GAUGE THEORY AND THE CHERN-SIMONS ACTION

RYAN O'FARRELL

Abstract. This paper builds up the mathematical foundations necessary for understanding Chern-Simons theory, a type of field theory fundamental to modern quantum condensed matter physics.

Contents

Introduction

Why is the universe the way that it is? While this is a question that has been asked for as long as humans have been asking questions, the best answer physicists have been able to come up with so far is *gauge theory*, which suggests that the fundamental particles and interactions of the universe ultimately arise from *local symmetries* of the universe.

However, collections of interacting particles can produce *emergent* behaviors not seen at the microscopic level. These can be described by *effective* gauge theories. In recent decades, it has become apparent that many of these theories are sensitive to topological properties of spacetime. In this paper, we will describe one kind of topological quantum field theory, *Chern-Simons theory*, and see how considerations from gauge theory and topology give rise to the *(integer) quantum Hall effect*.

1. Bundles on Spacetime

1.1. Principal Bundles.

Local symmetries are described mathematically by the action of *Lie groups* on *principal bundles*.

Definition 1.1. A *Lie group* is a topological group with the topology of a smooth manifold such that group operations (multiplication and inversion) are smooth.

Example 1.2. Translation invariance ($\mathbb{R}^n$), rotation invariance ($SO(n)$) and phase invariance ($U(1) \simeq S^1$) are common symmetries described by Lie groups.

For simplicity, every topological space in this paper will be a smooth manifold. Additionally, we will only consider *matrix Lie groups*, i.e. those that are Lie subgroups of $GL(n, \mathbb{C})$ for some n, as these are the ones that appear most commonly in physics. As it turns out, we don't lose much.

Theorem 1.3. *Every compact Lie group is a matrix group.*

Proof. This is Theorem 4.2 in [2] and is an immediate consequence of the Peter-Weyl theorem. □

A general gauge theory requires a spacetime manifold M and some internal information about G at each point of M. This is captured by a *fiber bundle*.

Definition 1.4. A *fiber bundle*[1] is a space E equipped with a continuous surjective map π to the *base space* M such that for every $x \in M$

- $\pi^{-1}(x)$ is homeomorphic to the *fiber* F
- There exists a neighborhood U of x such that there is a *trivialization*, that is, a diffeomorphism
$$\phi_U : E_U \to U \times F$$
from a subset $E_U = \pi^{-1}(U)$ of E satisfying
$$\pi_1 \circ \phi_U = \pi$$
where $\pi_1 : U \times F \to U$ is the canonical projection of the first element.

Fiber bundles as objects are generalizations of product spaces. Of course, to the aspiring category theorist, what really matters is morphisms *between* bundles.

Definition 1.5. Between two fiber bundles (F_1, E_1, π_1, M) and (F_2, E_2, π_2 M) over the same base space, a *bundle map* ϕ is a smooth map $E_1 \to E_2$ such that $\pi_2 \circ \phi = \pi_1$

A symmetry of the bundle under a Lie group G implies some kind of action of G on our bundles. For this, we need to ensure that it is a *principal bundle*.

Definition 1.6. A *principal* G-*bundle* P is a fiber bundle with fiber G and two additional pieces of structure

- A right action of G on P that restricts to an action $\pi^{-1}(x) \times G \to \pi^{-1}(x)$ on each fiber. Moreover, this action is *transitive* in that the map $G \to \pi^{-1}(x)$ given by $g \mapsto xg$ is a bijection.

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[1] Sometimes spelled "fibre bundle", but in general English is nonabelian.

- The existence of a *principal bundle atlas*: charts $\phi_i : P_{U_i} \to U_i \times G$ satisfying
$$\phi_i(p \cdot g) = \phi_i(p) \cdot g$$
where g acts on the second element of $\phi_i(p)$.

We will denote by $r : P \times G \to P$ the action $(p, g) \mapsto pg$. We will denote by $r_g : P \to P$ the action $p \mapsto pg$. We will denote by $Orb_p : G \to P$ the *orbit map* $g \mapsto pg$. The following theorem will not be proven here. It is neat, though.

