

THE BANACH-TARSKI PARADOX

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ABSTRACT. This paper offers an overview of the Banach-Tarski paradox. We will begin by discussing what it means for a set to be paradoxical under a group action. Then, we will prove the Banach-Tarski paradox and its more general forms. In the second half of the paper, we will investigate on what conditions does a group action on a set result in a paradox. We will conclude this paper by creating an equivalence relation between non-paradoxical groups, amenable groups, and groups that satisfy the Følner condition.

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1. INTRODUCTION

The Banach-Tarski paradox states that a ball is equidecomposable with two copies of itself via rotations. In section 2 we will begin by investigating equidecomposable sets and the useful properties they hold. Then in section 3 we will look to prove that the hollow sphere and ball are paradoxical under rotations in $\mathbb{R}^3$. We will prove a more general form of the Banach-Tarski paradox under the set of isometries in $\mathbb{R}^3$. In section 4 we will begin to construct conditions for non-paradoxical groups. Finally in section 5, we will complete the equivalence relation between non-paradoxical groups, amenable groups, and groups that satisfy the Følner condition.

2. EQUIDECOMPOSITIONS

We devote this section to understanding equidecomposable sets. From a more simple geometric interpretation, this concept can be thought of as creating equivalence classes between sets that can be cut into a finite number of pieces, rearranged, and glued back together in order to form each other. Our first main goal will be to show that we can remove a countable number of points from a circle without destroying this relation.

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Definition 2.1. Let a group Γ act on the set X . We say that $A, B \subset X$ are Γ -*equidecomposable* if there exist sets $C_1, \dots, C_k$ and elements $\gamma_1, \dots, \gamma_k \in \Gamma$ such that

$$A = \bigsqcup_{i \leq k} C_i$$

$$B = \bigsqcup_{i \leq k} \gamma_i \cdot C_i$$

If A and B are equidecomposable, we use the notation $A \approx B$.

Definition 2.2. Suppose $\Gamma \curvearrowright X$ and $A, B \subset X$. If there exists $B' \subset B$ such that $A \approx B'$, then we say that A is Γ -*embeddecomposable* into B and use the notation $A \preceq B$.

Theorem 2.3. Suppose $\Gamma \curvearrowright X$. Then Γ -*equidecomposability* forms an equivalence relation.

Proof. Reflexivity and symmetry are satisfied by the identity element and inverse elements of Γ . If $A \approx B$, then there exist subsets $C_1, \dots, C_k \subset A$ and elements $\gamma_1, \dots, \gamma_k \in \Gamma$ such that $\bigsqcup C_i = A$ and $\bigsqcup \gamma_i[C_i] = B$. If $B \approx C$, then there exist subsets $D_1, \dots, D_m \subset B$ and elements $\delta_1, \dots, \delta_m \in \Gamma$ such that $\bigsqcup D_j = B$ and $\bigsqcup \delta_j[D_j] = C$. Construct subsets $G_{ij} = \{x \in C_i : \gamma_i \cdot x \in D_j\}$. Then

- $\bigsqcup_i \bigsqcup_j G_{ij} = \bigsqcup_i C_i = A$
- $\bigsqcup_j \bigsqcup_i (\delta_j \cdot \gamma_i) \cdot G_{ij} = \bigsqcup_j \delta_j \cdot \bigsqcup_i \gamma_i \cdot G_{ij} = \bigsqcup_j \delta_j \cdot D_j = C$

□

Lemma 2.4. Suppose A is a circle in $\mathbb{R}^2$ and C is a countable subset of A . If

$$\Theta = \{\theta \in [0, 2\pi] : r_\theta^n(C) \cap C = \emptyset \quad \forall n \in \mathbb{N}\}$$

then Θ is non-empty.

Proof. As $[0, 2\pi]$ is uncountable, it suffices to prove that $[0, 2\pi] - \Theta$ is countable. Fix $n \in \mathbb{N}$ and $a, b \in C$. Let

$$B_n^{\{a,b\}} = \{\theta \in [0, 2\pi] : r_\theta^n(a) = b\}$$

where r_θ is a rotation taken around the center point of A . Note that

$$\bigcup_{a \in C} \bigcup_{b \in C} \bigcup_{n \in \mathbb{N}} B_n^{\{a,b\}} = [0, 2\pi] - \Theta$$

Two distinct elements of $B_n^{\{a,b\}}$ cannot have the same number of total rotations and all elements must have a number of total rotations of at most n . Thus, $B_n^{\{a,b\}}$ has cardinality of at most n . Therefore

$$\bigcup_{a \in C} \bigcup_{b \in C} \bigcup_{n \in \mathbb{N}} B_n^{\{a,b\}} \quad \text{is countable}$$

□

Theorem 2.5. Suppose that $A = \{x \in \mathbb{R}^2 : d(x, 0) = r\}$ for some positive $r \in \mathbb{R}$ and that C is a countable subset of A . Then $A \approx A - C$ via rotations.

Proof. By Lemma 2.4, let $\theta \in \Theta$. Suppose $m, n \in \mathbb{N}$ and $m < n$. Then

$$\begin{aligned} r_\theta^m [C] \cap r_\theta^n [C] &= r_\theta^m [C] \cap r_\theta^m (r_\theta^{n-m} [C]) \\ &= r_\theta^m (C \cap r_\theta^{n-m} [C]) \\ &= \emptyset \end{aligned}$$

So all rotations of C by θ are disjoint. To show that $A \approx A - C$ we will construct subsets $D_0 = \bigsqcup_{n \in \mathbb{N}} r_\theta^n [C]$ and $D_1 = A - D_0$. Then

- $A = D_0 \sqcup A - D_0$
- $A - C = r_\theta [D_0] \sqcup A - D_0$

□

This method of transforming some subsets to “fill in the gaps” of other subsets we wish to remove will be a key tool in many of our proofs involving equidecomposable sets. We finish this section on equidecompositions with a final property established through the Schröder-Bernstein theorem.

Theorem 2.6 (Schröder-Bernstein). *Suppose that A and B are sets and there are injective functions $f : A \rightarrow B$, $g : B \rightarrow A$. Then there exists a partition $A = A_0 \sqcup A_1$ such that $f \upharpoonright A_0 \cup g^{-1} \upharpoonright A_1$ is a bijection.*

Proof. Let $h = f \upharpoonright A_0 \cup g^{-1} \upharpoonright A_1$, and set

- $B' = B - f[A]$
- $A_1 = g[\bigcup_{n \in \mathbb{N}} (f \circ g)^n [B']]$
- $A_0 = A - A_1$

We will first prove that h is surjective by considering two cases. If $x \in f[A_0]$, then by definition $x \in h[A]$. Alternatively, suppose $x \in B' \cup f[A_1]$. Note that

$$(2.7) \quad B' \cup f[A_1] = B' \cup f \left[g \left[\bigcup_{n \in \mathbb{N}} (f \circ g)^n [B'] \right] \right] = g^{-1} [A_1]$$

Therefore $x \in g^{-1} [A_1] \subseteq h[A]$ and so we conclude h is surjective.

