

GEOMETRY AND PHYSICS OF BLACK HOLES

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ABSTRACT. This paper develops the mathematical and physical framework necessary to understand black hole spacetimes in general relativity. Beginning with the geometric preliminaries of manifolds, we introduce the basic rules of General Relativity. Minkowski spacetime is presented as the example illustrating causal structure and Penrose diagrams. We then turn to stationary black holes where we introduce the Schwarzschild solution. After we study the basic properties, we turn to rotating black holes with the Kerr solution, where angular momentum and the horizon structure are analyzed. Building on this foundation, the paper concludes with a discussion of black hole thermodynamics and their resemblance to traditional thermodynamic relations. In this way, the paper connects rigorous geometric constructions to the physical interpretation of black hole spacetimes, culminating in their thermodynamic behavior.

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1. INTRODUCTION

Black holes are one of the most striking predictions of general relativity. As exact solutions of Einstein's field equations, they not only describe regions of spacetime from which nothing can escape, but also reveal deep connections between geometry, causality, and even thermodynamics. Understanding black holes requires both a firm grasp of the mathematical framework of general relativity and a careful study of the physical consequences of the theory.

This paper is structured to build toward that goal. We begin with the geometric preliminaries of manifolds and curvature following [1], after which we introduce the notational conventions used in physics. Then we shall give an introduction to general relativity, following [2]. We begin by showing a derivation of the Einstein

field equations through the Bianchi identities. With this foundation established, we introduce the Minkowski spacetime as a sample solution, providing a baseline for understanding causal structures. We then turn to stationary black holes, examining the Schwarzschild solution, its symmetries, and the causal structure revealed by Penrose diagrams. Extending these ideas, we study rotating black holes described by the Kerr solution, exploring angular momentum, the ergosphere, and horizon structure. Finally, we conclude with a discussion of black hole thermodynamics, presenting a proof of the first law and reflecting on the parallels between these laws and classical thermodynamics.

In this way, the paper aims to highlight how the mathematical language of general relativity weaves physical insights about black holes, from their basic causal properties to their surprising thermodynamic behavior.

2. GEOMETRIC PRELIMINARIES

One of the most well-known and astonishing realizations in relativity is the fact that our space itself need not be flat, but is curved in a way similar to the Earth being a globe rather than a flat sheet of paper. To study the consequences of curvature in spacetime, we need first a framework where curvature can be rigorously studied. Riemannian geometry serves an important role here. Unfortunately, mathematicians and physicists have quite different notations for the same concepts. In this paper we shall first introduce the rigorous definitions and then connect to the physics notations with an emphasis on calculations.

2.1. Mathematical Definitions. To speak meaningfully about smooth functions and coordinate systems, we first need a topological structure. A *topological space* is a set X equipped with a collection of subsets called *open sets*. Open sets tell us which points in the set are “close” to each other, which allows us to define continuity and local neighbourhoods, which are essential to manifolds.

Definition 2.1. A *differentiable manifold* of dimension n is a topological space M that locally looks like $\mathbf{R}^n$ with smooth coordinate transitions, i.e. if x_α and x_β are functions from open subsets in $\mathbf{R}^n$ to M and their images overlap, then the function $x_\alpha^{-1} \circ x_\beta$ is differentiable. [1, p. 2]

On a manifold, we can establish a set of coordinates at least locally.

Since we are given smooth parametrizations of manifolds, we can now have differentiable mappings between manifolds. In fact, a function $\varphi : M_1 \rightarrow M_2$ is called differentiable (or smooth) if the function $x_2^{-1} \circ \varphi \circ x_1$ is a differentiable (or smooth) function, where x_1 and x_2 are local parametrizations of M_1 and M_2 . [1, p. 5]

Definition 2.2. Let M be a differentiable manifold with a smooth curve given by $\alpha : (-\epsilon, \epsilon) \rightarrow M$. Suppose $\alpha(0) = p \in M$, and let $\mathcal{D}$ be the set of functions differentiable at p . The *tangent vector to the curve α at $t = 0$* is a function $\alpha'(0) : \mathcal{D} \rightarrow \mathbf{R}$ given by:

$$\alpha'(0)(f) = \left. \frac{d(f \circ \alpha)}{dt} \right|_{t=0}, f \in \mathcal{D}.$$

A *tangent vector* at p is the tangent vector at $t = 0$ of some curve $\alpha : \mathbf{R} \rightarrow M$ with $\alpha(0) = p$. The set of all tangent vectors to M at p is denoted $T_p M$.

