_0\|^{p+1}_{L^{p+1}}$$

is negative, then there exists a time $T < \infty$ such that $\lim_{t\to T}\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = \infty$. (The solution blows up in finite time.)

(2) If the initial mass

$$M(u_0) = \|u_0\|_{L^2(\mathbb{R}^d)}$$

is less than $\|Q\|_{L^2(\mathbb{R}^d)}$*, then* $I = \mathbb{R}$ *and* $\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1(\mathbb{R}^d)} < \infty$*. (The solution exists globally in time.)*

2. Basic Properties of the NLS Equation

In this section, we will cover basic and essential properties of the nonlinear Schrodinger equation in more depth. We will mainly follow the conventions in [6], but the ideas here can also be found in [7] and [8]. We will also learn why the focusing case is more interesting than the defocusing case, and how the focusing nonlinearity leads to the existence of solitons.

2.1. Conservation and Criticality.

Consider a solution $u(t,x)$ to (1.1). We can define the following conserved quantities:

$$M(u(t)) = \int_{\mathbb{R}^d} |u(t,x)|^2\, dx, \tag{2.1}$$

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(t,x)|^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^d} |u(t,x)|^{p+1}\, dx. \tag{2.2}$$

where the first is the *mass* and the second is the *energy* of the solution. One can show these are conserved in time by differentiating under the integral sign. The two terms in the energy are the kinetic and potential energies, respectively, with the kinetic energy being the $\dot{H}^1$ seminorm of u.

Additionally, for any $\lambda > 0$, $u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$ is also a solution to the equation. This scaling invariance allows us to define notions of criticality. Specifically, a homogeneous Sobolev norm scales like

$$\|u_\lambda\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{\frac{2}{p-1}+s-\frac{d}{2}} \|u\|_{\dot{H}^s(\mathbb{R}^d)}.$$

The *critical regularity* s_c is the exponent for which the $\dot{H}^{s_c}$ norm is invariant under the scaling:

$$s_c = \frac{d}{2} - \frac{2}{p-1}.$$

- The problem is *mass-critical* exactly when $s_c = 0$, or when $p = 1 + \frac{4}{d}$. In this case, the conserved mass $M(u)$, the L^2 norm, is scale-invariant. It is constant under the rescaling $u \mapsto u_\lambda$.
- The problem is *energy-critical* exactly when $s_c = 1$, or when $p = 1 + \frac{4}{d-2}$ for $d \geq 3$. Under the scaling $u \mapsto u_\lambda$,

  $$\int |\nabla u_\lambda|^2 \mapsto \lambda^{\frac{4}{p-1}+2-d} \int |\nabla u|^2, \qquad \int |u_\lambda|^{p+1} \mapsto \lambda^{\frac{2(p+1)}{p-1}-d} \int |u|^{p+1}.$$

  These two powers coincide precisely when $s_c = 1$, i.e., when $p = 1 + \frac{4}{d-2}$. Thus, in the energy-critical case the two terms in $E(u)$ scale the same; equivalently, the energy is invariant under the scaling that leaves the $\dot{H}^1$ norm invariant.

Criticality is important since it can indicate what kind of behavior we expect. The subcritical case will have the best behavior – intuitively, if the quantity that is conserved by the scaling is coercive and can be used to control other norms such as the H^1 norm, then we can use it to prevent or prove blow-up or global existence. Furthermore, the energy subcritical case can often be easier to analyze since we might have access to tools like the Rellich-Kondrachov compactness theorem that we wouldn't in the energy (H^1) critical case.

The critical case delicately balances the two regimes and can be harder to analyze, but also more interesting with the possibility of blow-up. In particular, the mass critical case (which is still energy subcritical) is fascinating as algebraic simplifications will allow for elegant proofs of thresholds for blow-up and global existence in Section 6. We can expect ill-posedness in the supercritical case.

The energy also hints us at one essential difference between the focusing and defocusing nonlinearities: in the defocusing case, the energy is always non-negative since the second term in (2.2) would have a plus sign instead of the given minus sign. In that case, conservation of mass and energy would give us a uniform bound on the H^1 norm of u. In the focusing case, however, the energy can be negative, which is more interesting and allows the possibility of blowup in finite time.

2.2. **Solitons.** In this paper, we choose to consider the focusing nonlinearity which involves a minus sign in front of the second term in (1.1) as well as in the energy (2.2). As we saw earlier, the Laplacian term Δu in (1.1) is dispersive and tends to spread out and regularize the solution, while the focusing nonlinearity tends to concentrate the solution. In the focusing case, there exist balanced solutions where these two effects are in equilibrium, defined as follows:

Definition 2.3 (Soliton)**.** u is a *soliton* solution to the focusing NLS equation if, for some $Q \in H^1(\mathbb{R}^d_x)$ and $\tau > 0$, we have

$$u(x,t) = Q(x)e^{it\tau}.$$

Intuitively, these solitons are wave packets that maintain their shape over time[1]. The exponential factor above only contributes a time-dependent phase, so the spatial profile of the solution given by Q doesn't disperse or change its shape.

By plugging this into (1.1), we can see that the soliton Q must satisfy

$$\Delta Q + |Q|^{p-1}Q - \tau Q = 0.$$

We can now show that we don't have any soliton solutions in the defocusing case where the second term above would instead have a minus sign (keeping in mind our convention that $\tau > 0$):

Lemma 2.4. *Consider a solution $Q \in H^1(\mathbb{R}^d)$ to the equation*

$$\Delta Q + \alpha|Q|^{p-1}Q = \beta Q,$$

for $d \geq 3, \frac{3}{2} - \frac{2}{p-1} < 1$. If there exists a solution Q with $\alpha \leq 0 \leq \beta$, then Q must vanish.

Proof. We will use the energy method here, which involves multiplying the equation by Q and integrating by parts. Specifically, we have

$$\int \Delta Q \cdot Q + \int \alpha|Q|^{p-1}Q^2 = \int \beta Q^2,$$

and after integration by parts we get

$$\int |\nabla Q|^2 = \alpha \int |Q|^{p+1} - \beta \int |Q|^2.$$

If $\alpha \leq 0 \leq \beta$, then the right-hand side is nonpositive, so $\int |\nabla Q|^2 = 0$. Thus, Q is constant a.e., and by plugging this into the equation, we find that Q must be zero.

[1]There are also more general *traveling* solitons that preserve their form while moving through space, but we will focus on the standing wave case given here.