To show that h is injective, it suffices to prove that $f[A_0] \cap g^{-1} [A_1] = \emptyset$. This statement follows from (2.7) as f is injective on A . □

Theorem 2.8. *Suppose $\Gamma \curvearrowright X$ and $A, B \subset X$. If $A \preceq B$ and $B \preceq A$, then $A \approx B$.*

Proof. Let f and g be the injective functions formed under Γ such that for partitions $\{C_i\}_{i=0}^n$, $\{D_j\}_{j=0}^k$ of A and B respectively, $\bigsqcup f[C_i] \subset B$, $\bigsqcup g[D_j] \subset A$. By Theorem 2.6, we form the bijective function $f \upharpoonright A_0 \cup g^{-1} \upharpoonright A_1$ under the partition $A_0 \sqcup A_1 = A$. Consider the partition $\{A_0 \cap C_i : i < n\} \cup \{A_1 \cap g[D_j] : j < k\}$. By applying the bijective function $f \upharpoonright A_0 \cup g^{-1} \upharpoonright A_1$ on the family, we get a partition of B . Thus, $A \approx B$. □

It is important to note that in many future cases we may refer to a equidecomposition’s relating bijective function. This refers to the bijective function constructed in the same manner as in the previous theorem. Now that we have defined equidecompositions and proven some ease-of-use properties, we are ready to investigate the Banach-Tarski paradox.

3. THE BANACH-TARSKI PARADOX

In this section, we will begin by defining what it means for a set to be paradoxical in the Banach-Tarski manner, as well as showing some different forms in which paradoxical sets take place. Then, we will construct the necessary groundwork to prove the Banach-Tarski paradox and its more general forms.

Definition 3.1. Suppose $\Gamma \curvearrowright X$ and $A \subset X$. We say that A is *paradoxical* if there exists a partition $A = A_0 \sqcup A_1$ such that $A_0 \approx A$ and $A_1 \approx A$.

Theorem 3.2. Suppose A_0, A_1 are disjoint subsets of A . If $A_0 \approx A$ and $A_1 \approx A$, then A is paradoxical.

Proof. Let $A'_1 = A - A_0$. Then $A = A_0 \sqcup A'_1$. As A_1 and A_0 are disjoint, $A_1 \preceq A'_1$ and $A \approx A_1$, thus $A \preceq A'_1$. As $A'_1 \subset A$, $A'_1 \preceq A$. Thus, $A'_1 \approx A$ and A is paradoxical under the disjoint sets A'_1 and A_0 . $\square$

Theorem 3.3. Suppose Γ acts on X and $A \subset X$. If there exists a partition $A_1 \sqcup A_2 = A$ such that $A_1 \approx A$, $A_2 \approx A$, then A_1 and A_2 are paradoxical.

Proof. As $A_1 \approx A$ there exists a partition $\bigsqcup_{i \leq n} C_i = A_1$ and $\gamma_1, \dots, \gamma_n \in \Gamma$ such that $\bigsqcup \gamma_i(C_i) = A$. As $A_1 \subset A$, $A_1 \subset \bigsqcup \gamma_i(C_i)$. Let $D_i = \{x \in C_i : \gamma_i(x) \in A_1\}$ and define $D = \bigsqcup D_i$. Similarly, let $G_i = \{x \in C_i : \gamma_i(x) \in A_2\}$ and define $G = \bigsqcup G_i$. Then

- $\bigsqcup \gamma_i(D_i) = A_1$. Thus, $D \approx A_1$.
- $\bigsqcup \gamma_i(G_i) = A_2$. Thus, $G \approx A_2 \approx A \approx A_1$.

As $D, G \subset A_1$, by Theorem 3.2 A_1 is paradoxical. The proof of A_2 being paradoxical follows by the same process. $\square$

In the next theorem, we will see that Theorem 3.2 allows us to make any finite number of copies of paradoxical sets. This theorem will prove useful in allowing smaller paradoxical sets to become embeddecompositions of larger sets.

Theorem 3.4. Suppose $\Gamma \curvearrowright X$ and $A \subset X$. If A is paradoxical, then for all $n \in \mathbb{N}$ there exist disjoint subsets $\{C_i\}_{i=1}^n \subset \mathcal{P}(A)$ such that $C_i \approx A$ for all $i \leq n$.

Proof. Let A be paradoxical. Then there exists a partition $A_0 \sqcup A_1 = A$ such that $A_0 \approx A_1 \approx A$. By Theorem 3.3, there exist partitions $A_{00} \sqcup A_{01} = A_0$ and $A_{10} \sqcup A_{11} = A_1$ such that $A_{00} \approx A_{01} \approx A_0$ and $A_{10} \approx A_{11} \approx A_1$. Therefore, $A_{00} \approx A_{01} \approx A_{10} \approx A_{11} \approx A$. Continuing in this manner, we can construct such subsets for all n . $\square$

Now that we have established the various properties of paradoxical sets, we look to find groups that generate such sets. We will begin our search by investigating free groups generated by two elements. First, let us construct the framework of free groups.

Definition 3.5. Suppose S is a set of characters.

- Let S^+ be a set containing S such that for all $s \in S$, there exists some $s^{-1} \in S^+$ where $(s^{-1})^{-1} \sim s$.
- We define an S -word to be a finite sequence $\{s_i\}_{i=0}^k$ where $s_i \in S^+$ for all $i \leq k$. We further say that an S -word is *reduced* if for all $i \leq k$, $s_i \neq s_{i-1}^{-1}$.

- Let $\mathbb{F}_s$ be the set of all S -words in reduced form.
- Let $\mathbb{F}_2$ denote the set of all S -words in reduced form where S has cardinality 2.

Theorem 3.6. *Consider the following operation on a set of reduced S -words:*

$$(s_0 \dots s_k)(t_0 \dots t_l) = s_0 \dots s_{k-m} t_m \dots t_l$$

where m is the smallest integer such that the resulting S -word is reduced. If $\mathbb{F}_s$ is endowed with the operation, then it forms a group.

Proof. The empty S -word suffices for the identity element. If $\{s_i\}_{i=0}^n \in F_s$, then $\{s_{n-i}^{-1}\}_{i=0}^n \in F_s$ is the inverse element. Finally we must prove the operation is associative. For $s \in S^+$ define

$$\begin{aligned} \lambda_s: F_s &\rightarrow F_s \\ w &\mapsto sw \end{aligned}$$

where $\lambda_s(w)$ is in its reduced form. If $s = \{s_0, \dots, s_k\}$, then $\lambda_s = \lambda_{s_0} \circ \dots \circ \lambda_{s_k}$. Then by function composition, $\lambda_s \circ \lambda_t = \lambda_{st}$. Therefore

$$\lambda_{(st)u} = (\lambda_s \circ \lambda_t) \circ \lambda_u = \lambda_s \circ (\lambda_t \circ \lambda_u) = \lambda_{s(tu)}$$

Finally, note that

$$(s \cdot t) \cdot u = \lambda_{(st)u}(\emptyset) = \lambda_{s(tu)}(\emptyset) = s \cdot (t \cdot u)$$

□

Theorem 3.7. $\mathbb{F}_2$ is paradoxical under the left-multiplication action $\mathbb{F}_2 \curvearrowright \mathbb{F}_2$.

