THE BIGGEST OPEN PROBLEM IN EUCLIDEAN RAMSEY THEORY

NIKHIL PATEL

Abstract. We call a finite set $X \subset \mathbb{R}^d$ *Ramsey* if, for any $r \in \mathbb{N}$, we can choose an n sufficiently large such that every r-coloration of $\mathbb{R}^n$ must contain a congruent copy of X. It is an open problem to find an equivalent characterization of all Ramsey sets. This paper presents some results on specific and general cases of Ramsey-ness, the current conjectured characterizations, and a few productive directions that future research could follow.

Contents

1. Introduction

Ramsey Theory stems from a theorem of Frank P. Ramsey [Ram30], which is equivalent to the following statement (though originally phrased in the language of formal logic).

Theorem 1.1. *For all $r, d \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, any r-coloring of the edges of the complete graph K_n must contain a monochromatic copy of K_d as a subgraph.*

For example, we know that any 2-coloring of the edges of K_6 must contain three vertices all connected by edges of the same color. More broadly, Ramsey Theory generalizes the idea behind Theorem 1.1 to other contexts, finding examples of structures that must contain certain monochromatic substructures under arbitrary colorings. Because of this, the field is often characterized as revealing the inevitability of order among chaos [Pr5]. Euclidean Ramsey Theory is the study of Ramsey-like phenomena in Euclidean space. Central to this topic is the concept of *Ramsey* sets.

Date: August 21, 2025.

Definition 1.2. For a given $r \in \mathbb{N}$, a finite set $X \subset \mathbb{R}^d$ is called *r-Ramsey* if every r-coloration of $\mathbb{R}^n$ (i.e. every possible partition of $\mathbb{R}^n$ into r disjoint subsets) must contain a monochromatic congruent copy of X for sufficiently large n. Graham [Gra17] uses the following notation: $\mathbb{R}^n \xrightarrow{r} X$ if X is *r-Ramsey* in $\mathbb{R}^n$. X is *Ramsey* if it is r-Ramsey for all $r \in \mathbb{N}$.

Importantly, we only consider *congruent* copies of X, i.e. sets X' that are the image of X under some isometry (distance-preserving transformation, e.g. reflections, rotations, and translations). Scaled copies of X are disregarded in this definition.

This is a natural way to extend Ramsey's theorem to Euclidean space, replacing isomorphic subgraphs of K_n with isometric subsets of $\mathbb{R}^n$.

To get a sense for what Ramsey sets can look like, we will show that all regular simplices (2-point sets, equilateral triangles, tetrahedra, etc.) are Ramsey. Note that when we say a particular polytope is Ramsey, we are referring to its vertex set.

Definition 1.3. A *simplex* is a point configuration $\{x_0, x_1, \ldots, x_d\} \subset \mathbb{R}^d$ such that the vectors $\{x_i - x_0\}_{i\in[d]}$ span $\mathbb{R}^d$. A *regular simplex* is one whose points are pairwise unit distance apart.

Note that, by this definition, all simplices are non-degenerate (i.e. their convex hulls have non-zero measure). For example, we cannot have a 4-point simplex in any 2-dimensional subspace of $\mathbb{R}^3$.

Proposition 1.4. *The $(d+1)$-point regular simplex $S_d \subset \mathbb{R}^d$ is Ramsey for all $d \in \mathbb{N}$.*

Proof. Let $r \in \mathbb{N}$ (recall that r is the number of colors we allow ourselves when coloring the space). Consider an arbitrary r-coloring of $\mathbb{R}^{rd}$, and specifically consider a particular instance of $S_{rd} \subset \mathbb{R}^{rd}$. By the Pigeonhole Principle, because the $rd+1$ vertices of the simplex are partitioned into r subsets, there must exist one such subset with at least $d+1$ vertices. Since these points are all pairwise unit distance apart, they form a monochromatic copy of S_d. Thus, $\mathbb{R}^{rd} \xrightarrow{r} S_d$ for all $r \in \mathbb{N}$, so S_d is Ramsey. ∎

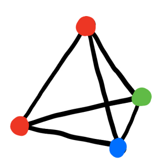

FIGURE 1. A 3-colored S_3, illustrating a proof that $\mathbb{R}^3 \xrightarrow{3} S_1$, where S_1 is the two-point simplex in $\mathbb{R}$.

This technique of limiting our view to colorings of only a finite subset of the space is extremely useful since the space of all possible r-colorings of $\mathbb{R}^n$ is unfathomably large and complex (after all, there are $r^{2^{\aleph_0}}$ of them).

In Section 2, we will see that the condition of regularity can be dropped, i.e. all simplices are Ramsey, but the proof becomes significantly more complex and goes beyond the scope of this paper.

Relative to the case of regular simplices, there are equally simple examples of sets that are not Ramsey. For example, Erdos et al. [EGM+73] provide what is arguably the simplest case of a non-Ramsey set:

Proposition 1.5. *Let ℓ_3 be the three-term arithmetic progression $\{x-\delta, x, x+\delta\} \subset \mathbb{R}$ for some $x \in \mathbb{R}$ and $\delta > 0$. ℓ_3 is not Ramsey.*

Proof. Assume without loss of generality that $\delta = 1$; as we will see with Proposition 2.1, this assumption suffices to prove the statement for all $\delta > 0$. We claim that for any n, there exists a 4-coloring $\chi : \mathbb{R}^n \to \{0, 1, 2, 3\}$ that avoids monochromatic copies of ℓ_3.