Theorem 1.7. *If two manifolds M and N are homotopy equivalent, then their isomorphism classes of principal bundles are in bijection.*

Proof. Lots of theorems about principal bundles can be proven much easier with the theory of *classifying spaces*, which will not be discussed here. This is one of them, so I will leave a reference [3]. □

1.2. **Aside: The Hopf Bundle.**

In light of the following corollary, it may seem a little pointless to introduce the general theory so quickly.

Corollary 1.8. *If M is contractible, then every principal bundle over M is trivial (isomorphic to $M \times G$).*

Proof. Contractible spaces are homotopy equivalent to a point, and the point only has one G-principal bundle isomorphism class over it. □

Physicists typically only care about the case where $M = \mathbb{R}^4$, which is contractible. However, the general theory becomes important when stranger spacetimes are considered (e.g. in string theory). Additionally, quantum mechanics hosts an unexpectedly well-known example of a principal bundle.

Consider unit vectors in the Hilbert space $\mathbb{C}^2$, the space of which is topologically S^3. If we identify vectors up to global phase, i.e. make the identification
$$(c, d) \sim (e^{i\theta}c, e^{i\theta}d), \theta \in \mathbb{R}$$
then we get the space of *qubits*, which is topologically the *Bloch sphere* S^2. With this "phase-forgetting" projection map $\pi : S^3 \to S^2$, we exhibit S^3 as a nontrivial $U(1)$-principal bundle over S^2. This is known as the *Hopf bundle*.

1.3. **Tangent Bundles and Forms.**

Equipped to every manifold M is a special bundle called the *tangent bundle* TM, which is the fiber bundle whose fibers are the tangent vector spaces at every point.

If we have a map $\phi : M \to N$ between manifolds, the derivative $D\phi$ assigns to each point of M a linear transformation between vector spaces. This means the derivative is a map $D\phi : TM \to TN$ between their corresponding tangent bundles.

The tangent space to a Lie group is special.

Definition 1.9. A *Lie algebra* $\mathfrak{g}$ is the tangent space to a Lie group G at a point.

Lie algebras are equipped with an operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ known as the *Lie bracket*. There is a lot that can be said about the algebraic structure of Lie algebras, of which very little is relevant to this paper. Since we're only working with matrix Lie groups, a Lie algebra is just some vector subspace of the matrix group $\mathrm{Mat}(n \times n, \mathbb{C})$. $\mathfrak{g}$ has a natural action from G by the *adjoint representation*.

Definition 1.10. The *adjoint representation* of G is the map $G \to Aut(\mathfrak{g})$ given by the action of conjugation. We use $g \mapsto Ad_g$ for this map, where

$$Ad_g(M) = gMg^{-1}$$

That $Ad_g(M)$ is in $\mathfrak{g}$ follows from it being the derivative of the map $h \mapsto ghg^{-1}$ at $h = \mathrm{Id}$.[2] This is not to be confused with the *fundamental representation*

Definition 1.11. The *fundamental representation* of G is the mapping $G \to GL(n, \mathbb{C}) = Aut(\mathbb{C}^n)$ given in the definition of G

Example 1.12. Consider the matrix Lie group $U(1)$, which is defined by

$$\{U \in GL(1, \mathbb{C}), U^\dagger U = \mathrm{Id}\}$$

The fundamental representation is the "obvious" injection $U(1) \hookrightarrow GL(1, \mathbb{C}) \simeq \mathbb{C}^\times$. However, since the group is abelian, the adjoint representation is trivial.

A map $\omega : TM \to \mathbb{R}$ is called a *(differential) 1-form.* Correspondingly, a map $\omega : TM \to \mathfrak{g}$ is called a $\mathfrak{g}$*-valued 1-form.* In general, we can write a $\mathfrak{g}$-valued 1-form as a tensor product $\omega \otimes X$, where ω is an $\mathbb{R}$-valued 1-form and $X \in \mathfrak{g}$.

A particularly natural $\mathfrak{g}$*-valued 1-form* on G itself is the *Maurer-Cartan form*

$$\mu_G = g^{-1} dg$$

dg is the identity map $TG \to TG$, and g^{-1} translates a vector from T_gG to $T_eG \simeq \mathfrak{g}$. Elements of the Lie algebra act like infinitesimal versions of elements of the Lie group. This is seen by the following definition.