Proof. Suppose WLOG that $\mathbb{F}_2$ is formed where $S = \{a, b\}$. Define $W_a \subset \mathbb{F}_2$ to be the set of all reduced S -words whose left-most-term is a . Define $W_{a^{-1}}, W_b, W_{b^{-1}}$ similarly. Note that

$$\begin{aligned} a \cdot W_{a^{-1}} &= \{e\} \cup W_b \cup W_{b^{-1}} \\ b \cdot W_{b^{-1}} &= \{e\} \cup W_a \cup W_{a^{-1}} \end{aligned}$$

Therefore

$$\begin{aligned} W_a \sqcup a \cdot W_{a^{-1}} &= \mathbb{F}_2 \\ W_b \sqcup b \cdot W_{b^{-1}} &= \mathbb{F}_2 \end{aligned}$$

Thus $W_a \cup W_a^{-1} \approx \mathbb{F}_2$ and $W_b \cup W_b^{-1} \approx \mathbb{F}_2$ □

Informally, we find that $\mathbb{F}_2$ is paradoxical by “peeling” back one subset to construct all remaining subsets. We will now work to show that an equivalent group exists within a subset of the rotational transformations in $\mathbb{R}^3$.

Definition 3.8. Let $\Gamma \curvearrowright X$ and $\alpha, \beta \in \Gamma$. We say that α, β form a *ping pong family* if there exist disjoint non-empty sets $X_\alpha, X_\beta \subset X$ such that

- $\forall g \in \{\alpha^z : z \in \mathbb{Z}\} - \{e\}, g \cdot X_\alpha \subset X_\beta$
- $\forall g \in \{\beta^z : z \in \mathbb{Z}\} - \{e\}, g \cdot X_\beta \subset X_\alpha$

Definition 3.9. For a group Γ with elements $\alpha, \beta \in \Gamma$ we define the subgroup $\langle \alpha, \beta \rangle$ such that

- $\alpha, \beta \in \langle \alpha, \beta \rangle$
- $\alpha^{-1}, \beta^{-1} \in \langle \alpha, \beta \rangle$
- $e \in \langle \alpha, \beta \rangle$
- $\forall p, q \in \langle \alpha, \beta \rangle, p \cdot q \in \langle \alpha, \beta \rangle$

Theorem 3.10. Suppose that $\Gamma \curvearrowright X$, Γ has infinite order, and that $\alpha, \beta \in \Gamma$ form a ping pong family. Then $\langle \alpha, \beta \rangle \cong \mathbb{F}_2$.

Proof. Consider the natural function

$$f: \langle \alpha, \beta \rangle \rightarrow \mathbb{F}_2$$

$$\alpha^{n_0} \beta^{n_1} \dots \beta^{n_{k-1}} \alpha^{n_k} \mapsto a^{n_0} b^{n_1} \dots b^{n_{k-1}} a^{n_k}$$

It suffices to prove that f is injective. As f is a homomorphism, we will do this by showing that the kernel is trivial. Suppose for contradiction there exists $w \in \mathbb{F}_2$ where $w \neq e$ and $w \mapsto e$. Then $(a^n w a^{-n}) \mapsto e$ for all $n \in \mathbb{N}$. Thus, we may assume $w = a^{n_0} \cdot b^{n_1} \dots b^{n_{k-1}} \cdot a^{n_k}$ where $\{n_i\}_{i=0}^k \subset \mathbb{Z} - \{0\}$. Therefore, $w \mapsto \alpha^{n_0} \cdot \beta^{n_1} \dots \beta^{n_{k-1}} \cdot \alpha^{n_k}$, however this element moves X_α into X_β and cannot be the identity element. $\square$

Theorem 3.11. Let $M_3(\mathbb{Q})$ be the set of all 3-by-3 invertible matrices with entries in $\mathbb{Q}$. If

$$\alpha = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

then $\langle \alpha, \beta \rangle \cong \mathbb{F}_2$.

Proof. By Theorem 3.10, it suffices to prove that $\langle \alpha, \beta \rangle$ form a ping pong family. Let

$$X_\alpha = \left\{ \begin{bmatrix} \frac{x}{5^k} \\ \frac{y}{5^k} \\ \frac{z}{5^k} \end{bmatrix} : k, x, y, z \in \mathbb{Z} \text{ and } \begin{array}{l} x \equiv 0 \\ y \not\equiv 0 \\ z \equiv \pm 3y \end{array} \right\}$$

$$X_\beta = \left\{ \begin{bmatrix} \frac{x}{5^k} \\ \frac{y}{5^k} \\ \frac{z}{5^k} \end{bmatrix} : k, x, y, z \in \mathbb{Z} \text{ and } \begin{array}{l} x \equiv \pm 3y \\ y \not\equiv 0 \\ z \equiv 0 \end{array} \right\}$$

Let $v_\alpha \in X_\alpha$ and $v_\beta \in X_\beta$. Then the proof follows by inductively checking that $\alpha^n v_\alpha, \alpha^{-n} v_\alpha \in X_\beta$ and $\beta^n v_\beta, \beta^{-n} v_\beta \in X_\alpha$ for all n . $\square$

Thus, we have discovered a group in the rotational transformations of $\mathbb{R}^3$ that is paradoxical under the left-multiplication action. We will now push this further to show that hollow spheres are paradoxical under the group $\langle \alpha, \beta \rangle$.

Definition 3.12. We say that an action $\Gamma \curvearrowright X$ is *free* if for all $x \in X$, $\gamma \cdot x = x$ if and only if $\gamma = e$.

Definition 3.13. Suppose $\Gamma \curvearrowright X$. We say that $x, y \in X$ have an *orbit equivalence* if $\gamma \cdot x = y$ for some $\gamma \in \Gamma$. We denote this as $xE_\Gamma^X y$.

Lemma 3.14. Suppose the action $\mathbb{F}_2 \curvearrowright X$ is free. Then X is paradoxical.