Let χ be defined as $\chi(x) = \lfloor |x|^2 \rfloor \pmod 4$. We will see that such "concentric shell" colorings prove to be a powerful counterexample technique in Euclidean Ramsey Theory.

Assume for the sake of contradiction that there exists a monochromatic $\{x - u, x, x + u\}$, where $x, u \in \mathbb{R}^n$ and $|u| = 1$. Let $r := \chi(x - u) = \chi(x) = \chi(x + u)$. We know that there exist constants $a_1, a_2, a_3 \in \mathbb{Z}$ and $\theta_1, \theta_2, \theta_3 \in [0, 1)$ such that:

(1) $|x - u|^2 = 4a_1 + r + \theta_1$
(2) $|x|^2 = 4a_2 + r + \theta_2$
(3) $|x + u|^2 = 4a_3 + r + \theta_3$

We know that $|a + b|^2 = |a|^2 + 2(a \cdot b) + |b|^2$ for any $a, b \in \mathbb{R}^n$, where $(a \cdot b)$ is the dot product of the two vectors. Thus, $|x + u|^2 - 2|x|^2 + |x - u|^2 = (|u|^2 + 2(x \cdot u)) + (|u|^2 - 2(x \cdot u)) = 2 = 4(a_3 + a_1 - 2a_2) + \theta_3 + \theta_1 - 2\theta_2$. However, since $\theta_3+\theta_1-2\theta_2 \in (-2, 2)$, the lattermost equality is impossible. Thus, by contradiction, the statement is proven. □

Note that we had to construct a coloring valid in any arbitrarily high-dimensional space. Thus, this construction of concentric monochromatic spherical shells took on a less geometric flavor, instead leaning on vector algebra and modular arithmetic. In Section 2, we will see how the authors applied this approach to prove the non-Ramsey-ness of a significant class of sets.

At the heart of Euclidean Ramsey Theory is the question of exactly which sets are Ramsey. Beyond what we have already seen, we have many examples of Ramsey and non-Ramsey sets. However, a general characterization still eludes us.

2. A Toolkit of Results and Examples

This central question has been studied from many angles, and various authors have developed tools to generate new Ramsey sets or to prove that a particular set is not Ramsey. In this section, we summarize what is known about Ramsey-ness and discuss instances of these principles being applied to generate concrete examples of Ramsey and non-Ramsey sets.

To begin, we have a few straightforward results that follow quickly from Definition 1.2.

Definition 2.1. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is an *isometry* if, for the standard ℓ_2 (Euclidean) distance metric d, $d(x_1, x_2) = d(f(x_1), f(x_2))$ for all $x_1, x_2 \in \mathbb{R}^n$.

Definition 2.2. Let X and Y be non-empty subsets of $\mathbb{R}^n$. X and Y are *congruent* if there exists an isometry mapping X onto Y. More generally, X and Y are *similar* if X is congruent to some scaled copy aY, where $a \neq 0$.

Note that any distance-preserving mapping between X and Y as metric spaces can be extended to an isometry of the ambient space.

Proposition 2.3. *Let X be a Ramsey set.*

(1) Any subset $Y \subset X$ is also Ramsey.
(2) Any set X' similar to X (also known as a homothetic copy of X) is also Ramsey.

Proof.

(1) This follows from the fact that any monochromatic copy of X must contain a monochromatic copy of Y.
(2) Let $r \in \mathbb{N}$ and let n be sufficiently large such that $\mathbb{R}^n \xrightarrow{r} X$. Let $\mathcal{C} := \{C_i\}_{i\in[r]}$ be an r-coloring of $\mathbb{R}^n$. Since X' is similar to X, it is congruent to aX for some $a > 0$. Scale $\mathcal{C}$ by a^{-1} to get a new coloring $\mathcal{C}^* := \{a^{-1}C_i\}_{i\in[r]}$. Because X is Ramsey, this coloring must have a monochromatic congruent copy X^* of X. Thus, $\mathcal{C}$ must have a monochromatic copy of aX, i.e. of X'.

□

2.1. **Spheres.**

In the paper in which the concept of Ramsey sets was introduced, Erdös et al. [EGM+73] also provide the strongest currently known criterion for a set *not* being Ramsey.

Definition 2.4. A set $X \subset \mathbb{R}^d$ is *spherical* if it lies on the surface of some d-sphere, or more precisely, if there exist $c \in \mathbb{R}^d$ and $r > 0$ such that $X \subset \{x \in \mathbb{R}^d \mid |x - c| = r\}$.

Theorem 2.5. *If X is not spherical, then it is not Ramsey.*

To prove this theorem, the authors extend their argument for Proposition 1.5 using the following lemmas, translating the problem into the language of algebra and then proving a result on non-monochromaticity in this new language.