Definition 1.13. For an element $X \in \mathfrak{g}$, the *fundamental vector field* $\tilde{X}$ associated to X is defined at $p \in P$ to be $D_{\mathrm{Id}} Orb_p(X)$

In this way, each element of the Lie algebra determines an "infinitesimal action" on P.

1.4. **Ehresmann connections.** The action of G provides a way to move *along* fibers. But what's the natural way to move *between* fibers? As it turns out, there's no canonical way to do this, so we have to be a bit clever. This will eventually lead to the notion of a "field on spacetime".

If we take the derivative of the projection map $\pi : P \to M$ we get a map $D\pi : TP \to TM$ between tangent bundles. The kernel of this map picks out a subbundle $VP \subset TP$, called the *vertical bundle.* At each point $p \in P$, we can therefore decompose the tangent vector space as a direct sum

$$T_p = V_pP \oplus H_pP$$

where H_pP is the *horizontal bundle.* Besides ensuring that V_pP and H_pP are complementary, we're very free to pick H_pP.

Definition 1.14. An *Ehresmann connection* is a smooth choice of H_pP for each point $p \in P$.

There's an alternate definition of a connection. Observe that instead of defining H_pP directly at each point, we can first define a map $\omega_p : T_pP \to V_pP$ and define $H_p = \ker(\omega_p)$. But note that V_pP is basically the same as the tangent space to our Lie group T_pG. We can therefore define an Ehresmann connection entirely as the

[2]This is the definition of the adjoint representation for non-matrix Lie groups.

kernel of a map $\omega : TP \to \mathfrak{g}$, a Lie algebra-valued 1-form. In order to ensure that we get a smooth definition of H_pP, we put two restrictions on ω.

- $\omega \circ Dr_g = Ad_{g^{-1}} \circ \omega$. This asserts that horizontal subspaces can be related by the action of G.
- $\omega(\tilde{X}) = X$. This asserts that ω acts like the identity on the vertical subspace.

We call ω a *connection 1-form.* Note that the Maurer-Cartan form is a connection 1-form of the principal bundle where M is a single point. In fact, it is the *unique* such connection!

2. Gauge Transformations

2.1. **Sections.** We can move from space-time M to this "higher" space E with a choice of *section.*

Definition 2.1. A *(local) section on $U \subseteq M$* is a map $s : U \to P$ satisfying $\pi \circ s \equiv id_U$.

A choice of section is called *fixing a gauge*, and a change of section is called a *gauge transformation.*

Observation 2.2. If P is a principal bundle, then given two sections s, s', we can always write $s' = r_{g(x)} \circ s$, where $g(x)$ applies the transitive group action to every fiber to "slide" everything to its correct image.

The derivative of a section s gives us a map $Ds : TU \to TP$. Combined with the connection ω, we can associate to each section a Lie algebra-valued 1-form on M known as the *local connection 1-form $A = \omega \circ Ds$.*

Theorem 2.3. *Under the transformation $s \mapsto r_{g(x)} \circ s$, A transforms as*

$$A \mapsto Ad_{g^{-1}} \circ A + g^{-1} \cdot dg$$

Proof. We must compute $\omega \circ D(r \circ (s, g))$. Observe that with

$$D_{(x,g)}r : T_xM \oplus T_gG \to T_{xg}M$$

by linearity we have

$$\begin{aligned} D_{(x,g)}r(X, Y) &= D_{(x,g)}r(X, 0) + D_{(x,g)}r(0, Y) \\ &= (D_x r_g)(X) + (D_g Orb_x)(Y) \end{aligned}$$

Hence, by the chain rule

$$D_{s(m),g(m)}(r \circ (s, g)) = (D_{s(m)} r_g) \circ D_m s + (D_{g(m)} Orb_m) \circ D_m g$$