(Note that the boundary term from integration by parts vanishes here, since Q is in $H^1(\mathbb{R}^d) = H_0^1(\mathbb{R}^d)$, so we can approximate it by $C_c^\infty(\mathbb{R}^d)$ functions.) □

2.3. **Ground States and the Weinstein Functional.** It turns out that there is an alternate way to characterize solitons. In particular, consider $1 < p < 1 + \frac{4}{d-2}$ for $d \geq 3$ (energy subcritical), then define the Weinstein functional [17] as:

$$W(u) \coloneqq \frac{\int_{\mathbb{R}^d} |u|^{p+1}}{\left(\int_{\mathbb{R}^d} |u|^2\right)^{1-\frac{(d-2)(p-1)}{4}} \left(\int_{\mathbb{R}^d} |\nabla u|^2\right)^{\frac{d(p-1)}{4}}}. \tag{2.5}$$

As covered in the introduction, the maximum $W_{max} = \sup\{W(u) \,:\, 0 \neq u \in H^1(\mathbb{R}^d)\}$ is exactly the best constant $C_{d,p}$ in the Gagliardo-Nirenberg inequality (1.3). It turns out that such a maximizer will also satisfy an equation of the precise form we need.

Lemma 2.6. *A maximizer Q of the Weinstein functional W satisfies the following equation for some α, β:*

$$\Delta Q + \alpha |Q|^{p-1} Q = \beta Q.$$

Proof Sketch. We will derive the Euler-Lagrange equation for the Weinstein functional $W(u)$ in the standard way, by computing its first variation. By scaling invariance, we can normalize the maximizer Q such that $\|Q\|_{L^2} = 1$ and $\|\nabla Q\|_{L^2} = 1$, simplifying the denominator of $W(Q)$ to 1.

Now, consider the perturbation $Q^\epsilon = Q + \epsilon u$ and set the derivative at $\epsilon = 0$ to zero. The derivatives of the numerator $N(u)$ and denominator $D(u)$ are:

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} N(Q^\epsilon) = (p+1) \int |Q|^{p-1} Q\, u$$
$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} D(Q^\epsilon) = 2a \int Q\, u + 2b \int \nabla Q \cdot \nabla u$$

The maximizer condition $\frac{d}{d\epsilon} W(Q^\epsilon)|_{\epsilon=0} = 0$ simplifies to $N'(Q) - N(Q) D'(Q) = 0$, which yields:

$$(p+1) \int |Q|^{p-1} Q\, u - N(Q) \left[2a \int Q\, u + 2b \int \nabla Q \cdot \nabla u \right] = 0.$$

We then use integration by parts on the last term to move the derivative from the test function u onto Q. The integral equation we get will be of the form $\int [\dots] u \, \mathrm{d}x = 0$, and since it must hold for all u, the integrand must be zero. This gives the pointwise equation:

$$(p+1) |Q|^{p-1} Q - 2a\, N(Q)\, Q + 2b\, N(Q)\, \Delta Q = 0.$$

Rearranging this expression and defining constants α and β gives the equation we want. □

This is quite a surprising result, that the optimal constant in the Gagliardo-Nirenberg inequality is attained by a function that satisfies the same equation as soliton solutions to the NLS equation! A heuristic explanation of this is that solitons arise from the balance of the kinetic and potential energy terms in the energy (2.2). The kinetic energy term is related to the $\dot{H}^1$ norm, while the potential energy term is related to the L^{p+1} norm, and conservation of mass keeps the L^2 norm constant.

This is more clearly linked to the structure of the Gagliardo-Nirenberg inequality, which controls the L^{p+1} norm in terms of the L^2 and $\dot{H}^1$ norms.

3. Concentration Compactness Principle

This section will develop some important lemmas and ideas we will need in order to prove the existence of a maximizer to the Weinstein functional.

As we saw in Figure 1 in the introduction, sequences of functions can weakly converge but still exhibit many types of behavior other than concentration. Concentration compactness was originally introduced by P.L. Lions in [9] to simplify analyzing the limiting behavior of such sequences of functions. An overview of the abstract concentration compactness lemma is covered in Section 3 of [10], with the main idea being that a sequence of functions of constant L^p norm must have a subsequence that either vanishes, concentrates (compactness), or splits into two or more parts (dichotomy). In this paper, rather than using this lemma as a black box, we will take a more modern approach inspired by [3] and develop concentration compactness as a broader strategy: to prove that a sequence has a convergent subsequence, we will first rule out two possible obstructions to convergence, vanishing and dichotomy (splitting into multiple bubbles).

To rule out vanishing, we will introduce the idea of *weak L^2 mass*, which measures the largest possible L^2 mass weak limits can have under certain transformations, and relate it to other forms of vanishing (for example, vanishing in certain L^p norms). If we are able to rule out some equivalent form of vanishing, we will know the weak L^2 mass is nonzero, which will allow us to extract a nontrivial weak limit after taking a subsequence and applying some transformations.

Dichotomy, in turn, will be ruled out by forcing a contradiction with subadditivity properties of the functional we are maximizing. This will leave us with only one remaining possibility: the sequence is tight (up to transformations), and we can extract a convergent subsequence, which will converge to a maximizer.

We begin by considering a very important lemma that will also aid us in ruling out dichotomy. When we have a sequence of functions f_k (bounded in $L^p(U)$) converging pointwise to some limit f, Fatou's lemma can give us a lower bound on the limit of the integrals or norms of the functions: $\|f\|_{L^p(U)} \le \liminf_{k\to\infty} \|f_k\|_{L^p(U)}$. However, it turns out that a stronger result holds, due to Brezis-Lieb [16]: in the limit, f_k's L^p norm splits into the L^p norm of the limit f and the L^p norm of the difference $f_k - f$.

Lemma 3.1 (Brezis-Lieb)**.** *Let f_k be a sequence of functions converging almost everywhere to f, such that f_k is uniformly bounded in $L^p(U)$. Then, we have that*

$$\lim_{k\to\infty} \left(\|f_k\|^p_{L^p(U)} - \|f_k - f\|^p_{L^p(U)} \right) = \|f\|^p_{L^p(U)}.$$

Proof. It suffices to prove that

$$\lim_{k\to\infty} \int \left| |f_k|^p - |f_k - f|^p - |f|^p \right| dx = 0.$$

Following the approach in [2], we define

$$g_k^\varepsilon := \left(\left| |f_k|^p - |f_k - f|^p - |f|^p \right| - \varepsilon |f_k - f|^p \right)^+$$

which is nonnegative and converges to 0 a.e. By the triangle inequality, we have that

$$g_k^\varepsilon \le \left(\left| |f_k|^p - |f_k - f|^p \right| + |f|^p - \varepsilon |f_k - f|^p \right)^+ .$$

We then use the inequality

$$\left| |a+b|^p - |a|^p \right| \le \varepsilon |a|^p + C(\varepsilon, p) |b|^p$$

which can be proven by taking cases on whether $b < \delta a$ or $b \ge \delta a$ for some δ depending on ε. We get that

$$\left| |f_k|^p - |f_k - f|^p \right| \le \varepsilon |f_k - f|^p + C(\varepsilon, p) |f|^p.$$