Lemma 2.6. *A set $X = \{x_0, x_1, \ldots, x_k\} \subset \mathbb{R}^d$ is not spherical if and only if there exist scalars $c_1, c_2, \ldots, c_k$ not identically zero such that:*

$$\sum_{i=1}^{k} c_i(x_i - x_0) = 0 \tag{2.1}$$

and

$$\sum_{i=1}^{k} c_i(|x_i|^2 - |x_0|^2) \neq 0. \tag{2.2}$$

Lemma 2.7. *Let $c_1, \ldots, c_k, b \in \mathbb{R}$ with $b \neq 0$. There exists an $r \in \mathbb{N}$, and some r-coloring of the real numbers, such that there is no monochromatic solution $X = \{x_0, \ldots, x_n\}$ to the equation*

$$\sum_{i=1}^{k} c_i(x_i - x_0) = b. \tag{2.3}$$

Using these lemmas, we can sketch out the authors' proof of Theorem 2.5.

Proof sketch. For a given finite non-spherical configuration $X \subset \mathbb{R}^d$, the proof of Theorem 2.5 uses the constants $\{c_i\}_{i=0\ldots k}$ from Lemma 2.6 and the resultant coloring $\chi : \mathbb{R} \to [r]$ from Lemma 2.7 to construct a coloring $\chi^* : \mathbb{R}^n \to [r]$ defined by $\chi^*(x) = \chi(|x|^2)$ for any $n \geq d$. By Lemma 2.7, X is not monochromatic under this coloring. Finally, the authors show that Equations 2.1 and 2.2 remain valid for any congruent copy of X with the same choice of constants, so by applying Lemma 2.7 again, we can see that no monochromatic congruent copy of X exists under this coloring. This completes the proof that X is not Ramsey. □

Theorem 2.5 allows us to immediately see that ℓ_3 is not Ramsey (as claimed in Section 1) since there is no hypersphere in any dimension whose surface contains three collinear points. More generally, this technique of translating the geometric concept of spherical sets into an algebraic one is very useful in proving negative results in Euclidean Ramsey Theory. For example, recently, Conlon and Führer [CF24] use this technique to prove that for any non-spherical set, there is an $m \in \mathbb{N}$ such that any red-blue coloring of any $\mathbb{R}^n$ has neither a red copy of the set nor a blue copy of ℓ_m, the set of m collinear points at unit distance apart.

2.2. **Boxes.**

Later in the same paper, Erdös et al. develop another powerful tool, this one producing examples of Ramsey sets.

Theorem 2.8. *If X and Y are Ramsey sets, then the Cartesian product $X \times Y$ is Ramsey as well.*

From this result along with the $n = 1$ case of Proposition 1.4, we can immediately derive a significant class of Ramsey sets.

Definition 2.9. An *n-brick* is a set that can be expressed as the Cartesian product of a collection of n two-point sets in $\mathbb{R}$.

Corollary 2.10. *All n-bricks are Ramsey.*

We can use Theorem 2.8 along with Proposition 2.3 to construct a variety of Ramsey sets.

Proposition 2.11. *Any isosceles triangle with side lengths c, d, and d for $d > c$ is Ramsey.*

Proof. Let $T' = \Delta ABC$ be an equilateral triangle of side length c in $\mathbb{R}^2$. Consider the triangular prism $T^* := T' \times \{0, \alpha\}$ for some $\alpha > 0$. Take the copies A_0, B_0 of A, B from $T' \times \{0\}$ and the copy C_α of C from $T' \times \{\alpha\}$. The distance between A_0 and B_0 is still c, and the distance between A_0 and C_α is $\sqrt{c^2 + \alpha^2}$, which can equal any $d > c$ depending on the value of α. By Theorem 2.8, T^* is Ramsey, and because the vertex set of $\Delta A_0 B_0 C_\alpha$ is a subset of T^*, we can apply Proposition 2.3 to conclude that it must also be Ramsey. □

This technique (which we can call *lifting*) is powerful, but it has its limits. With the tools we have now, it is still not possible to prove the Ramsey-ness of a shape with any obtuse angles. Thus, the following two results require an even more powerful tool.

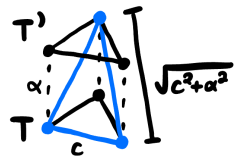

FIGURE 2. A lifted equilateral triangle, illustrating the proof of Proposition 2.11.

2.3. **Symmetry Groups.**

Theorem 2.12. *All (non-degenerate, i.e. positive-measure) triangles are Ramsey* [FR86].

Theorem 2.13. *All (non-degenerate isosceles) trapezoids are Ramsey* [Kř92].

Note that both of these theorems are consistent with the condition that all Ramsey sets are spherical, since both of these classes of sets can be embedded as subsets of circles in $\mathbb{R}^2$. Also, note that Theorem 2.13 implies Theorem 2.12, since for any triangle ΔABC, we can construct a trapezoid $ABCC'$, where C' is the reflection of C about the perpendicular bisector of $\overline{AB}$.

Both of these theorems were originally proven using relatively ad hoc combinatorial and analytic methods with little resemblance to each other. However, shortly after having submitted his proof of Theorem 2.13, Křiž [Kř91] developed a very powerful algebraic theorem that covers not only these two cases but also many others.

Definition 2.14. Let X be a finite point configuration spanning a d-dimensional subspace $S \subset \mathbb{R}^n$.