By the first constraint on connection 1-forms, we have that

$$\omega \circ (D_{s(m)} r_g) = Ad_{g^{-1}} \circ \omega$$

For the second term, we have that

$$(D_{g(m)} Orb_m) \circ D_m g = (D_{Id} Orb_m) \circ \mu_G \circ D_m g$$

which is just the fundamental vector field associated to $\mu \circ D_m g$. Again using the defining property of a connection 1-form,

$$\omega \circ (D_{g(m)} Orb_m) \circ D_m g = \mu_G \circ D_m g = g^{-1} dg$$

Hence we get the full transformation law

$$A \mapsto Ad_{g^{-1}} \circ A + g^{-1} \cdot dg$$

□

2.2. **Aside: Electromagnetism as a U(1) gauge theory.** Consider a $U(1)$ gauge theory. We can identify $\mathfrak{u}(1) \simeq i\mathbb{R}$ and define an $\mathbb{R}$-valued 1-form A where $\omega = iA$. The transformation law tells us that under a change of section given by $g(x) = e^{i\phi(x)}$, A transforms as

$$A' = A + d\phi$$

which is exactly the conventional gauge transformation of electromagnetism. We can pick a set of coordinates and take derivatives to turn $A : TM \to \mathbb{R}$ into a map $\tilde{A} : M \to \mathbb{R}$ that we can identify as a typical "field on spacetime" called the *gauge boson field*. In the $(3+1)$-dimensional $U(1)$ case $\tilde{A}$ is the *electromagnetic 4-potential*, and from it all other interesting quantities in electromagnetism can be derived.

Observe that we get one gauge boson field for every generator of our Lie algebra. Since $\mathfrak{u}(1)$ is 1-dimensional, there is only one photon. The full Standard Model is an $SU(3) \times SU(2) \times U(1)$ gauge theory, whose Lie algebra has 12 generators: 8 gluons, 3 weak force bosons ($W^{\pm}$ and Z^0) and 1 photon (although this is complicated by Higgs mechanism considerations).

2.3. **Curvature.** Given a connection, we can define its corresponding *curvature 2-form*. Unlike for $\mathbb{R}$-valued forms, the normal way of defining the wedge product does not work for Lie algebra-valued forms since Lie algebras do not have a notion of "multiplication". We do, however, have the Lie bracket. Given two $\mathfrak{g}$-valued 1 forms

$$a = \sum_{n=1}^{\dim \mathfrak{g}} a^n \otimes T_n \qquad b = \sum_{n=1}^{\dim \mathfrak{g}} b^n \otimes T_n$$

where $\{T_n\}$ is a basis for the Lie algebra, we can define the Lie bracket

$$[a, b] = \sum_{m,n=1}^{\dim \mathfrak{g}} (a^n \wedge b^n) \otimes [T_a, T_b]$$

which is a $\mathfrak{g}$-valued 2-form. We can then use this to define the *curvature 2-form* of a connection ω

$$F_\omega = d\omega + \frac{1}{2}[\omega, \omega]$$

Just like the local connection 1-form, we can define a *local curvature* $F_A = F_\omega(Ds, Ds)$. Under changes of section (gauge transformations), F_A transforms as

$$F_A \mapsto Ad_{g^{-1}} \circ F_A$$

In physics this is also known as the "field strength tensor". In the $U(1)$ case, this is known the *Faraday tensor* and is manifestly gauge invariant since $Ad_{g^{-1}}$ is trivial.

3. Chern-Weil Theory

There are conventional ways of constructing gauge invariant theories from gauge fields and their corresponding curvatures (e.g. Yang-Mills theories). We are not interested in those.

3.1. The Chern-Weil Homomorphism.

Recall that the space Ω of *differential forms* on M is a graded algebra with a product given by the *wedge product* $\wedge$. Additionally, we have the *exterior derivative* map $d : \Omega \to \Omega$ satisfying $d^2 = 0$. A form $\omega \in \Omega$ is *closed* if it is in the kernel of d (i.e. $d\omega = 0$) and *exact* if it is in the image of d (i.e. $\omega = d\beta$ for $\beta \in \Omega$).

Definition 3.1. The *de Rham cohomology* of M is the quotient of the closed forms on M by the subgroup of exact forms on M. It is a graded ring with product given by the conventional wedge product $\wedge$.

On any vector space we can define

Definition 3.2. A *symmetric* k-*linear function* is a map $Q : V^k \to \mathbb{R}$ that is invariant under the reordering of variables.