Thus, we have that

$$g_k^\varepsilon \le (C(\varepsilon, p) + 1) |f|^p,$$

so for fixed ε, $\{g_k^\varepsilon\}$ is bounded by an integrable function. By the dominated convergence theorem,

$$\lim_{k\to\infty} \int g_k^\varepsilon \, dx = 0.$$

Finally, we have that

$$\begin{aligned} \limsup_{k\to\infty} \int \left| |f_k|^p - |f_k - f|^p - |f|^p \right| dx &\le \limsup_{k\to\infty} \int g_k^\varepsilon \, dx + \varepsilon \limsup_{k\to\infty} \int |f_k - f|^p \, dx \\ &= \varepsilon \limsup_{k\to\infty} \int |f_k - f|^p \, dx = O(\varepsilon), \end{aligned}$$

so we are done by letting $\varepsilon \to 0$.

□

3.1. **Vanishing estimates and weak L^2 mass.** We now introduce the idea of weak L^2 mass, which will help us rule out vanishing.

Definition 3.2. Define the weak L^2 mass of a bounded sequence $(u_n) \in L^2$ by

$$\mathbf{m}_2((u_n)) = \sup \left\{ \int_{\mathbb{R}^d} |u|^2 \ : \ \text{there exists } (z_k) \subset \mathbb{R}^d, n_k \in \mathbb{N}, \text{ such that } u_{n_k}(\cdot - z_k) \xrightarrow{L^2} u \right\}.$$

This represents the largest possible mass of a weak limit of the sequence under translations.

The main use of this definition is the following theorem, which gives us equivalent ways to characterize vanishing of a sequence. Below, 2^* indicates the Sobolev conjugate exponent, defined as $2^* = \frac{2d}{d-2}$ for $d \ge 3$.

Theorem 3.3 (Vanishing equivalence in L^2). *Let (u_n) be a bounded sequence in $H^1(\mathbb{R}^d)$ $(d \ge 3)$. Then $\mathbf{m}_2((u_n)) = 0$ if and only if $u_n \to 0$ strongly in $L^p(\mathbb{R}^d)$ for all $2 < p < 2^*$.*

If $\mathbf{m}_2((u_n)) = 0$, then no nontrivial L^2 mass can be recovered from any translated subsequence. This theorem tells us that this local loss of mass forces the sequence to vanish in every subcritical L^p norm. Note that the restriction on p to $(2, 2^*)$ is necessary and sharp: for example, if we took $p = 2$, then we could spread out a constant L^2 mass into bumps that are farther and farther apart to force a contradiction. If we instead took $p = 2^*$, then since the Sobolev embedding $H^1 \hookrightarrow L^{2^*}$ is scale-invariant, we can create a sequence of bubbles that dilates with constant L^{2^*} and H^1 norm while the local L^2 mass goes to zero.

With p strictly in between 2 and 2^*, we can successfully interpolate between the local L^2 smallness and the global H^1 bound to prove vanishing in L^p as in our theorem. Before this, we will need the following estimate:

Lemma 3.4. *If u_n is bounded in $H^1(\mathbb{R}^d)$ $(d \geq 3)$, there exists $C = C(d)$ such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^{2+4/d} \leq C\, \mathbf{m}_2((u_n))^{2/d} \limsup_{n\to\infty} \|u_n\|^2_{H^1(\mathbb{R}^d)}.$$

Proof. Let Q be an integer cube of length 1 in $\mathbb{R}^d$. By Holder's inequality, we have

$$\|u_n\|_{L^q(Q)} \leq \|u_n\|^{\theta}_{L^2(Q)} \|u_n\|^{1-\theta}_{L^{2^*}(Q)} \tag{3.5}$$

for θ such that $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{2^*}$. Taking the q-th power, we get:

$$\int_Q |u_n|^q \leq \left(\int_Q |u_n|^2\right)^{q\theta/2} \left(\int_Q |u_n|^{2^*}\right)^{q(1-\theta)/2^*}.$$

We choose q such that $q(1-\theta) = 2$. A direct calculation yields $q = 2 + \frac{4}{d}$ (and indeed, this value of q is always less than $2^* = \frac{2d}{d-2} = 2 + \frac{4}{d-2}$). For this value of q, the exponent on the first term is $q\theta/2 = q/2 - 1 = \frac{2}{d}$. By the Gagliardo-Nirenberg-Sobolev inequality, $\|u_n\|_{L^{2^*}(Q)} \leq C(d)\, \|u_n\|_{H^1(Q)}$. Substituting this into our inequality gives:

$$\begin{aligned}\int_Q |u_n|^{2+4/d} &\leq \left(\int_Q |u_n|^2\right)^{2/d} \left(\left(C(d)\, \|u_n\|_{H^1(Q)}\right)^{2^*}\right)^{2/2^*}\\ &= C(d)^2 \left(\int_Q |u_n|^2\right)^{2/d} \|u_n\|^2_{H^1(Q)}.\end{aligned}$$

Let's relabel the constant as $C(d)$. Now, we sum over a lattice of disjoint integer cubes $\{Q_i\}$ that tile $\mathbb{R}^d$:

$$\begin{aligned}\int_{\mathbb{R}^d} |u_n|^{2+4/d} &= \sum_i \int_{Q_i} |u_n|^{2+4/d}\\ &\leq C(d) \sum_i \left(\int_{Q_i} |u_n|^2\right)^{2/d} \|u_n\|^2_{H^1(Q_i)}\\ &\leq C(d) \left(\sup_i \int_{Q_i} |u_n|^2\right)^{2/d} \sum_i \|u_n\|^2_{H^1(Q_i)}\\ &= C(d) \left(\sup_i \int_{Q_i} |u_n|^2\right)^{2/d} \|u_n\|^2_{H^1(\mathbb{R}^d)}.\end{aligned}$$

Taking the $\limsup_{n\to\infty}$ of both sides and using the property that $\limsup(a_n b_n) \leq (\limsup a_n)(\limsup b_n)$ for non-negative sequences a_n, b_n, we get:

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^{2+4/d} \leq C(d) \left(\limsup_{n\to\infty} \sup_i \int_{Q_i} |u_n|^2\right)^{2/d} \left(\limsup_{n\to\infty} \|u_n\|^2_{H^1(\mathbb{R}^d)}\right). \tag{3.6}$$

Let $A := \limsup_{n\to\infty} \sup_i \int_{Q_i} |u_n|^2$. It suffices to show that $A \leq \mathbf{m}_2((u_n))$.