Proof. Under the axiom of choice, let $T \subset X$ be a transversal of E_Γ^X classes containing exactly one point from each equivalence class. Construct the subsets $\{\gamma \cdot T : \gamma \in \mathbb{F}_2\}$. We find that these subsets form a partition of X , as all points have orbit equivalence with themselves and $\mathbb{F}_2 \curvearrowright X$ is free. Finally, allow $W_a, W_{a^{-1}}, W_b, W_{b^{-1}}$ to be defined as in Theorem 3.7. Then

- $(W_a \cdot T \sqcup W_{a^{-1}} \cdot T) \sqcup (W_b \cdot T \sqcup W_{b^{-1}} \cdot T) = X$
- $W_a \cdot T \sqcup a \cdot W_{a^{-1}} \cdot T = \bigcup \{\gamma \cdot T : \gamma \in \mathbb{F}_2\} = X$
- $W_b \cdot T \sqcup b \cdot W_{b^{-1}} \cdot T = \bigcup \{\gamma \cdot T : \gamma \in \mathbb{F}_2\} = X$

□

Theorem 3.15. Suppose that $A \in M_3(\mathbb{Q})$ where A is orthogonal, $\det(A)$ equals 1, and A stabilizes at least three points of a set. Then A is the identity transformation.

Proof. Let x, y be fixed points of A where $y \neq -x$ and thus are linearly independent. Therefore, there exists some v such that $\langle x, v \rangle = \langle y, v \rangle = 0$. As A is orthogonal, $\langle x, v \rangle = \langle x, Av \rangle = 0$ and $\langle y, v \rangle = \langle y, Av \rangle = 0$. Therefore, $Av = \pm v$. If $Av = -v$, then $\det(A) = -1$. Thus, $Av = v$. As x, y, v form a basis, A is the identity transformation. □

Lemma 3.16. Suppose S is a hollow sphere. There is a countable $\mathbb{F}_2$ -invariant set $C \subset S$ such that the action $\mathbb{F}_2 \curvearrowright S - C$ is free.

Proof. Define α, β as in Theorem 3.11. Then

- $\alpha^T \alpha = \alpha^{-1} \alpha = \beta^T \beta = \beta^{-1} \beta = I$
- $\det(\alpha) = \det(-\alpha) = \det(\beta) = \det(-\beta) = 1$.

Thus by Theorem 3.15, if $\gamma \in \langle \alpha, \beta \rangle$ where γ is not the identity transformation, then γ has at most two fixed points. Let $C = \{x \in S : \exists \gamma \in \mathbb{F}_2 - \{e\} \cdot \gamma \cdot x = x\}$. As $\mathbb{F}_2$ is countable and each element of $\mathbb{F}_2 - \{e\}$ has at most two fixed points, C is countable. Fix $x \in C$. Then there exists γ_0 where $\gamma_0 \cdot x = x$. If $\gamma \in \mathbb{F}_2$, then

$$\begin{aligned} (\gamma \cdot \gamma_0 \cdot \gamma^{-1}) \cdot \gamma \cdot x &= \gamma \cdot (\gamma_0 \cdot \gamma^{-1} \cdot \gamma) \cdot x \\ &= \gamma \cdot (\gamma_0 \cdot x) \\ &= \gamma \cdot x \end{aligned}$$

Therefore $\gamma \cdot x \in C$ and so C is $\mathbb{F}_2$ invariant. Thus, by construct of C , $\mathbb{F}_2 \curvearrowright S - C$ is free. □

Lemma 3.17. Suppose S is a hollow sphere. For all countable $C \subset S$, $S \approx S - C$ via rotations.

Proof. As C is countable, there exists some $z \in S$ such that $z, -z \notin C$. We then consider rotations around the axis $\{z, -z\}$. By applying a similar method as in Lemma 2.4, we find an angle θ such that $r_\theta^n(C) \cap C = \emptyset$ for all n . Consider subsets $D_0 = \bigsqcup_{n \in \mathbb{N}} r_\theta^n[C]$ and $D_1 = S - D_0$. Then

- $S = D_0 \sqcup S - D_0$
- $S - C = r_\theta[D_0] \sqcup A - D_0$

□

Theorem 3.18. *Suppose S is a hollow sphere. Then S is paradoxical.*

Proof. By Lemma 3.16 there exists a countable, $\mathbb{F}_2$ invariant set $C \subset S$ such that $\mathbb{F}_2 \curvearrowright S - C$ is free. By Lemma 3.14, there exists a partition $A_0 \sqcup A_1 = S - C$ such that $A_0 \approx S - C$ and $A_1 \approx C - S$. By Lemma 3.17, $C - S \approx S$ thus $A_0 \approx S$ and $A_1 \approx S$. $\square$

Now that we have proven that hollow spheres are paradoxical under $M_3(\mathbb{Q})$, we move towards proving the Banach-Tarski Paradox.

Theorem 3.19. *Suppose B is a ball of radius s . Then B is paradoxical.*

Proof. Let S be a hollow sphere of radius s . By Theorem 3.18, there exists a partition $A_0 \sqcup A_1 = S$ such that for subsets $C_1, \dots, C_k, D_1, \dots, D_m \subset S$ and rotations $\gamma_1, \dots, \gamma_k, \delta_1, \dots, \delta_m \in M_3(\mathbb{Q})$, $\bigsqcup C_i = A_0$, $\bigsqcup D_j = A_1$, and $\bigsqcup \gamma_i(C_i) = \bigsqcup \delta_j(D_j) = S$. For $0 < r \leq s$, let $rS = \{x \in \mathbb{R}^3 : d(x, 0) = r\}$. Then

$$B - \{0\} = \bigsqcup_r rS = \left(\bigsqcup_r rA_0 \right) \sqcup \left(\bigsqcup_r rA_1 \right)$$

Note that

- $\bigsqcup_r rA_0 = \bigsqcup_i \bigsqcup_r r \cdot C_i$
- $\bigsqcup_r rA_1 = \bigsqcup_j \bigsqcup_r r \cdot D_j$
- $B - \{0\} = \bigsqcup_r rS = \bigsqcup_i \gamma_i \cdot (\bigsqcup_r r \cdot C_i) = \bigsqcup_j \delta_j \cdot (\bigsqcup_r r \cdot D_j)$

Therefore $B - \{0\}$ is paradoxical. Finally, by applying Theorem 2.5, we can see that $B - \{0\} \approx B$. Thus, B is paradoxical. $\square$

Our final goal is to prove a more general form of the Banach-Tarski paradox under the group formed from the isometries in $\mathbb{R}^3$. First, we show a useful property of paradoxical sets.

Theorem 3.20. *Suppose r, s are positive real numbers and let*

- $B_r = \{x \in \mathbb{R}^3 : d(x, 0) \leq r\}$
- $B_s = \{x \in \mathbb{R}^3 : d(x, 0) \leq s\}$

Then $B_r \approx B_s$.

Proof. If $r = s$, then the proof is clear using the identity transformation. WLOG, assume that $r < s$. Then $B_r \subset B_s$ and so $B_r \preceq B_s$ by the identity transformation. Fix a finite set $F \subset M_3(\mathbb{Q})$ such that $B_s \subset \bigcup_{\gamma \in F} \gamma \cdot B_r$. By Theorem 3.4, we can construct disjoint subsets $C_\gamma \subset B_r$ for each $\gamma \in F$ such that $B_r = \bigsqcup C_\gamma$ and $B_r \approx C_\gamma$ for all $\gamma \in F$ under an associated bijective function g_γ . Therefore $B_s \subset \bigcup_{\gamma \in F} \gamma \cdot g_\gamma[C_\gamma]$ and $B_s \preceq B_r$. Thus by Theorem 3.2, $B_r \approx B_s$. $\square$

Informally, we can think of this as allowing paradoxical sets to “grow” as large as necessary.