(1) A *group of isometries* G of X is a set of isometries in S mapping X onto itself that is closed under the group operation of composition.
(2) G *acts transitively* on X if for each pair of points $x_1, x_2 \in X$, there is an $f \in G$ such that $f(x_1) = x_2$. If X has a group of isometries that acts transitively, then we call it a *transitive* set.
(3) A subgroup H of G is *normal* if for all $x \in G$ and $y \in H$, $xyx^{-1} \in H$. We notate this as $H \triangleleft G$.
(4) G is *solvable* (or *soluble*) if there is a finite sequence of normal subgroups $1 = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_k = G$ of G such that each quotient G_i/G_{i-1} is an Abelian (i.e. commutative) group.
(5) Let $x \in X$. The *orbit* of x under G is the set $\{gx \mid g \in G\}$. Observe that X can be partitioned into one or more orbits, all disjoint from one another.

Theorem 2.15. *Let X and G be a configuration and one of its isometry groups, as in Definition 2.14. If G acts transitively and is solvable, then X is Ramsey. More generally, if G acts transitively and has a solvable subgroup with at most two distinct orbits, then X is Ramsey.*

Importantly, G need not be the *full* group of isometries of X; so long as it is a group satisfying the requirements, it can be used to show that X is Ramsey. As we will see in the Appendix, this allows us to prove Ramsey-ness without having to worry about accounting for all possible symmetries of X.

With Theorem 2.15, we are able to construct many more examples of Ramsey sets.

Definition 2.16. A *regular n-gon* is an equilateral and equiangular polygon with n sides. More generally, for $\alpha, \beta \in \mathbb{R}$ such that $0 \leq \alpha < \beta$, an (α, β) *semi-regular 2n-gon* is an equiangular polygon with $2n$ sides whose lengths alternate between two values, α and β.

Remark 2.17. We can see that α, β, and n uniquely determine the radius (or the distance from the center to each vertex) of a semi-regular polygon. There is no simple formula relating radius to α, β, and n, but we do not need one for our purposes. Also, note that in the case when $\alpha = 0$, a semi-regular $2n$-gon is just a regular n-gon, and that there is no way for a semi-regular $2n$-gon to be a regular $2n$-gon since α is strictly less than β. Finally, observe that semi-regular polygons are just truncated regular polygons.

Corollary 2.18. *All semi-regular 2n-gons are Ramsey.*

Proof. Let P be an (α, β) semi-regular 2n-gon. The group of isometries of P is the group of rotations and reflections mapping the set of β-edges onto itself, which acts transitively on the vertices of P. This is exactly the dihedral group D_n. The cyclic group C_n is both Abelian and a subgroup of D_n. Since C_n is Abelian, it is trivially solvable (with the subnormal series $1 \triangleleft C_n$). Also, C_n has two orbits on the vertex set of P; specifically, considering all points with respect to the β edge it is incident to, we can partition the points into the class on the clockwise ends of these edges and those on the counterclockwise ends. Thus, by Theorem 2.15, P is Ramsey. $\square$

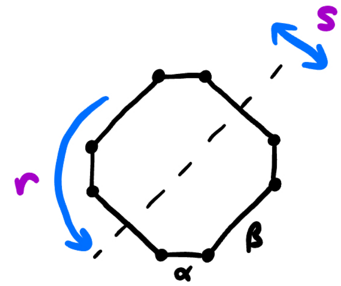

FIGURE 3. A semi-regular 8-gon, along with its two generating symmetries.

Semi-regular polygons can be used to construct sets satisfying Theorem 2.15 containing arbitrary trapezoids (and by extension, arbitrary triangles), thus proving Theorems 2.12 and 2.13. See Appendix A for a full construction and proof of Theorem 2.13. Interestingly, though this proof is a great example of the power of Křiž's result, it does not seem to appear anywhere in the literature; Leader et al. [LRW10] state that such a proof is possible but do not explicitly provide it.

In 1990, Frankl and Rödl [FR90] generalized their result in Theorem 2.12 to higher-dimensional configurations. Recall Definition 1.3: a *simplex* is a point configuration $\{x_0, x_1, \dots, x_d\} \subset \mathbb{R}^d$ such that the vectors $\{x_i - x_0\}_{i \in [d]}$ span $\mathbb{R}^d$ (which necessarily means that when interpreted as polyhedra rather than point configurations, they are non-degenerate).

Theorem 2.19. *All simplices are Ramsey.*

Again, since this result was published before that of Křiž, their proof is combinatorial and relatively complex. Karamanlis [Kar22] provides an alternate, (somewhat) simpler proof of Theorem 2.19 using Křiž's result.

3. Two Rival Conjectures

Based on Theorem 2.5, Graham [Gra17] poses the following conjecture, and, in keeping with the Erdös tradition, offers $1000 for a proof or counterexample.

Conjecture 3.1. *All spherical sets are Ramsey.*

Later, Leader et al. [LRW10] observed that in all known proofs of a class of sets being Ramsey, the authors embed the set in a larger, often very complex and high-dimensional transitive set, followed by an ad hoc combinatorial argument to show that the desired set must be Ramsey. Thus, they pose the following rival conjecture.

Definition 3.2. A set X is called *subtransitive* if it is a subset of a (potentially higher-dimensional) finite transitive set.