By definition of lim sup, there exists a subsequence u_{n_k} and a sequence of cubes Q_k (with translations z_k from a fixed cube Q_0) such that

$$\lim_{k\to\infty} \int_{Q_k} |u_{n_k}(x)|^2 dx = A.$$

Let $v_k(x) = u_{n_k}(x + z_k)$. The sequence v_k is bounded in $H^1(\mathbb{R}^d)$, so there exists a further subsequence (which we still denote by v_k) that converges weakly in H^1 to some $v \in H^1(\mathbb{R}^d)$. By the Rellich-Kondrachov compactness theorem, since Q_0 is a bounded domain, $v_k \to v$ strongly in $L^2(Q_0)$. Therefore,

$$\int_{Q_0} |v|^2 = \lim_{k\to\infty} \int_{Q_0} |v_k|^2 = \lim_{k\to\infty} \int_{Q_k} |u_{n_k}|^2 = A.$$

By the definition of $\mathbf{m}_2((u_n))$, any weak limit v of a translated subsequence must satisfy $\int_{\mathbb{R}^d} |v|^2 \le \mathbf{m}_2((u_n))$. Thus,

$$A = \int_{Q_0} |v|^2 \le \int_{\mathbb{R}^d} |v|^2 \le \mathbf{m}_2((u_n)).$$

Substituting this back into inequality (3.6) completes the proof. □

Now, we can return to proving our theorem on vanishing.

Proof of Theorem 3.3.
$\Longrightarrow$ Direction: Suppose that (u_n) is a bounded sequence in $H^1(\mathbb{R}^d)$ and that $\mathbf{m}_2((u_n)) = 0$. We want to show that $u_n \to 0$ strongly in $L^p(\mathbb{R}^d)$ for all $p \in (2, 2^*)$. First, we will show strong convergence for the specific exponent $q = 2 + \frac{4}{d}$. After that, we will use an interpolation argument to cover the entire range $p \in (2, 2^*)$.

By assumption, the sequence (u_n) is bounded in $H^1(\mathbb{R}^d)$ ($\|u_n\|_{H^1(\mathbb{R}^d)} \le M$), so $\limsup_{n\to\infty} \|u_n\|^2_{H^1(\mathbb{R}^d)}$ is finite. Lemma 3.4 provides the estimate:

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^{2+4/d} \, \mathrm{d}x \le C\, \mathbf{m}_2((u_n))^{2/d} \limsup_{n\to\infty} \|u_n\|^2_{H^1(\mathbb{R}^d)}.$$

Substituting $\mathbf{m}_2((u_n)) = 0$ yields that

$$\limsup_{n\to\infty} \|u_n\|^q_{L^q(\mathbb{R}^d)} = \limsup_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^q \, \mathrm{d}x = 0,$$

where $q = 2 + \frac{4}{d}$, so $u_n \to 0$ strongly in $L^q(\mathbb{R}^d)$.

Now, let p be an arbitrary exponent in the open interval $(2, 2^*)$. We consider two cases based on the position of p relative to q.

Case 1: $2 < p < q$**.** We can express p as an interpolation between 2 and q. There exists a unique $\alpha \in (0,1)$ such that $\frac{1}{p} = \frac{\alpha}{2} + \frac{1-\alpha}{q}$. By the standard Holder L^p interpolation inequality we have

$$\|u_n\|_{L^p} \le \|u_n\|^{\alpha}_{L^2} \|u_n\|^{1-\alpha}_{L^q}.$$

The boundedness of (u_n) in $H^1(\mathbb{R}^d)$ implies its boundedness in $L^2(\mathbb{R}^d)$: $\|u_n\|_{L^2} \le M$. From above, we know $\|u_n\|_{L^q} \to 0$ as $n \to \infty$. Taking the limit, we get

$$\lim_{n\to\infty} \|u_n\|_{L^p} \le \lim_{n\to\infty} (M)^{\alpha} \|u_n\|^{1-\alpha}_{L^q} = (M)^{\alpha} \cdot 0 = 0.$$

Thus, $u_n \to 0$ strongly in $L^p(\mathbb{R}^d)$.

Case 2: $q \le p < 2^*$**.** Similarly, we interpolate between q and $2^* = \frac{2d}{d-2}$. There exists a unique $\beta \in [0,1)$ such that $\frac{1}{p} = \frac{\beta}{q} + \frac{1-\beta}{2^*}$. The interpolation inequality yields

$$\|u_n\|_{L^p} \le \|u_n\|^{\beta}_{L^q} \|u_n\|^{1-\beta}_{L^{2^*}}.$$

As before, we know that $\|u_n\|_{L^q} \to 0$. Furthermore, the Sobolev embedding theorem ($H^1(\mathbb{R}^d) \hookrightarrow L^{2^*}(\mathbb{R}^d)$) says that the boundedness of (u_n) in $H^1(\mathbb{R}^d)$ implies its boundedness in $L^{2^*}(\mathbb{R}^d)$. That is, $\|u_n\|_{L^{2^*}} \leq C\|u_n\|_{H^1}$ for some constant C. Taking the limit as $n \to \infty$, we find

$$\lim_{n\to\infty} \|u_n\|_{L^p} \leq \lim_{n\to\infty} \|u_n\|_{L^q}^{\beta}(CM)^{1-\beta} = 0 \cdot (CM)^{1-\beta} = 0.$$

Thus, $u_n \to 0$ strongly in $L^p(\mathbb{R}^d)$.

Therefore, combining both cases, we have that $u_n \to 0$ strongly in $L^p(\mathbb{R}^d)$ for any $p \in (2, 2^*)$.

$\Longleftarrow$ Direction: Conversely, suppose $u_n \to 0$ strongly in $L^p(\mathbb{R}^d)$ for all $2 < p < 2^*$. We want to show that any translated subsequence $v_n := u_{n_k}(\cdot - z_k)$ can only weakly converge to 0 in $H^1(\mathbb{R}^d)$. Fix some $q \in (2, 2^*)$. Since translations don't change the L^q norm, we have that $\|v_n\|_{L^q(\mathbb{R}^d)} = \|u_{n_k}\|_{L^q(\mathbb{R}^d)} \to 0$, so $v_n \to 0$ strongly in $L^q(\mathbb{R}^d)$.

Now, suppose this subsequence v_n converges weakly in H^1 to some u. By the Sobolev embedding theorem, we have a continuous embedding $H^1(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ for all $q \in [2, 2^*]$. Thus, v_n must also converge weakly to u in $L^q(\mathbb{R}^d)$. However, we already established that $v_n \to 0$ strongly in $L^q(\mathbb{R}^d)$, so the only possible weak limit is $u = 0$. Then, the supremum in Definition 3.2, taken over all such possible weak limits, must be 0 as needed.