Furthermore, a *vector field* X on a manifold M associates to each point $p \in M$ a vector $X(p) \in T_p M$. [1, pp. 7-8]

It can be shown that in local coordinates, $\{\frac{\partial}{\partial x_i}, \quad i \in \{1, \dots, n\}\}$ is a basis of the tangent space $T_p M$ [1, p. 8]. This matches our intuition on $\mathbf{R}^n$ and functions on it. In fact, a vector field is called *differentiable* or *smooth* if the components in this basis are differentiable or smooth functions.

In physics, the concept of covectors is quite essential as well. A covector ω is an object of the dual space $T_p^* M$ and is a linear map $\omega : T_p M \rightarrow \mathbf{R}$. In our case, a differential 1-form of the form $\omega = \sum_{i=1}^n \omega_i dx_i$ would serve as a covector.

Covector fields can be defined in a similar manner.

Note, however, that in general, vector fields do not commute, i.e. $XYf \neq YXf$. We define the bracket $[X, Y] = XY - YX$ as a measure of the non-commutativity.

We can then differentiate curves to get velocity vectors. For a curve $c : I \rightarrow M$, we can define the *velocity field* of the curve as the vector field $\frac{dc}{dt}$ such that for a smooth function f on M , we have $\frac{dc}{dt} f = \frac{d}{dt}(f \circ c)(t)$

To further study the properties of such manifolds and curves, we need to equip them with metrics. These metrics need to have some nice properties.

Definition 2.3. A *Riemannian metric* on a manifold M associates to each point p of M an inner product $\langle, \rangle_p$ that is symmetric, bilinear, and positive-definite on the tangent space $T_p M$. Furthermore, in a given parametrization in a neighborhood U of p , $\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_q = g_{ij}(x_1, \dots, x_n)$ is a differentiable function for all $q \in U$. [1, p. 38]

In relativity, although the metric is not positive-definite, we require that the metric is *non-degenerate*, i.e. if $\langle X, Y \rangle$ vanishes for all Y , then $X = 0$.

With a metric, we can then define the length of curves in a similar way to Euclidean spaces as $l_a^b(c) = \int_a^b \langle \frac{dc}{dt}, \frac{dc}{dt} \rangle^{1/2} dt$, where $c : I \supset [a, b] \rightarrow M$.

Let us now denote the set of smooth vector fields on M by $\mathcal{X}(M)$.

Definition 2.4. A *affine connection* ∇ on a manifold M is a mapping

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$$

denoted by $(X, Y) \rightarrow \nabla_X Y$ that satisfies: [1, p. 50]

- (1) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$;
- (2) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$;
- (3) $\nabla_X (fY) = f\nabla_X Y + X(f)Y$.

The *Levi-Civita* connection is the unique connection that satisfies:

- (1) $\nabla_X Y - \nabla_Y X = [X, Y]$, $X, Y \in \mathcal{X}(M)$ (Symmetric);
- (2) $X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$ (Compatible with the metric). [1, pp. 53-56]

This operation is also called the *covariant derivative* of Y in the direction of X . It measures how much Y changes in the direction of X . We should note that since vectors at different points in the manifold belong to different tangent spaces, two vectors cannot be directly compared with each other. In standard relativity we assume the connection to be Levi-Civita.

The connection can be extended to covector fields as well. We can define

$$(2.5) \quad (\nabla_X \omega)(Y) = X \cdot [\omega(Y)] - \omega(\nabla_X Y).$$

On a manifold with the Levi-Civita connection, we also have the following equation:

$$(2.6) \quad \frac{d}{dt} \langle V, W \rangle = \langle \frac{DV}{dt}, W \rangle + \langle V, \frac{DW}{dt} \rangle$$

A proof can be found in [1, pp. 51-52].