Conjecture 3.3. *A set is Ramsey if and only if it is subtransitive.*

It is clear that this conjecture is a stronger version of Křiž's theorem, since it simply removes the solvability criterion. We can further situate it within the hierarchy of known results with the following claim, whose proof is alluded to in [LRW10].

Proposition 3.4. *Every subtransitive set is spherical.*

Proof. Let $X \subset \mathbb{R}^d$ be the relevant transitive superset of a subtransitive set S. First, we show that there exists a unique closed ball $B(X) \subset \mathbb{R}^d$ enclosing X with minimal radius. To see this, consider the function $r : \mathbb{R}^d \to \mathbb{R}$ defined as $r(p) = \max_{x \in X} |x - p|$, which represents the minimum possible radius of a p-centered closed ball enclosing X. Since the function is continuous, non-negative, and goes to $+\infty$ in all directions, it must have at least one global minimizer. Assume for the sake of contradiction that it has two distinct global minimizers p_1 and p_2, where $r(p_1) = r(p_2) = r^* \geq 0$. Each $|x - p|$ component is convex, so their maximum r is as well. Thus, all (uncountably infinitely many) points p on the line between p_1 and p_2 are also global minimizers. To each of these points corresponds an $x \in X$ that is exactly r^* distance away. However, each point in X can be a given distance away from at most two points in a line, implying that X is also uncountably infinite, which contradicts our implicit assumption that X is finite. Thus, a global minimum $r^* := \min_{p \in \mathbb{R}^n} r(p)$ occurs at exactly one point $p^* \in \mathbb{R}^n$, which are the radius and center, respectively, of $B(X)$.

By definition, an isometry f of X maps X onto itself, so $f(X) = X$. Thus, since $B(X)$ is unique and $B(X)$ and $B(f(X))$ are the same ball, any isometry of X must map $B(X)$ onto itself.

X is spherical if and only if all of its points lie on the surface of $B(X)$. Assume for the sake of contradiction that X is non-spherical, meaning that there is a point $x^* \in X$ in the interior of $B(X)$. Let $x \in X$ be a point on the surface of $B(X)$. Because X's isometry group acts transitively, there must be an isometry f mapping x onto x^*. Since the center p^* of the ball is a fixed point under any isometry of X, f reduces the distance between x and p^*, contradicting the assumption that it is an isometry. Thus, X must be spherical. $\square$

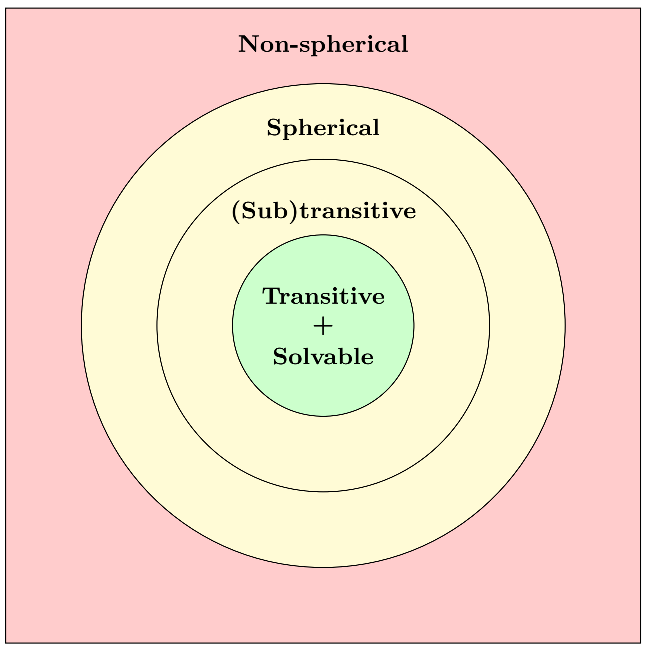

FIGURE 4. A diagram of what is known so far about which sets are and are not Ramsey.

Though Leader et al. were unable to prove either direction of Conjecture 3.3, they offered some evidence in the "subtransitive $\implies$ Ramsey" direction by providing a series of conjectures translating this statement into the language of algebra and then that of combinatorics, and proving that this combinatorial statement is very similar to the Hales-Jewett Theorem, a foundational result in Ramsey Theory. Later, Kanellopoulos and Karamanlis [KK20] show that this property holds for finite solvable groups, effectively recovering Křiž's result.

Also, they show that their conjecture is distinct from that of Graham by proving that there must exist spherical sets that are not subtransitive. In [LRW10], they provided a non-constructive proof that almost all 16-point subsets of a circle are not subtransitive, and in [LRW11], they showed that the same is true for 4-point subsets. They also provided an explicit example of a very simple class of these sets, namely the cyclic "kite" $\{(-1, 0), (a, \sqrt{1 - a^2}), (1, 0), (a, -\sqrt{1 - a^2})\}$ for any transcendental $a \in (-1, 1)$. We will refer to these as *transcendental kites*.

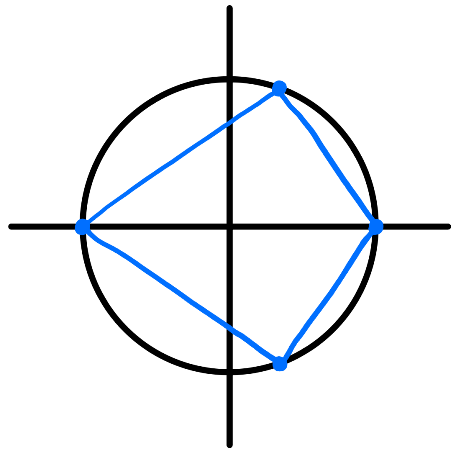

FIGURE 5. A kite with $a = \frac{1}{\pi}$, a non-subtransitive, spherical 4-point set.