$\square$

4. Existence of Ground States

In this section, we will discuss the existence of ground states in the focusing NLS equation. We will mainly follow the approach in [3], which uses concentration compactness and vanishing estimates (such as those we covered in the previous section) to prove the existence of a maximizer to the Weinstein functional W. We will focus on the energy subcritical case, which is simpler to handle due to the compactness of the Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$ on bounded domains. Note also that the mass-critical exponent $p_m = 1 + \frac{4}{d}$ is strictly less than the energy-critical exponent $p_e = 1 + \frac{4}{d-2}$ for $d \geq 3$, so the mass-critical case is also included in the following theorem.

Theorem 4.1 (Existence of ground states, energy-subcritical)**.** *Let $p \in (1, 1 + \frac{4}{d-2})$. Then, there exists a maximizer $Q \in H^1(\mathbb{R}^d)$ to the Weinstein functional W.*

The rest of this section will be devoted to proving this theorem. However, instead of directly studying the Weinstein functional in its quotient form as in (2.5), it is easier to first consider the following minimization problem, which we will soon relate back to the Weinstein functional:

$$J(u) = \|u\|_{H^1(\mathbb{R}^d)}^2 = \int |\nabla u(x)|^2 \, \mathrm{d}x + \int |u(x)|^2 \, \mathrm{d}x,$$

$$I(\lambda) := \inf\left\{ J(u) \, : \, u \in H^1(\mathbb{R}^d), \|u\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} = \lambda \right\}.$$

First, we will use scaling to reduce the problem to $I(1)$. Indeed, let v be a minimizer to $I(1)$, and let $u = \lambda^{\frac{1}{p+1}} v$. Then, $\|u\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} = \lambda$, and we can compute that $J(u) = \lambda^{\frac{2}{p+1}} J(v)$. Thus, since we can do the same in reverse, we have that

$$I(\lambda) = \lambda^{\frac{2}{p+1}} I(1), \tag{4.2}$$

so it suffices to solve $I(1)$.

Next, we will relate this auxiliary minimization problem to our original problem of maximizing (2.5).

Lemma 4.3. *If W_{max} is the optimal constant in the Gagliardo-Nirenberg inequality, then*
$$I(1) = \frac{1}{\theta^\theta(1-\theta)^{1-\theta}W_{max}}$$
where $\theta = \frac{d(p-1)}{2(p+1)}$ (so that $\frac{1}{p+1} = \frac{1-\theta}{2} + \frac{\theta}{2^*}$).

Proof Sketch. ($\geq$) We get the lower bound by applying a weighted arithmetic-geometric mean inequality (derived from Young's inequality) to the terms of the H^1 norm:
$$\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 \geq \frac{\|\nabla u\|_{L^2}^{2(1-\theta)}\|u\|_{L^2}^{2\theta}}{\theta^\theta(1-\theta)^{1-\theta}}.$$
If we take the infimum over all u with $\|u\|_{L^{p+1}} = 1$ and apply the Gagliardo-Nirenberg inequality to the numerator, we get the following lower bound (which also shows that $I(1)$ is positive):
$$I(1) \geq \frac{1}{\theta^\theta(1-\theta)^{1-\theta}W_{max}}.$$
($\leq$) For any $u \in H^1$ with unit L^{p+1} norm, let $u_\alpha(x) = \alpha^{d/(p+1)}u(\alpha x)$ which also has unit L^{p+1} norm. The energy of this function is
$$J(\alpha) = \alpha^{2\theta}\|\nabla u\|_{L^2}^2 + \alpha^{-2(1-\theta)}\|u\|_{L^2}^2.$$
Since $I(1) \leq J(\alpha)$ for any α, we can minimize $J(\alpha)$ with respect to α. A direct calculation shows the minimizer and minimum value are
$$\alpha^* = \sqrt{\frac{1-\theta}{\theta}}\frac{\|u\|_{L^2}}{\|\nabla u\|_{L^2}}, \quad \inf_{\alpha>0} J(\alpha) = \frac{\|\nabla u\|_{L^2}^{2(1-\theta)}\|u\|_{L^2}^{2\theta}}{\theta^\theta(1-\theta)^{1-\theta}}.$$
Taking the infimum over all admissible u on the right-hand side again invokes the Gagliardo-Nirenberg inequality, and we get the desired upper bound. □

Remark 4.4. The relation established in Lemma 4.3 not only relates the optimal values of $I(1)$ and $W_{\max}$, but also shows that their optimizers correspond. Indeed, if v is a minimizer of $I(1)$, then we can show the Gagliardo-Nirenberg inequality must be sharp and that it is automatically a maximizer for W. Conversely, any maximizer Q of W yields a minimizer for $I(1)$ by normalizing to $u = Q/\|Q\|_{L^{p+1}}$ and then using the above scaling, i.e. $u_{\alpha^*}(x) = (\alpha^*)^{d/(p+1)}u(\alpha^* x)$.

With this equivalence established, we can now focus on proving the existence of minimizers for $I(1)$. Our strategy will be to take a minimizing sequence $\{u_n\}$ for $I(1)$, show that this sequence cannot be vanishing in $\mathbf{m}_2((u_n))$ and thus extract a nontrivial weak limit. Then, we will use the Brezis-Lieb lemma (Lemma 3.1) to rule out dichotomy and show that this weak limit is indeed a minimizer.

Proof of Theorem 4.1. Consider a minimizing sequence $\{u_n\}$, i.e., let $J(u_n) = \|u_n\|_{H^1}^2 \downarrow I(1)$ with $\|u_n\|_{L^{p+1}} = 1$. Then, u_n is bounded in $H^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$, so we can define $\mathbf{m}_2((u_n))$ as in Definition 3.2. In addition, note that if $\mathbf{m}_2((u_n)) = 0$, then by Theorem 3.3, the sequence would vanish in L^{p+1}, contradicting $\|u_n\|_{L^{p+1}} = 1$. Thus, $\mathbf{m}_2((u_n)) > 0$.

Thus, there exist some subsequences and translations n_k, x_k such that $u_{n_k}(\cdot + x_k) \rightharpoonup u$ weakly in $H^1(\mathbb{R}^d)$ for some $u \neq 0$. Since our functional J and our constraint only depend on norms (which are invariant to translation), $v_n := u_{n_k}(\cdot + x_k)$ is also a minimizing sequence of J, and $v_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^d)$. On any bounded ball $B_R(0)$, since the Rellich-Kondrachov compactness theorem says that $H^1(B_R(0)) \subset\subset L^2(B_R(0))$, we have a subsequence of v_n that converges strongly to u in $L^2(B_R(0))$.