Definition 3.21. We say that a set $A \subset \mathbb{R}^3$ is *bounded* if $\sup\{d(x, o) : x \in A\} < \infty$.

Definition 3.22. We say that a set $A \subset \mathbb{R}^3$ has *nonempty interior* if there exists $x \in \mathbb{R}^3$ and $\epsilon > 0$ such that $\{y \in \mathbb{R}^3 : d(y, x) < \epsilon\} \subset A$.

Theorem 3.23. *Any two bounded subsets of $\mathbb{R}^3$ with nonempty interior are equidecomposable.*

Proof. Let A be a bounded subset of $\mathbb{R}^3$ with nonempty interior. By Theorem 3.20, it suffices to prove that A is equidecomposable with a ball. As A is bounded, there exists some s such that $A \subset B_s$. Therefore, $A \preceq B_s$. As A has nonempty interior, there exists ϵ such that $B_\epsilon \subset A$. Therefore, $B_\epsilon \preceq A$. By Theorem 3.20, $B_s \approx B_\epsilon \approx A$. $\square$

This powerful theorem raises a question that will guide us through the coming sections; under what conditions does a group generate or not generate paradoxical sets?

4. AMENABLE GROUPS

In this section, we define what it means for groups to be amenable. We will prove that all sets are not paradoxical when acted upon by an amenable group. We will then construct a framework through filters and ultralimits to find a relation between amenable groups and groups that satisfy the Følner-condition.

Definition 4.1. A *finitely additive probability measure* (fapm) on a nonempty set X is a mapping $m : \mathcal{P}(X) \rightarrow [0, 1]$ where

- $m(X) = 1$
- m is additive, i.e. $m(A \cup B) = m(A) + m(B)$ for $A, B \in \mathcal{P}(X)$ disjoint.

Definition 4.2. An action $\Gamma \curvearrowright X$ is *amenable* if there exists a fapm m such that for all $\gamma \in \Gamma$ and $A \subset X$, $m(\gamma \cdot A) = m(A)$.

Theorem 4.3. Suppose m is a Γ -invariant fapm on X . If $A \approx B$, then $m(A) = m(B)$.

Proof.

$$m(A) = \sum m(C_i) = \sum m(\gamma \cdot C_i) = \sum m(B)$$

$\square$

Theorem 4.4. Suppose $\Gamma \curvearrowright X$ is amenable and $A \subset X$. If $m(A) > 0$, then A is not paradoxical under Γ .

Proof. Suppose for contradiction A is paradoxical under a partition $A_0 \sqcup A_1 = A$. As $A_0 \approx A$, by Theorem 4.3 $m(A_0) = m(A)$. Therefore $m(A_1) = 0$. However, $m(A_1) = m(A)$. $\square$

Definition 4.5. A group Γ is *amenable* if the left multiplication action $\Gamma \curvearrowright \Gamma$ is amenable.

In the following theorem, we show that this second definition of amenability provides a more general form of the one provided previously.

Theorem 4.6. If Γ is amenable and X is a nonempty set, then every action $\Gamma \curvearrowright X$ is amenable.

Proof. Let Γ be amenable under a fapm m . Fix $x \in X$ and define $f : \Gamma \rightarrow X$ such that $f(\gamma) = \gamma \cdot x$. Let

$$\begin{aligned} m' : X &\rightarrow [0, 1] \\ A &\mapsto m(f^{-1}(A)) \end{aligned}$$

where $f^{-1}(A) = \{\gamma \in \Gamma : \gamma \cdot x \in A\}$. We first will show that m' is a fapm on X .

- $m'(X) = m(f^{-1}(X)) = m(\Gamma) = 1.$

- Let A, B be disjoint subsets of X . Then

$$m'(A \cup B) = m(f^{-1}(A \cup B)) = m(f^{-1}(A)) + m(f^{-1}(B)) = m'(A) + m'(B)$$

Finally, we will show that m' is Γ -invariant. Fix $\gamma_0 \in \Gamma$. Note that

$$f^{-1}(\gamma_0 \cdot A) = \{\gamma \in \Gamma : \gamma \cdot x \in \gamma_0 \cdot A\} = \gamma_0 \cdot \{\gamma \in \Gamma : \gamma \cdot x \in A\} = \gamma_0 \cdot f^{-1}(A)$$

Therefore

$$m'(\gamma_0 \cdot A) = m(\gamma_0 \cdot f^{-1}(A)) = m'(A)$$

□

From this, we find that when an amenable group acts upon any set, it cannot create a paradox. In this next part, we provide a formal introduction to ultralimits in order to prove groups that satisfy the Følner condition are amenable.

Definition 4.7. A *proper filter* on X is a family $\mathbf{F} \subset \mathcal{P}(X)$ that satisfies the following conditions

- $\emptyset \notin \mathbf{F}$
- $X \in \mathbf{F}$
- If $A, B \in \mathbf{F}$, then $A \cap B \in \mathbf{F}$
- If $A \in \mathbf{F}$ and $A \subset B$, then $B \in \mathbf{F}$

Definition 4.8. Let $\mathbf{F}$ be a proper filter on X . If for all $A \in \mathbf{F}$ either $A \in \mathbf{F}$ or $A^c \in \mathbf{F}$, then $\mathbf{F}$ is an ultrafilter.

Theorem 4.9. Every proper filter on X extends to an ultrafilter on X .

Proof. Let $\mathbf{F}$ be a proper filter on X . Suppose $A \in \mathbf{F}$. As $A \cap A^c = \emptyset$, $A^c \notin \mathbf{F}$. Suppose $A, A^c \notin \mathbf{F}$. Then we can simply add A to $\mathbf{F}$. Continuing in this manner, we extend $\mathbf{F}$ to an ultrafilter. □

Definition 4.10. Let $x \in X$. We say that $\mathcal{U}_x$ is a *principal ultrafilter* at x if for all $A \subset X$ where $x \in A$, $A \in \mathcal{U}_x$.

Theorem 4.11. If $\mathcal{U}$ is a non principal ultrafilter on X , then all subsets of X whose complement is finite are in $\mathcal{U}$.

Proof. Let $x \in X$. As $\mathcal{U}$ is a non principal ultrafilter, $X - \{x\} \in \mathcal{U}$. Consider the finite subset $\{x_i\}_{i=1}^n \subset X$.

$$X - \{x_i\}_{i=1}^n = \bigcap_{i=1}^n X - \{x_i\}$$

As $\mathcal{U}$ is closed under finite intersections, $X - \{x_i\}_{i=1}^n \in \mathcal{U}$. □

Definition 4.12. A *topology* on X is a family $\tau \subset \mathcal{P}(X)$ where

- $\emptyset \in \tau$
- $X \in \tau$
- If $A, B \in \tau$, then $A \cap B \in \tau$
- If $\mathcal{A} \subset \tau$, then $\bigcup \mathcal{A} \in \tau$

Definition 4.13. Let τ be a topology on X and $x \in X$. We define the *neighborhood filter* $\mathcal{N}_x$ on X such that if there exists $O \in \tau$ where $x \in O$ and $O \subset A$, then $A \in \mathcal{N}_x$.