It is also worth noting that it is merely a conjecture that we can remove Křiž's solvability condition. I have not encountered a result in the literature showing that there exist subtransitive sets not satisfying the conditions of Theorem 2.15, but there is also no proof that such a set cannot exist.

4. Directions for Further Research

Aside from proving either of the two main conjectures directly, we can break these large conjectures down into simpler cases that may shed some light on a better approach. For example, after having proven that all three-point subsets of the circle (i.e. all non-degenerate triangles) are Ramsey, it is natural to pose the following conjecture.

Conjecture 4.1. *All 4-point subsets of the circle are Ramsey.*

This statement, although seemingly simple and a natural extension of Theorem 2.12, has proven extremely elusive to researchers. Leader et al's result shows us why: unlike triangles, which are all subtransitive, almost all 4-point subsets of the circle are *not* subtransitive. Thus, we are entirely out of tools to work with since all known results start with either transitively acting, solvable symmetry groups or non-sphericality, neither of which apply in this case. Thus, a major next step could be:

Task 4.2. *Find a novel criterion for determining whether or not a set is Ramsey.*

As motivation for the discovery of such a new criterion, one can pursue the following conjecture of Leader et al. [LRW11]:

Conjecture 4.3. *All transcendental kites are not Ramsey.*

If we are in the business of disproving one of the main conjectures, then proving the Ramsey-ness or non-Ramsey-ness of even one of these kites would suffice. If we can find a Ramsey transcendental kite, then we will have found a Ramsey set that is spherical but not subtransitive, thus disproving the subtransitivity conjecture, and if we can find a non-Ramsey example, then we will have disproven the sphericality conjecture.

As mentioned in Section 3, although I have not encountered an example of a set that is subtransitive but does not satisfy the conditions of Theorem 2.15, I have also not seen a proof that such a set cannot exist. Thus, the following task would help differentiate between Křiž's result and Leader et al.'s conjecture.

Task 4.4. Find an example of a transitive configuration that does not generally satisfy the conditions of Theorem 2.15, or prove that there is no such example.

If such a set proved to be Ramsey, then this would provide further evidence toward Leader et al.'s conjecture, but if not, then both major conjectures would be disproven, and it would be warranted to conjecture that Křiž's conditions are necessary *and* sufficient.

However, finding such a set is certainly non-trivial. In this search, we can no longer rely on geometric intuition since all readily accessible examples of low-dimensional transitive sets in the literature also satisfy the (weaker) solvability condition. In $\mathbb{R}^2$, all transitive sets are dihedrally symmetric, and in $\mathbb{R}^3$, all *semi-regular* polyhedra (vertex-transitive and all faces regular polygons) have symmetry groups $D_n \times \mathbb{Z}/2$, tetrahedral symmetry, octahedral symmetry, and icosahedral symmetry [Sch17], all of which also satisfy the condition. All examples of non-semi-regular transitive polyhedra I have seen also had one of the aforementioned symmetry groups. As far as I can tell, it is still an open problem to find all transitive sets in $\mathbb{R}^3$. Thus, a natural and noble next step could be:

Task 4.5. Identify all possible isometry group structures for transitive sets in $\mathbb{R}^d$ for any/all $d \in \mathbb{N}$.

Finally, we briefly discuss a few generalizations of the concept of a Ramsey set, each producing a slew of interesting open problems. Graham's 2017 review [Gra17] discusses many of these generalizations in more detail.

A related and commonly studied notion is that of a *sphere-Ramsey* set, which essentially replaces Euclidean space with a finite-radius sphere in Definition 1.2. This concept was introduced in Erdös et al.'s original paper [EGM+73] and has been extensively studied since. Furthermore, one could study the phenomenon of Ramsey sets on arbitrary manifolds and metric spaces (essentially adding a "non-" to the term "Euclidean Ramsey Theory"); so long as a space is equipped with a distance metric, one can define a notion of congruence with which we can extend the idea of Ramsey sets.

Later, Erdös et al. [EGM+75] introduce the concept of *edge-Ramsey* sets. In this case, we consider all colorings of all *edges* in $\mathbb{R}^n$ (i.e. all pairs of points) and ask what monochromatic configurations of edges must exist in all colorings.

Appendix A. Proof of Theorem 2.13

Definition A.1. Consider a point configuration obtained by embedding an (α, β) semi-regular $2n$-gon P into $\mathbb{R}^3$, placing a copy P' "above" P such that they lie on parallel planes and share axes of rotation, and rotating P' about this axis by an angle $\theta \in [0, \frac{2\pi}{n})$. If $\theta = 0$, we call this a *drum*, and if $\theta = \frac{\pi}{n}$, we call this an *anti-drum*. Finally, in the specific case when $\theta \notin \{0, \frac{\pi}{n}\}$ and $\alpha = 0$, we call this a *skew-drum*. Note that a drum is just a P-based prism, and an anti-drum is analogous to an anti-prism, with the shorter edges of P aligned with the longer edges of P'. Also, note that the condition that $\alpha = 0$ in skew-drums is important because otherwise, as we will see, they are not transitive.