Using a standard diagonal argument, we can construct a new subsequence (which we will still call v_n for simplicity), that converges strongly to u in $L^2_{\text{loc}}(\mathbb{R}^d)$ and thus pointwise almost everywhere. First, we take the first element of the subsequence converging strongly in $L^2(B_1(0))$, then take the second element of the further subsequence which converges strongly in $L^2(B_2(0))$, and so on. This diagonal subsequence converges strongly to u in $L^2(B_R(0))$ for every $R > 0$, which means it converges strongly to u in $L^2_{\text{loc}}(\mathbb{R}^d)$, and thus a further subsequence converges pointwise a.e. to u as well.

Note that v_n is bounded in $H^1(\mathbb{R}^d)$ since it is weakly convergent in $H^1(\mathbb{R}^d)$. $H^1(\mathbb{R}^d)$ embeds into $L^2(\mathbb{R}^d)$ by definition, and into $L^{2^*}(\mathbb{R}^d)$ by Sobolev inequality, so by interpolation, v_n is also bounded in $L^q(\mathbb{R}^d)$ for any $q \in (2, 2^*)$. Hence, we can now apply Lemma 3.1 to get

$$1 = \int |v_n|^{p+1} = \underbrace{\int |u|^{p+1}}_{:=\lambda} + \int |v_n - u|^{p+1} + o(1). \tag{4.5}$$

Lemma 3.1 applied again with $p = 2$ to v_n and ∇v_n then gives us that

$$\begin{aligned}\|v_n\|^2_{H^1(\mathbb{R}^d)} &= \|u\|^2_{H^1(\mathbb{R}^d)} + \|v_n - u\|^2_{H^1(\mathbb{R}^d)} + o(1)\\ &\geq \|u\|^2_{H^1(\mathbb{R}^d)} + I\Big(\int |v_n - u|^{p+1}\Big) + o(1)\end{aligned}$$

Now, we take the limit as $n \to \infty$ using (4.5) to get that

$$I(1) \geq \|u\|^2_{H^1(\mathbb{R}^d)} + I(1-\lambda) \geq I(\lambda) + I(1-\lambda),$$

where we used that $\|u\|^{p+1}_{L^{p+1}} = \lambda$ and the continuous dependence of I on λ from (4.2). If we plug in this scaling relation, we find that

$$I(1) \geq I(\lambda) + I(1-\lambda) = I(1)\left(\lambda^{\frac{2}{p+1}} + (1-\lambda)^{\frac{2}{p+1}}\right). \tag{4.6}$$

Suppose $\lambda \in (0,1)$ for the sake of contradiction. Then, we would have that

$$\left(\lambda^{\frac{2}{p+1}} + (1-\lambda)^{\frac{2}{p+1}}\right) > 1$$

which would force a contradiction in (4.6). We also know $\lambda \neq 0$ since $u \neq 0$, so we must have $\lambda = 1$. Therefore, $I(1) = \|u\|^2_{H^1(\mathbb{R}^d)}$, so u is a minimizer to $I(1)$. Then, Lemma 4.3 and Remark 4.4 imply it is also a maximizer to the Weinstein functional W, completing the proof. □

Remark 4.7. The energy-critical case (where $p = 1 + \frac{4}{d-2}$, or $p + 1 = 2^*$) is tougher to handle due to the lack of compactness in the Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^{2^*}(\mathbb{R}^d)$. Indeed, we now have lack of compactness due to dilations (as well as translations, like before), and since the Sobolev inequality doesn't involve the L^2 norm, we can't work in H^1 anymore. Instead, we would need to revamp our definitions of weak mass to account for this new lack of compactness due to dilations, and work in the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^d)$ instead. [3] covers some of these necessary changes in Section 6, as well as an interesting proof of existence of optimizers for the Sobolev inequality in the critical case.

5. Theory of local well-posedness

In this section, we will discuss the theory of local well-posedness for the focusing NLS equation, following the approach in [12] which fixes $d = 3$.

The general idea is to turn the PDE into an integral equation with the linear propagator $S(t) = e^{it\Delta}$, which is the solution operator for the free/linear Schrodinger equation.[2]

However, we must also control the nonlinear term in NLS, which we will do using Strichartz estimates [13]. Combining these with Holder's inequality and Sobolev embeddings, we can bound the nonlinear term on short time intervals. Then, we can use a standard contraction mapping argument to prove local well-posedness (in particular, guaranteeing existence of the solution for short times).

Definition 5.1 (Admissible pairs). Let a pair (q, r) be *admissible* in dimension d if

$$2 \le q, r \le \infty, \qquad \frac{2}{q} + \frac{d}{r} = \frac{d}{2},$$

excluding the endpoint $(q, r, d) = (2, \infty, 2)$.

Note that this describes a line segment in the $(\frac{1}{q}, \frac{1}{r})$-plane connecting the points $(0, \frac{1}{2})$ and $\frac{d}{4}, 0$.

Proposition 5.2 (Strichartz estimates). *Let (q, r) and $(\tilde{q}, \tilde{r})$ be admissible pairs. For the linear propagator $S(t) = e^{it\Delta}$ we have the homogeneous, dual and inhomogeneous estimates*

$$\|S(t)u_0\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u_0\|_{L_x^2}, \tag{5.3}$$

$$\Big\| \int_{\mathbb{R}} S(-t')F(t')\,dt' \Big\|_{L_x^2} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{5.4}$$

$$\Big\| \int_{t_0}^{t} S(t-t')F(t')\,dt' \Big\|_{L_t^q L_x^r([t_0,t_1]\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}([t_0,t_1]\times\mathbb{R}^d)}. \tag{5.5}$$

Proof Ideas. The first two key facts used in the proof are as follows. First, we have the dispersive estimate

$$\|S(t)f\|_{L_x^\infty} \lesssim \frac{1}{|t|^{d/2}} \|f\|_{L_x^1}$$

for all $t \ne 0$, which can be proved by explicitly computing the integral kernel of $S(t)$ using the Fourier transform: $S(t)u_0(x) = K_t * u_0(x)$ for

$$K_t(x) \coloneqq \mathcal{F}^{-1}\left(e^{-4\pi^2 it|\xi|^2}\right)(x) = \frac{e^{i|x|^2/4t}}{(4\pi it)^{d/2}}.$$

Second, we have the unitarity of $S(t)$ on L_x^2:

$$\|S(t)f\|_{L_x^2} = \left(\int_{\mathbb{R}^d} |e^{-it|\xi|^2}\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} = \|\widehat{f}\|_{L_\xi^2} = \|f\|_{L_x^2}.$$

[2]This comes from taking the Fourier transform in space in the linear Schrodinger equation $i\partial_t u + \Delta u = 0$ and solving for $u(t,x) = \mathcal{F}^{-1}\left(e^{-it|\xi|^2}\hat{u}_0(\xi)\right)(x) =: \left(e^{it\Delta}u_0\right)(x)$.