Invariant k-linear functions form a graded ring with the *symmetric product* $\circ$, which we denote $I^*(G)$

$$(Q_1 \circ Q_2)(v_1, \ldots, v_{k+\ell}) = \frac{1}{(k+\ell)!} \sum_{\sigma \in \mathbb{S}_{k+\ell}} T_1(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdots T_2(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})$$

The adjoint representation gives us a natural action of G on the vector space $\mathfrak{g}^k$ by applying the adjoint action to each copy of $\mathfrak{g}$.

Definition 3.3. An *invariant* k-*linear function*[3] Q is symmetric k-linear function that is invariant under the adjoint action.

We can let V be the space of $\mathfrak{g}$-valued 2-forms by defining for $\omega_i \in \Omega(M)$, $X_i \in \mathfrak{g}$

$$Q(\omega_1 \otimes X_1, \ldots, \omega_n \otimes X_n) := (\omega_1 \wedge \cdots \wedge \omega_n) Q(X_1, \ldots, X_n)$$

and extending by linearity to all $\mathfrak{g}$-valued 2-forms. Alternatively, we can define a product-like operation on two $\mathfrak{g}$-valued 2-forms

$$(\omega_1 \otimes X_1) \wedge (\omega_2 \otimes X_2) = (\omega_1 \wedge \omega_2) \otimes (X_1 X_2)$$

Note that $X_1 X_2$ is not typically in $\mathfrak{g}$, but generically takes values in some matrix group. Q now assigns to every curvature 2-form a corresponding 2k-form on E by writing

$$Q_\omega := Q(F_\omega, \ldots, F_\omega)$$

Observe that a choice of section turns Q_ω into a $2k$-form on M. Since F_A transforms under the adjoint representation with respect to changes of section and Q is invariant under the adjoint action, Q_ω can be identified with a unique $2k$-form on M. We call this form **the** *characteristic form* associated to Q, a name justified by the following surprising theorem.

Theorem 3.4. *(Chern-Weil)*

- *Q_ω is closed.*

[3] Sometimes confusingly called an "invariant polynomial"

- *The cohomology class $w(E;Q) = [Q_\omega]$ does not depend on the choice of connection.*
- *The map $w(E,-) : (I^*(G), \circ) \to (H^*_{dR}(M), \wedge)$ is a ring homomorphism (the Chern-Weil Homomorphism).*

Proof. All of these are found in [4]. □

An immediate corollary is the following.

Corollary 3.5. *For any two connections ω_1, ω_2, the form $Q(F^k_{\omega_1}) - Q(F^k_{\omega_2})$ is exact and can be written as dCS_Q for some form CS_Q*

We call CS_Q the *Chern-Simons form* associated to Q. Note that we have

Lemma 3.6. *(Poincaré) If M is contractible then $H^k_{dR}(M)$ is trivial for $k > 0$.*

As a result, in order to construct interesting Chern-Simons forms, we should look at spacetimes with nontrivial topology.

3.2. **The Chern-Simons 3-form.** Let's take a look at a specific Chern-Simons form. One classic example of an invariant polynomial on the space of matrices is

$$Q(A, B) = tr(AB)$$

Observe that it is invariant under permutation and the adjoint action.

$$tr(AB) = tr(BA) \qquad tr((GAG^{-1})(GBG^{-1})) = tr(AB)$$

Chern-Weil theory then tells us we should look at

$$tr(F_A \wedge F_A) = 8\pi^2 c_2$$

and this form is independent of the choice of section used to define F_A. This turns out to be $8\pi^2$ times the *second Chern class*, an important object in the general study of *characteristic classes* [5]. What's relevant here is that Chern-Weil theory tells us it must have a corresponding Chern-Simons 3-form

$$CS = tr(dA \wedge A + \frac{2}{3} A \wedge A \wedge A)$$

such that on local patches

$$c_2 = dCS$$

Observe that under gauge transformations

$$A^g = g^{-1}Ag + g^{-1}dg$$

we get after a lot of algebra that the Chern-Simons term transforms as

$$CS(A^g) = CS(A) + \int_M d(tr(Adg\, g^{-1}) - \frac{1}{3}\int_M tr((g^{-1}dg)^3)$$