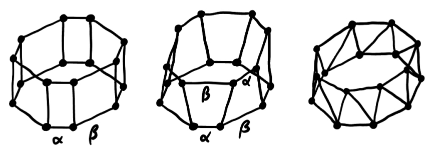

FIGURE 6. A drum, and anti-drum, and a skew-drum, respectively.

Corollary A.2. *All drums, anti-drums, and skew-drums are Ramsey.*

Proof. The Ramsey-ness of drums follows immediately from the Ramsey-ness of semi-regular polygons along with the Cartesian Product theorem. However, to demonstrate an application of Kříž's result, we provide an alternate proof here. Consider a drum X constructed from a semi-regular $2n$-gon P and its copy, P'. We claim that the group G of X's isometries is isomorphic to $D_n \times \mathbb{Z}/2$. Every isometry f of X corresponds to exactly one element of $D_n \times \mathbb{Z}/2$. This is because f can be uniquely decomposed into functions g and h, where g is an isomorphism mapping P and P' onto themselves and h is either the identity map or the vertical reflection swapping P and P'. The group of all possible maps g is isomorphic to D_n, and the group of possible maps h is isomorphic to $\mathbb{Z}/2$, so our claim is proven. Since D_n is solvable and a normal subgroup of G with $G/D_n = \mathbb{Z}/2$ Abelian, G is solvable. Thus, by Theorem 2.15, X is Ramsey.

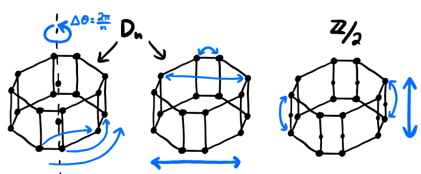

FIGURE 7. The generating symmetries of a drum.

Now, consider an anti-drum X. In this case, we claim that the group G of isometries of X is isomorphic to D_{2n}. The dihedral group is generated by two actions, r and s, corresponding to a rotation mapping adjacent edges of a $2n$-gon onto each other and a reflection flipping an edge on itself, respectively. Number the set of β-edges of both P and P' from e_1 through e_{2n} as they appear counterclockwise around the central axis of rotation. In this situation, r represents a vertical reflection and a rotation such that edges are mapped like $e_1 \to e_2 \to e_3 \to \ldots \to e_{2n} \to e_1$, and s represents a horizontal reflection mapping $e_1 \leftrightarrow e_1$, $e_2 \leftrightarrow e_{2n}$, $e_3 \leftrightarrow e_{2n-1}$, and so on. Together, these two actions generate a transitively acting group, since for

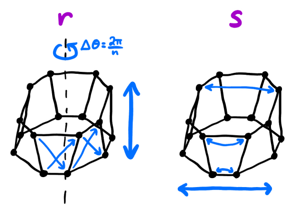

FIGURE 8. The generating symmetries of an anti-drum.

$x_1, x_2 \in X$, if x_1 is incident to edge n_1 and x_2 is incident to edge n_2 (WLOG assume $n_1 \leq n_2$), then we simply need to apply either the action $r^{n_2-n_1}$ or the action $r^{n_2}sr^{-n_1}$ to map x_1 onto x_2. Since any dihedral group is solvable, X is Ramsey.

Finally, consider a skew-drum X. Here, we claim that the group G of isometries of X is isomorphic to D_n. Skew-drums have bases P, P' which are both regular n-gons. Thus, r still represents a rotation of X by $\frac{2\pi}{n}$ about its axis of rotation. However, since X is chiral, G cannot contain any reflections. We can get around this by observing that there is a rotation s of $\mathbb{R}^3$ that swaps P' and P and reverses each one's orientation; this simulates the reflection action in the traditional dihedral group. Each point in P can reach all other points in P through repeated applications of r and can reach all points in P' through an s followed by r's. Note that this transitivity would not hold if $\alpha > 0$ since every semi-regular polygon has two orbits of vertices under rotational symmetry alone, and the pseudo-reflection in the skew-drum does not bridge the gap between these two orbits.

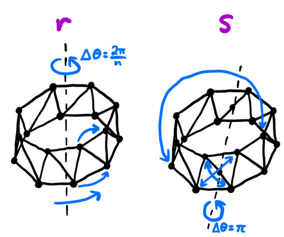

FIGURE 9. The generating symmetries of a skew-drum.

□

From this result, we can construct a Ramsey set that can be used to prove Theorem 2.13 (and by extension, Theorem 2.12) using Křiž's theorem.

Proof. Let T be a trapezoid with parallel side lengths α and β and height $h > 0$. WLOG, assume $\alpha < \beta$. Our goal is to construct an anti-drum with top and bottom faces P, P', where an α-edge in P and the β-edge directly above it form a copy of T.