The remainder of the proof interpolates these estimates via Riesz-Thorin, and then relies on a "TT^*" argument and the Hardy-Littlewood-Sobolev inequality; see [8] or [18] for details. ☐

For a time interval I, we define the Strichartz (solution) space $\mathcal{S}^0(I)$ to control a function across all admissible spacetime norms at the same time, and its dual (nonlinear) space $\mathcal{N}^0(I)$:

$$\mathcal{S}^0(I) := \sup_{\text{admissible } (q,r)} \|\cdot\|_{L_t^q L_x^r(I\times\mathbb{R}^d)}, \qquad \mathcal{N}^0(I) := (\mathcal{S}^0(I))^*.$$

Hence $\|F\|_{\mathcal{N}^0(I)} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\mathbb{R}^d)}$ for any admissible $(\tilde{q},\tilde{r})$. Since our initial data is in H^1, we will also need these natural lifted norms:

$$\|u\|_{\mathcal{S}^1(I)} := \|u\|_{\mathcal{S}^0(I)} + \|\nabla u\|_{\mathcal{S}^0(I)}, \qquad \|F\|_{\mathcal{N}^1(I)} := \|F\|_{\mathcal{N}^0(I)} + \|\nabla F\|_{\mathcal{N}^0(I)}.$$

By (5.3)–(5.5) we then have the standard Duhamel bound

$$\|u\|_{\mathcal{S}^1(I)} \lesssim \|u(t_0)\|_{H^1} + \left\||u|^{p-1}u\right\|_{\mathcal{N}^1(I)}, \tag{5.6}$$

for any $t_0 \in I$.

Our key estimate is the following, which deals with the energy-subcritical case in $d = 3$.

Proposition 5.7. *Let $d = 3$ and $2 \le p < 5$. Then the Cauchy problem for (1.1) is locally well-posed in $H^1(\mathbb{R}^3)$: there exists $T = T(\|u_0\|_{H^1}) > 0$ and a unique solution $u \in C([-T,T];H^1) \cap \mathcal{S}^1([-T,T])$.*

Proof Sketch. First, we rewrite (1.1) in its integral form using Duhamel's formula:

$$\Gamma u := S(t)u_0 - i\lambda \int_0^t S(t-t') \left(|u|^{p-1}\, u\right)(t')\, dt', \qquad S(t) = e^{it\Delta}.$$

To show that Γ is a contraction, we estimate the difference $\Gamma u - \Gamma v$. By the Strichartz estimates, using that ∇ commutes with $S(t)$, we have

$$\|\Gamma u - \Gamma v\|_{S_T^1} \lesssim \left\||u|^{p-1}\, u - |v|^{p-1}\, v\right\|_{N_T^1},$$

so it suffices to estimate the nonlinearity in the dual norm N_T^1. The key part of this norm involves the gradient of the nonlinearity. Via the complex chain rule, we can derive that $\nabla(|u|^{p-1}\, u) = O(|u|^{p-1}\, \nabla u)$. To handle the difference between u and v, we can add and subtract a term to isolate the differences $\nabla(u - v)$ and $(u - v)$:

$$\begin{aligned} |u|^{p-1}\, \nabla u - |v|^{p-1}\, \nabla v &= \left(|u|^{p-1}\, \nabla u - |u|^{p-1}\, \nabla v\right) + \left(|u|^{p-1}\, \nabla v - |v|^{p-1}\, \nabla v\right) \\ &= |u|^{p-1}\, \nabla(u-v) + \left(|u|^{p-1} - |v|^{p-1}\right) \nabla v. \end{aligned}$$

By the Mean Value Theorem, the difference $|u|^{p-1} - |v|^{p-1}$ is bounded by a term proportional to $\left(|u|^{p-2} + |v|^{p-2}\right)|u - v|$. This gives us the two main components to estimate:

$$\lesssim |u|^{p-1}\, |\nabla(u-v)| + \left(|u|^{p-2} + |v|^{p-2}\right)|u - v|\, |\nabla v|.$$

We estimate these terms in the dual admissible space $L_t^2 W_x^{1,\frac{6}{5}}$. By the Sobolev inequality, we have $\|f\|_{L_x^{\frac{5(p-1)}{2}}} \lesssim \|f\|_{W_x^{1,\frac{30}{13}}}$ for $p \le 5$. Then, by Hölder's inequality (with $\frac{1}{2} = \frac{5-p}{10} + \frac{p}{10}$ in time) and using the admissible Strichartz norms (i.e. that $(10, \frac{13}{30})$ is admissible), we derive that:

$$\begin{aligned}\left\| |u|^{p-1} u - |v|^{p-1} v \right\|_{L_t^2 W_x^{1,\frac{6}{5}}} &\lesssim T^{\frac{5-p}{10}} \left(\|u\|^{p-1}_{L_t^{10} W_x^{1,\frac{30}{13}}} + \|v\|^{p-1}_{L_t^{10} W_x^{1,\frac{30}{13}}} \right) \|u - v\|_{L_t^{10} W_x^{1,\frac{30}{13}}} \\ &\lesssim T^{\frac{5-p}{10}} \left(\|u\|^{p-1}_{S_T^1} + \|v\|^{p-1}_{S_T^1} \right) \|u - v\|_{S_T^1} .\end{aligned}$$

This shows that for some constant C,

$$\|\Gamma(u) - \Gamma(v)\|_{S_T^1} \lesssim C T^{\frac{5-p}{10}} \left(\|u\|_{S_T^1} + \|v\|_{S_T^1} \right)^{p-1} \|u - v\|_{S_T^1} .$$

Choose $R = 2\|u_0\|_{H^1}$ and $T > 0$ small depending on $\|u_0\|_{H^1}$ so that $C T^{\frac{5-p}{10}} R^{p-1} \ll 1$. Then, Γ is a contraction on the closed ball $\left\{ u : \|u\|_{S_T^1} \le R \right\}$, and the Banach fixed point theorem yields a unique solution $u \in S_T^1$. Standard continuation arguments can show that $u \in C([-T, T]; H^1)$ and prove continuous dependence on data. □

Note that in the defocusing case, this local existence automatically extends to global existence since the energy is coercive and controls the H^1-norm. In the focusing case, however, the energy is not coercive so we cannot guarantee global existence for all initial data. In the next section, we will discuss conditions that lead to either global existence or finite-time blow-up in the focusing case for the mass-critical exponent.

The proof above won't extend immediately to the energy-critical case $p = 5$ in $d = 3$ since the time-exponent becomes zero (Holder's inequality is not enough). However, instead of trying to get a small constant from the time power, we can instead make the linear part $S(t)u_0$ itself very small by restricting to a very short time interval, and viewing the nonlinear part as a small perturbation of that. A full proof of this can be found in [12].