Theorem 4.14. $\mathcal{N}_x$ is a proper filter.

Proof. Note that $\emptyset \notin \mathcal{N}_x$ and $X \in \mathcal{N}_x$. Suppose that $A, B \in \mathcal{N}_x$. Then there exists $O_1, O_2 \in \tau$ such that $x \in O_1$, $O_1 \subset A$, $x \in O_2$, and $O_2 \subset B$. Therefore, $x \in O_1 \cap O_2$ and $O_1 \cap O_2 \subset A \cap B$. Finally, suppose $A \in \mathcal{N}_x$. Then there exist O such that $x \in O$ and $O \subset A$. If $A \subset B$, then $O \subset B$ and thus $B \in \mathcal{N}_x$. $\square$

Definition 4.15. Given a topological space (X, τ) and a filter $\mathcal{F}$ on X , we say that $\mathcal{F}$ converges to $x \in X$ if $\mathcal{N}_x \subset \mathcal{F}$.

Definition 4.16. Given a function $f : X \rightarrow Y$ and a filter $\mathcal{F} \subset \mathcal{P}(\mathbb{X})$, we define the *push-forward filter* $f_*\mathcal{F}$ such that $A \in f_*\mathcal{F}$ if $f^{-1}(A) \in \mathcal{F}$.

We are now ready to define the ultralimit and prove some of its useful properties.

Definition 4.17. Let $f : X \rightarrow \mathbb{R}$ be bounded and fix an ultrafilter $\mathcal{U}$ on X . The ultralimit is defined such that $\lim_{\mathcal{U}} f = r$ if $f_*\mathcal{U}$ converges to r .

Theorem 4.18. For all bounded $f, g : X \rightarrow \mathbb{R}$, $\lim_{\mathcal{U}}(f + g) = \lim_{\mathcal{U}} f + \lim_{\mathcal{U}} g$.

Proof. Let $\lim_{\mathcal{U}} f = r$ and $\lim_{\mathcal{U}} g = s$. Then for all $\epsilon > 0$, $\{x \in X : |f(x) - r| < \frac{\epsilon}{2}\} \in \mathcal{U}$ and $\{x \in X : |g(x) - s| < \frac{\epsilon}{2}\} \in \mathcal{U}$. As the ultralimit is closed under finite intersections and all supersets,

$$\left\{x \in X : |f(x) - r| < \frac{\epsilon}{2} \text{ and } |g(x) - s| < \frac{\epsilon}{2}\right\} \subset \{x \in X : |f(x) + g(x) - (r + s)| < \epsilon\} \in \mathcal{U}$$

Therefore $\lim_{\mathcal{U}} f + g = r + s = \lim_{\mathcal{U}} f + \lim_{\mathcal{U}} g$. $\square$

Theorem 4.19. For all $a \in \mathbb{R}$ and bounded $f : X \rightarrow \mathbb{R}$, $\lim_{\mathcal{U}}(a \cdot f) = a \cdot \lim_{\mathcal{U}} f$.

Proof. Let $\lim_{\mathcal{U}}(a \cdot f) = r$ and fix $\epsilon > 0$. Then $\{x \in X : |a \cdot f(x) - r| < \epsilon\} \in \mathcal{U}$, thus $\{x \in X : |f(x) - \frac{r}{a}| < \frac{\epsilon}{a}\} \in \mathcal{U}$. As ϵ is arbitrary,

$$\left\{x \in X : |f(x) - \frac{r}{a}| < \epsilon\right\} = \left\{x \in X : |f(x) - \frac{r}{a}| < \frac{\epsilon}{a}\right\}$$

Therefore $\lim_{\mathcal{U}} f = \frac{r}{a}$ and $a \cdot \lim_{\mathcal{U}} f = r = \lim_{\mathcal{U}}(a \cdot f)$. $\square$

Definition 4.20. For a subset $A \subset X$, we define the *density function*

$$d_A : \mathbb{N} \rightarrow [0, 1]$$

$$n \mapsto \frac{|A \cap F_n|}{|F_n|}$$

where $\{F_n\}$ is a collection of non-empty finite subsets of X .

Theorem 4.21. For all disjoint $A, B \subset X$, $d_{A \cup B} = d_A + d_B$.

Proof.

$$\begin{aligned} d_{A \cup B}(n) &= \frac{|(A \cup B) \cap F_n|}{|F_n|} \\ &= \frac{|A \cap F_n|}{|F_n|} + \frac{|B \cap F_n|}{|F_n|} \\ &= d_A(n) + d_B(n) \end{aligned}$$

$\square$

Now that we have thoroughly introduced both the ultralimit and the density function, we are ready to use them to prove a relation between groups that satisfy the Følner condition and amenable groups.

Theorem 4.22. *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and $A \subset X$. Then $\lim_{\mathcal{U}} d_A(n)$ is a fapm.*

Proof. First note that $\lim_{\mathcal{U}} d_X(n) = \lim_{\mathcal{U}} 1 = 1$. Let A, B be disjoint. Then by Theorem 4.21, $m(A \cup B) = \lim_{\mathcal{U}} d_{A \cup B}(n) = m(A) + m(B)$. $\square$

Definition 4.23. Suppose that Γ is a group, $S \subset \Gamma$ is finite, and that $\epsilon > 0$. We say that a nonempty finite set $F \subset \Gamma$ is (S, ϵ) -Følner if

$$\forall \gamma \in S \quad \frac{|\gamma \cdot F \Delta F|}{|F|} < \epsilon$$

Definition 4.24. A group Γ satisfies the Følner condition if for all finite $S \subset \Gamma$ and $\epsilon > 0$ there is a (S, ϵ) -Følner set.

Now that we have proven $\lim_{\mathcal{U}} d_A(n)$ is a fapm, it suffices to prove that this function is Γ -invariant when Γ satisfies the Følner condition.