Two cases that are especially relevant to physics are when $G = SU(N)$ or $G = U(1)$. For $G = SU(N)$, we can choose $M = S^3$ as an "interesting" spacetime manifold. Since the second term is a total derivative, it vanishes. The third term, however, does *not* vanish. It turns out to equal $8\pi^2$ times the the degree of the map $g : S^3 \to G$! Therefore, this integral takes values in $\pi_3(G)$, and when $G = SU(N)$, we have a stunning result

Theorem 3.7. *If G is a compact connected simple Lie group (like $SU(N)$), then $\pi_3(G) = \mathbb{Z}$.*

Gauge transformations that are not homotopic to the identity are called *large gauge transformations*. Under these gauge transformations, the Chern-Simons term shifts by $8\pi^2 n$ for $n \in \mathbb{Z}$!

Another common gauge group in physics is $U(1)$. Even though $\pi_3(U(1)) = 0$, if we choose M differently, for example $M = S^1 \times S^2$, we get nontrivial results. ([6] describes this as compactifying space to S^2 and making time periodic as S^1.) Now, the relevant quantity is winding numbers of maps $S^1 \times S^2 \to U(1)$, which take values in $\pi_1(U(1)) = \mathbb{Z}$. We've seen from Section 2.2 that $U(1)$ gauge transformations take the form

$$A \mapsto A + d\phi$$

so

$$\int_M d(A + d\phi) \wedge (A + d\phi) = \int_M CS + \int_M dA \wedge d\phi$$

If ϕ does not wind around the S^1, the integral does not change. However, if ϕ is a large gauge transformation, there is an additional contribution of $2\pi \cdot 4\pi = 8\pi^2$ to the integral for each time the gauge transformation winds around $S^1 \times S^2$, so once again, large gauge transformations shift the Chern-Simons term by $8\pi^2 n$.

4. 2+1D $U(1)_k$ Chern-Simons Theory

As a treat, we will now see some of the unusual physical consequences of a system with an action containing a Chern-Simons term.

Definition 4.1. A *Chern-Simons theory* is a field theory whose Lagrangian density is proportional to a Chern-Simons form

This means they are governed by a *Chern-Simons action*

$$S = \frac{k}{4\pi} \int_M CS$$

k is called the *level* of the Chern-Simons theory. To be a *quantum field theory* means the system is governed by the *partition function*[4]

$$Z \sim e^{iS}$$

A rigorous treatment of quantization is not presented here, since it is more witchcraft than mathematics. The most striking feature of Chern-Simons theories is that, unlike most physical theories, they make no reference to a metric on M. They are purely *topological quantum field theories*!

The second most striking feature of Chern-Simons theories is that, in order for Z to be gauge invariant, k must be quantized to integer values only. Since gauge transformations send $S \mapsto S + 2\pi kn$, we must have this constraint in order for Z to remain invariant. This constraint ends up having observable consequences.

As above, consider a $G = U(1)$ Chern-Simons theory. $A = \sum_\mu a_\mu \, i dx_\mu$ is a 1-form with values in $\mathfrak{u}(1)$. Since the Lie algebra is abelian, we can simplify a lot

[4] Not to be confused with the partition function in statistical mechanics, which is somewhat related, or the partition function in number theory, which is completely unrelated.

and write the action (after a lot of algebra with forms) as a sum over permutations of (t,x,y)

$$S_{CS}[A] = \frac{k}{4\pi} \int_M \sum_{\sigma \in \mathbb{S}_3} \mathrm{sgn}(\sigma) a_{\sigma(1)} \partial_{\sigma(2)} a_{\sigma(3)}$$

Treating A as a kind of electromagnetic 4-potential, we can compute the current density in space

$$J_x = \frac{\delta S_{CS}[A]}{\delta a_i} = \frac{k}{2\pi} \left(\partial_y a_t - \partial_t a_y\right) \stackrel{\text{def}}{=} \frac{k}{2\pi} E_y$$

where E is the *electric field*. This describes a system with a *Hall conductivity* of $\frac{k}{2\pi}$. Strikingly, this *quantum Hall effect* has been observed in real materials, matching the predicted value from Chern-Simons theory to a precision of one part in one billion [6]!

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