6. Theory of global existence and finite-time blow-up

In this section, we will discuss the theory of global existence and finite-time blow-up for solutions to the focusing NLS equation. Specifically, we will consider the equation in the mass-critical case, i.e., for $p = p_m = 1 + \frac{4}{d}$. Recall that in this case, the mass (L^2 norm of u) is invariant under the natural scaling $u_\lambda(t, x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$ of the NLS equation.

We will begin by detailing an argument first due to Glassey [11] but also covered in [6], that shows that if the initial data has negative energy, then the $H^1(\mathbb{R}^d)$ norm of the solution must blow up in finite time. Then, we will discuss the mass (or $L^2(\mathbb{R}^d)$) threshold for global existence and blow-up, which is curiously related to the ground state soliton Q we found earlier.

6.1. Glassey's blow-up argument.

Define the *variance* of a solution $u(t, x)$ as

$$V(t) = \int_{\mathbb{R}^d} |x|^2 |u(t, x)|^2 \, dx.$$

Theorem 6.1. *Suppose we have initial data $u_0 \in H^1(\mathbb{R}^d)$ with negative energy $E(u_0) < 0$ and $|x|u_0 \in L^2(\mathbb{R}^d)$, and u is a solution to the focusing mass-critical NLS equation with maximal time of existence T. Then, $T < \infty$ and $\|u(t)\|_{H^1(\mathbb{R}^d)} \to \infty$ as $t \to T$. (The solution blows up in finite time.)*

Proof. We begin by considering the second derivative of the variance and relating it to the energy $E(u)$. By differentiating under the integral sign and using the equation, we find that

$$V''(t) = 8\int_{\mathbb{R}^d} |\nabla u(t,x)|^2 \, \mathrm{d}x - \frac{4d(p-1)}{p+1}\int_{\mathbb{R}^d} |u(t,x)|^{p+1} \, \mathrm{d}x,$$

and since $p = 1 + \frac{4}{d}$, this simplifies to

$$V''(t) = 16E(u(t)) = 16E(u_0) < 0$$

where we used conservation of energy. Note how the choice of mass-critical p allows the cancellations that produce a pure multiple of the energy here. This result is called a "virial identity," as it is analogous to the virial theorem in classical mechanics which also relates kinetic and potential energies.

Note that since $V''(t)$ is a negative constant, V is a concave quadratic function of t. We now claim the following inequality holds:

$$\|u(t)\|^2_{L^2(\mathbb{R}^d)} \leq \frac{2}{d}\|xu(t)\|_{L^2(\mathbb{R}^d)}\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}. \tag{6.2}$$

To prove this, we use integration by parts as follows:

$$\begin{aligned}\|u(t)\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |u(t,x)|^2 \, \mathrm{d}x \\ &= \frac{1}{d}\int (\nabla\cdot x)|u(t,x)|^2 \, \mathrm{d}x \\ &= -\frac{2}{d}\operatorname{Re}\int x\cdot\nabla u(t,x)\overline{u(t,x)} \, \mathrm{d}x.\end{aligned}$$

where we used that $\nabla\cdot x = d$. Then, by Cauchy-Schwarz, we get (6.2).

Lastly, note that conservation of mass gives $\|u(t)\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)} =: M$, a positive constant. Thus, by (6.2), we have that

$$M \leq \frac{2}{d}\sqrt{V(t)}\,\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}.$$

Since $V(t)$ is a concave quadratic satisfying $V''(t) < 0$, it must reach zero at some finite time $T < \infty$. As $V(t) \to 0$ when $t \to T$, the above inequality forces $\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \to \infty$. Therefore, $\|u(t)\|_{H^1(\mathbb{R}^d)}$ blows up as $t \to T$. □

6.2. **Mass threshold for global existence.** In the previous section, we saw that if the initial data has negative energy, then the solution must blow up in finite time. Now, we will develop a related sufficient condition for the solution to exist globally in time. We start by examining a ground state Q, a maximizer to the Weinstein functional as in Lemma 2.6. Consider a solution $u \in H^1(\mathbb{R}^d)$ to the focusing mass-critical NLS equation.

By the Gagliardo-Nirenberg inequality, we have that

$$\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^d)} \leq \alpha\|u\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)}\|\nabla u\|^2_{L^2(\mathbb{R}^d)}$$

where α is the sharp constant in the inequality. By definition of Q, we can write it as

$$\alpha = \frac{\|Q\|^{p+1}_{L^{p+1}(\mathbb{R}^d)}}{\|Q\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)}\|\nabla Q\|^2_{L^2(\mathbb{R}^d)}} = \frac{p+1}{2\|Q\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)}}$$

where we used the Pohozaev identities to simplify.[3]

Now, consider the energy $E(u)$ defined in (2.2). We can use the above inequality to bound the potential energy term as follows:

$$\begin{aligned}
E(u) &= \frac{1}{2}\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} \\
&\geq \frac{1}{2}\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 - \frac{1}{p+1}\cdot\frac{p+1}{2\|Q\|_{L^2(\mathbb{R}^d)}^{\frac{4}{d}}}\|u\|_{L^2(\mathbb{R}^d)}^{\frac{4}{d}}\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 \\
&= \frac{1}{2}\left(1-\left(\frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{\|Q\|_{L^2(\mathbb{R}^d)}}\right)^{\frac{4}{d}}\right)\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 .
\end{aligned}$$

where we used conservation of mass $\|u(t)\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)}$. Immediately, we can make the following observation.

Corollary 6.3. *Suppose our initial data $u_0 \in H^1(\mathbb{R}^d)$ satisfies the mass constraint $\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}$. Then, the energy $E(u(t))$ is always non-negative. In addition, $\|u(t)\|_{H^1(\mathbb{R}^d)}$ is bounded for all time.*

Therefore, given what we showed in Section 5, we can conclude that the solution $u(t)$ exists globally in time. This gives us a mass threshold for global existence: if the initial data has mass less than that of the ground state Q, then the solution exists for all time. This completes the proof of Theorem 1.4.

Acknowledgments

It is a pleasure to thank my mentor, David Bowman, for his brilliant guidance and support throughout this project. David provided very illuminating problems to work on as well as invaluable insights, not only related to the contents of this paper but across many areas of partial differential equations.

I would also like to thank Professor Peter May for organizing this REU program from which I have learned so much. In addition, I would like to thank Professor Carlos Kenig for his amazing lectures on the Calderon-Zygmund theorem and for his discussions with me. Lastly, I would like to thank Professor Sung-Jin Oh and Professor Bobby Wilson for recommending me to this program and for their continued support.

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