Theorem 4.25. *Suppose Γ is a countable group that satisfies the Følner condition. Then Γ is amenable.*

Proof. Let $\{\gamma_i\}$ be an enumeration of Γ . Define $S_n = \{\gamma_i : i < n\}$ and fix a sequence $\{\epsilon_n\}_{n=1}^{\infty}$ that converges to 0. By the Følner condition, there exists nonempty finite $F_n \subset \Gamma$ that are (S_n, ϵ_n) -Følner. Let $m(A) = \lim_{\mathcal{U}} d_A(n)$. We now will show that this fapm is amenable. Fix $\gamma \in \Gamma$ and $A \subset \Gamma$. Note that

$$\begin{aligned} |d_{\gamma \cdot A}(n) - d_A(n)| &= \left| \frac{|(\gamma \cdot A) \cap F_n| - |A \cap F_n|}{|F_n|} \right| \\ &= \left| \frac{|A \cap (\gamma^{-1} \cdot F_n)| - |A \cap F_n|}{|F_n|} \right| \\ &\leq \frac{|(\gamma^{-1} \cdot F_n) \Delta F_n|}{|F_n|} \end{aligned}$$

Fix $\epsilon > 0$. Then $\{n \in \mathbb{N} : \gamma^{-1} \in S_n \text{ and } \epsilon_n < \epsilon\}$ is cofinite and thus by the Følner condition

$$\left\{ n \in \mathbb{N} : \frac{|(\gamma^{-1} \cdot F_n) \Delta F_n|}{|F_n|} < \epsilon \right\} \text{ is cofinite}$$

Therefore, $\{n \in \mathbb{N} : |d_{\gamma \cdot A}(n) - d_A(n)| < \epsilon\}$ is also cofinite. By Theorem 4.11 $\mathcal{N}_0 \subset (d_{\gamma \cdot A} - d_A)_* \mathcal{U}$ and thus $m(\gamma \cdot A) - m(A) = \lim_{\mathcal{U}} d_{\gamma \cdot A} - d_A = 0$. $\square$

At this moment, we have proven a useful chain between groups that satisfy the Følner condition, amenable groups, and groups that when acting on sets will not generate paradoxes. In the next section, we will show that these three statements are in fact equivalent to one another.

5. THE FÖLNER CONDITION AND PARADOXICAL GROUPS

We now move to show that if a group Γ does not satisfy the Følner Condition, $\Gamma \curvearrowright \Gamma$ is paradoxical. In doing so, we will prove the equivalence relation between non-paradoxical groups, amenable groups, and groups that satisfy the Følner Condition. To begin, we must construct a subset of graphs that will be central in our final proof.

Definition 5.1. Given a set V , we say that a set $G \subset V^2$ is a *graph* on V if

- $\forall x, (x, x) \notin G$
- $\forall x, y$ if $(x, y) \in G$, then $(y, x) \in G$

Definition 5.2. Suppose G is a graph on V .

- If there exists a partition $X \sqcup Y = V$ such that $G \subset X \times Y \cup Y \times X$, then we say G is a *bipartite graph*.
- Let $A \subset V$. We define the G -neighbors of A to be the set $N_G(A) = \{w \in V : \exists a \in A (a, w) \in G\}$.
- Let $v \in V$. We say that v has G -degree equal to $|N_G(\{v\})|$.
- If all vertices of G have finite G -degree, then we say that G is *locally finite*.

Definition 5.3. Suppose G is a graph on V and let M be a subgraph of G .

- If every vertex of M has M -degree of at most 1, then we say M is a *matching*.
- Let M be a matching. We define the *domain* of M to be the set $\text{dom}(M) = \{v \in V : \exists w \in V (v, w) \in M\}$.
- Let M be a matching. We say that M is *perfect* if $\text{dom}(M) = V$.

Now, we consider the following condition that allow us to find matchings within graphs.

Definition 5.4. Suppose that G is a locally finite bipartite graph on $V = X \sqcup Y$. We say that G satisfies the *Hall condition* if

- for all finite $A \subset X$, $|A| \leq |N_G(A)|$
- for all finite $B \subset Y$, $|B| \leq |N_G(B)|$

Definition 5.5. Let $M \subset G$ be a matching and let $\{v_i\}_{i=1}^{n+1}$ be a set of vertices in G . We define an M -alternating path to be a sequence of connected edges $\{(v_i, v_{i+1})\}_{i=1}^n \subset G$ such that if i is odd, then $(v_i, v_{i+1}) \notin M$ and if i is even then $(v_i, v_{i+1}) \in M$. We further say the path is augmented if $v_{n+1} \notin M$.

Lemma 5.6. Let G be a locally finite bipartite graph on $V = X \sqcup Y$ where G satisfies the Hall condition. Then for all finite matching $M \subset G$ and $x \in V$ where $x \notin \text{dom}(M)$, there exists a finite matching $M' \subset G$ such that $\{x\} \cup \text{dom}(M) \subset \text{dom}(M')$.

Proof. Consider an M -alternating augmented path $\{(v_i, v_{i+1})\}_{i=1}^n$ where $v_1 = x$. We would like to find a path where $n \leq 2 \cdot |\text{dom}(M) \cap Y|$ in order to construct our M' . For sake of contradiction, suppose this path does not exist. We'll construct a family $\{B_i\}$ recursively such that $B_0 = N_G(\{x\})$ and $B_{i+1} = N_G(N_M(B_i) \cup \{x\})$. Thus for all $y \in B_i$, we can construct an alternating path from x to y of length at most $2i + 1$. By our initial assumption, $B_i \subset \text{dom}(M)$ for $i \leq m$. However by the Hall condition, $|B_{i+1}| \geq |B_i| + 1$. Thus, $i < |B_i|$. We now construct M' on such an

augmented path by flipping all edges on the path and adding all remaining edges of M back into M' . Then $\text{dom}(M') = \text{dom}(M) \cup \{x, v_{n+1}\}$. $\square$

From Lemma 5.6, we now reach the following theorem through induction.

Theorem 5.7. *Let G be a locally finite bipartite graph on $V = X \sqcup Y$ where G satisfies the Hall condition. Then for all finite $F \subset V$ there is a finite matching $M \subset G$ such that $F \subset \text{dom}(M)$.*

Now that we have found a condition for the existence of a matching within a bipartite graph, we must find such a matching that is perfect. To do this, we must first construct the cone filter.

Definition 5.8. Let X be a set.

- We define $\text{FIN}(V) = \{F \subset V : F \text{ is finite}\}$.
- Let $F \in \text{FIN}(V)$. We define $C_F = \{A \in \text{FIN}(V) : F \subset A\}$.
- We define the *cone filter* $\mathcal{F}$ on $\text{FIN}(V)$ such that if there exists $F \in \text{FIN}(V)$ where $C_F \subset P$, then $P \in \mathcal{F}$.

Theorem 5.9. *$\mathcal{F}$ is a proper filter on $\text{FIN}(V)$.*

Proof. We must prove the four conditions for a proper filter.

- Fix $F \in \text{FIN}(V)$. As $F \in C_F$, C_F is nonempty. Therefore, all $P \in \mathcal{F}$ are nonempty.
- Fix $F \in \text{FIN}(V)$. Then $C_F \subset \text{FIN}(V)$ and so $\text{FIN}(V) \in \mathcal{F}$.
- If $A, B \in \mathcal{F}$, then there exists $F, G \in \text{FIN}(V)$ such that $C_F \subset A$ and $C_G \subset B$. Therefore $C_{F \cap G} = C_F \cap C_G \subset A \cap B$. Thus, $A \cap B \in \mathcal{F}$.
- Let $A \in \mathcal{F}$. Then there exists $F \in \text{FIN}(V)$ such that $C_F \subset A$. If $A \subset B$, then $C_F \subset B$. Thus, $B \in \mathcal{F}$.