Consider an arbitrary anti-drum whose bases are (α, β) semi-regular $2n$-gons, and whose bases are a vertical distance d apart. Let e_α be an α-edge in P, and let e_β be the corresponding β-edge in P'. In the extreme case when $d = 0$, e_α and e_β are very close together, so the trapezoid formed by these two line segments has a relatively small height h_0. As $|d|$ goes to infinity, the bases of this trapezoid stay constant, while its height also goes to infinity, sweeping through every possible height in the interval (h_0, ∞). As shown in Corollary A.2, this collection of anti-drums is Ramsey, so by Proposition 2.3, each of these intermediate trapezoids must be as well. Thus, it just remains to show that we can achieve an $h_0 < h$ with the right n.

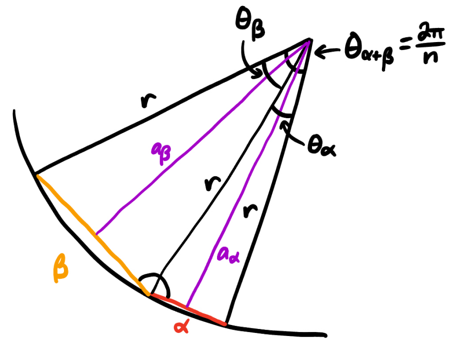

FIGURE 10. A slice of an (α, β) semi-regular $2n$-gon and its circumcircle.

Considering the $d = 0$ extreme, we can see that h_0 is the difference between the lengths of P's apothems (line segments from the center perpendicular to the polygon's sides) to α sides and β sides. Consider the diagram in Figure 10. The apothem lengths a_α and a_β correspond to central angles θ_α and θ_β, which add to a total central angle $\theta_{\alpha+\beta} = \frac{2\pi}{n}$. As we raise n to ∞, each of these central angles goes to 0. Looking at the isosceles triangle formed by the α edge and its two adjacent radii, we can also see that $a_\alpha = r \cos \frac{\theta_\alpha}{2} \to r$ as $n \to \infty$; the same holds for a_β and θ_β. Since $\beta > \alpha$, we can see that $\theta_\beta > \theta_\alpha$, implying that $a_\alpha > a_\beta$. Thus, $h_0 = a_\alpha - a_\beta > 0$ and $h_0 \to 0$ as $n \to \infty$, so for any h, we can choose a sufficiently large n such that $h_0 < h$, allowing us to construct a copy of T within the drum's vertex set. $\square$

As previously noted, Theorem 2.13 proves Theorem 2.12. However, a similar construction using skew-drums can be used to prove Theorem 2.12 directly.

Acknowledgments

I would like to thank my mentor, Prof. Alexander Razborov, for his guidance through the rich literature of (Euclidean) Ramsey Theory since January, and through the process of writing this paper. I would also like to thank Prof. Peter May along with all participants and lecturers in the REU for exposing me to so many fascinating ideas in mathematics and cultivating a wonderful learning environment this summer.

References

[CF24] David Conlon and Jakob Führer. *Non-spherical sets versus lines in Euclidean Ramsey theory*, Jun 2024.

[EGM+73] Paul Erdös, Ronald L. Graham, Peter Montgomery, Bruce L. Rothschild, Joel H. Spencer, and Ernst G. Straus. Euclidean ramsey theorems i. *Journal of Combinatorial Theory, Series A*, 14(3):341–363, May 1973.

[EGM+75] Paul Erdös, Ronald L. Graham, Peter Montgomery, Bruce L. Rothschild, Joel Spencer, and Ernst G. Straus. volume II, page 559–583. 1975.

[FR86] Peter Frankl and Vojtech Rödl. All triangles are ramsey. *Transactions of the American Mathematical Society*, 297(2):777–779, Oct 1986.

[FR90] Peter Frankl and Vojtech Rodl. A partition property of simplices in euclidean space. *Journal of the American Mathematical Society*, 3(1):1–7, Jan 1990.

[Gra17] Ronald L. Graham. *Euclidean Ramsey Theory*, page 281–297. CRC Press, 3 edition, 2017.

[Kar22] Miltiadis Karamanlis. Simplices and regular polygonal tori in euclidean ramsey theory. *The Electronic Journal of Combinatorics*, 29(3), Jul 2022.

[KK20] Vassilis Kanellopoulos and Miltiadis Karamanlis. A hales–jewett type property of finite solvable groups. *Mathematika*, 66(4):959–972, Aug 2020.

[Kř91] Igor Kříž. Permutation groups in euclidean ramsey theory. *Proceedings of the American Mathematical Society*, 112(3):899–907, Jul 1991.

[Kř92] Igor Kříž. All trapezoids are ramsey. *Discrete Mathematics*, 108:59–62, 1992.

[LRW10] Imre Leader, Paul A. Russell, and Mark Walters. Transitive sets in euclidean ramsey theory. *Journal of Combinatorial Theory*, 119(2):382–396, Feb 2010.

[LRW11] Imre Leader, Paul A. Russell, and Mark Walters. Transitive sets and cyclic quadrilaterals. *Journal of Combinatorics*, 2(3):457–462, 2011.

[Pr5] Hans Jürgen Prömel. Complete disorder is impossible: The mathematical work of walter deuber. *Combinatorics, Probability and Computing*, 14(1):3–16, Jan 2005.

[Ram30] Frank P. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, s2-30(1):264–286, 1930.

[Sch17] Egon Schulte. *Symmetry of Polytopes and Polyhedra*, pages 477–503. CRC Press, 3 edition, 2017.