$\square$

Theorem 5.10. *Suppose that G is a locally finite bipartite graph on $V = X \sqcup Y$ and G satisfies the Hall condition. Then G admits a perfect matching.*

Proof. By Theorem 5.7 for every $F \in \text{FIN}(V)$ there exists a finite matching $M_F \subset G$ such that $F \subset \text{dom}(M_F)$. By Theorem 4.9, we can extend the cone filter $\mathcal{F}$ to an ultrafilter $\mathcal{U}$ on $\text{FIN}(V)$. Define a graph M such that $(v, w) \in M$ if

$$\{F \in \text{FIN}(V) : (v, w) \in M_F\} \in \mathcal{U}$$

Fix some $(v, w) \in M$. As $\emptyset \notin \mathcal{U}$, $\{F \in \text{FIN}(V) : (v, w) \in M_F\}$ is nonempty. Thus there exists $F \in \text{FIN}(V)$ such that $(v, w) \in M_F$ where $M_F \subset G$. Thus $(v, w) \in G$ and so $M \subset G$. As a matching is symmetric, if $(v, w) \in M_F$ for some $F \in \text{FIN}(V)$, then $(w, v) \in M_F$. Therefore M is a symmetric subgraph of G . We will now prove that M is a perfect matching. For the sake of contradiction, suppose M is not a matching. Then there exists distinct $u, v, w \in V$ such that $(u, v), (u, w) \in M$. Thus

- $A = \{F \in \text{FIN}(V) : (u, v) \in M_F\} \in \mathcal{U}$
- $B = \{F \in \text{FIN}(V) : (u, w) \in M_F\} \in \mathcal{U}$

As filters are closed under finite intersections

$$A \cap B = \{F \in \text{FIN}(V) : (u, v), (u, w) \in M_F\} \in \mathcal{U}$$

However M_F is a matching and thus $A \cap B = \emptyset$. Therefore, M is a matching. Now fix $v \in V$. Then for all $F \in C_{\{v\}}$, there exists some $w \in N_G(\{v\})$ such that $(v, w) \in M_F$. As $C_{\{v\}} \subset \mathcal{U}$, there exists some $w \in N_G(\{v\})$ such that

$$\{F \in C_{\{v\}} : (v, w) \in M_F\} \in \mathcal{U}$$

However, $C_{\{v\}} \subset \text{FIN}(V)$, and thus $(w, v) \in M$. $\square$

Lemma 5.11. *Suppose Γ is a group that does not satisfy the Følner condition. Then there exists a finite subset $T \subset \Gamma$ such that if $F \subset \Gamma$ is finite, then $2|F| \leq |T \cdot F|$.*

Proof. As Γ does not satisfy the Følner condition there exists $S \subset \Gamma$ and $\epsilon > 0$ where for all nonempty finite sets $F \subset \Gamma$ there exists $\gamma \in S$ such that

$$(5.12) \quad \frac{|(\gamma \cdot F)\Delta F|}{|F|} \geq \epsilon$$

We'll construct a family $\{S_n\}$ recursively where $S_1 = S \cup S^{-1} \cup \{e\}$ and $S_{n+1} = S_1 \cdot S_n$. From (5.12) we can prove through induction that for all finite $F \subset \Gamma$, $(1 + \frac{\epsilon}{2})^n |F| \leq |S_n \cdot F|$. As $(1 + \frac{\epsilon}{2})$ is fixed there exists large enough N such that $(1 + \frac{\epsilon}{2})^N \geq 2$ and we can let $T = S_N$ to complete the proof. $\square$

Theorem 5.13. *Suppose Γ is a group where Γ does not satisfy the Følner condition. Then Γ is paradoxical.*

Proof. As Γ does not satisfy the Følner condition, by the previous lemma there exists a finite $T \subset \Gamma$ such that for all finite $F \subset \Gamma$, $2|F| \leq |T \cdot F|$. Let $X = \Gamma$ and $Y = \Gamma_0 \sqcup \Gamma_1$ where $\Gamma_i = \{(\gamma, i) : \gamma \in \Gamma\}$. Let G be a bipartite graph on $X \sqcup Y$ where $(\gamma, (\delta, i)) \in G$ if there exists $\tau \in T$ such that $\tau \cdot \gamma = \delta$. We will first prove that G satisfies the Hall Condition. Let $A \subset X$ be finite. Then if $\delta \in T \cdot A$, there exists $\gamma \in A$ and $\tau \in T$ such that $\tau \cdot \gamma = \delta$. Thus $(\delta, 0), (\delta, 1) \in N_G(A)$. Therefore $T \cdot A \times \{0\} \subset N_G(A)$. By the previous lemma

$$|A| \leq |T \cdot A \times \{0\}| = |T \cdot A| \leq N_G(A)$$

Let $B \subset Y$ be finite. Let $B_i = B \cap \Gamma_i$ and WLOG assume $|B_0| \geq |B_1|$. Define $C = \{\delta \in \Gamma : (\delta, 0) \in B_0\}$. Therefore if $\gamma \in T \cdot C$, there exists some $\delta \in \Gamma$ and $\tau \in T$ such that $\tau \cdot \delta = \gamma$, i.e. $\tau^{-1} \cdot \gamma = \delta$ and so $(\gamma, (\delta, i)) \in G$. Thus $T \cdot C \subset N_G(B_0)$. Therefore

$$|N_G(B)| \geq |T \cdot C| \geq 2|C| = 2|B_0| \geq |B_0 \cup B_1| = |B|$$

Thus, by the Hall Condition G admits a perfect matching $M \subset G$. Finally, define

$$\begin{aligned} A_0 &= \{\gamma \in \Gamma : M \text{ matches } \gamma \text{ to some } (\delta, 0) \in \Gamma_0\} \\ A_1 &= \{\gamma \in \Gamma : M \text{ matches } \gamma \text{ to some } (\delta, 1) \in \Gamma_1\} \end{aligned}$$

For $\tau \in T$, let $C_{i\tau} = \{\delta \in \Gamma : (\tau \cdot \gamma, i) \in M\}$. Then $\bigsqcup_{\tau \in T} C_{0\tau} = A_0$, $\bigsqcup_{\tau \in T} C_{1\tau} = A_1$, and $\bigsqcup_{\tau \in T} \tau(C_{0\tau}) = \bigsqcup_{\tau \in T} \tau(C_{1\tau}) = \Gamma_0$. Thus $A_0 \approx \Gamma_0$ and $A_1 \approx \Gamma_1$. Therefore $A_0 \approx \Gamma$ and $A_1 \approx \Gamma$. $\square$

We have now shown an equivalence between amenable groups, groups that satisfy the Følner condition, and non paradoxical groups.

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