Following the definition of connections, we can also uniquely define the covariant derivative of a vector field V along a curve $c : I \rightarrow M$ as $\frac{DV}{dt} = \nabla_{dc/dt} V$ if V is induced by a vector field $Y \in \mathcal{X}(M)$. This operation follows the rules $\frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt}$ and $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$, where W is a vector field on the curve and f is a function $I \rightarrow R$. With the Levi-Civita connection, we also have: [1, p. 53]

$$(2.7) \quad \frac{d}{dt}\langle V, W \rangle = \left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{DW}{dt} \right\rangle.$$

Now that we have rigorously defined covariant derivatives and obtained all the nice properties we need, we can finally touch on some of the intrinsic properties of manifolds. Let us begin with what “straight lines” look like in manifolds.

Definition 2.8. A parametrized curve $\gamma : I \rightarrow M$ is a *geodesic* at $t_0 \in I$ if $\frac{D}{dt}(\frac{d\gamma}{dt}) = 0$ at the point t_0 . If γ is a geodesic at t for all $t \in I$, γ is said to be a *geodesic*. [1, p. 61]

We notice that $\frac{d}{dt}\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle = 2\langle \frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle = 0$. This shows that the geodesic has constant velocity, which corresponds to the straight lines in Euclidean spaces. In fact, it can also be shown that geodesics locally minimize the curve length joining two points.

Moreover, we can now define the curvature of the metric on a manifold.

Definition 2.9. The *curvature* R of a manifold M associates to every pair $X, Y \in \mathcal{X}(M)$ a mapping $R(X, Y) : \mathcal{X}(M) \rightarrow \mathcal{X}(M)$ given by:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad Z \in \mathcal{X}(M)$$

The curvature measures the “non-commutativity of covariant derivatives.” In other words, it is related to the failure of a vector Z to return to its initial value after being parallel transported along two vector fields (X and Y), as shown in [3, pp. 37-38].

Now that we have set up a rigorous foundation with the mathematical language, let us introduce the notations common among physicists in order to better prepare for relativity.

2.2. Physicist Notations. The geometric structures introduced above - manifolds, metrics, connections, and curvature - can also be expressed in a coordinate-based formalism more familiar to physicists. In fact, after choosing local coordinates $\{x_i\}$, we arrive at a local basis for vector fields $\{\partial_i\}$, as discussed above. In this way, all the above definitions can be translated into formulas and equations in the local coordinates. In this language, vectors, covectors, and, as we shall soon define, tensors, are represented by indexed components, and geometric operations such as contraction and covariant differentiation are written using the Einstein summation convention. The following section reformulates the same ideas in this notation, which will be used throughout the discussion of Einstein’s field equations and black hole spacetimes.

Reflecting upon our definitions so far, we realize that many of the functions we defined are multilinear. The metrics, the connections, and the curvature are all linear functions on the vector fields. However, they cannot be simply represented by matrices as some of them map more than one vector to a vector. The connection, for example, maps two vector fields to one vector field, and the curvature takes three vector fields. It is then useful for us to develop a new tool that captures the

feature. Such a tool is called *tensors*. An (a, b) tensor is a multilinear map that takes a covectors and b vectors as input and outputs a differentiable function on M . A vector, for example, is a $(1, 0)$ tensor and a covector is a $(0, 1)$ tensor. The metric, on the other hand, is a $(0, 2)$ tensor as it is a function of two vectors.

Furthermore, as we shall soon discover, in local coordinates the calculations we make would involve a large amount of summing over the coordinates. For example,

the metric of two vectors $\langle x, y \rangle$ can be written as $\sum_{i=1}^n \sum_{j=1}^n g_{ij} x_i y_j$ for $g_{ij} \in \mathbf{R}$ because

it is linear in the vector fields. For simplicity, Einstein took inspiration from earlier geometers and developed his summation convention as follows:

(1) An (a, b) tensor is written as a variable name with a superscripts and b subscripts. For example, a vector is written as X^μ and a covector as ω_ν . This can also be used to denote the μ^{th} or ν^{th} component of the vector / covector in local coordinates.

(2) Whenever an index appears in both the superscript and the subscript, it represents a sum over the index. For example, the metric $\langle X, Y \rangle$ can now be rewritten

$$\text{as } g_{\mu\nu} X^\mu Y^\nu = \sum_{\mu=1}^n \sum_{\nu=1}^n g_{\mu\nu} X^\mu Y^\nu.$$

Note that we can also have free indices, i.e. indices that are not summed over. For example, the curvature defined in Definition 2.9, $W = R(X, Y)Z$ is usually written by physicists as $W^\mu = R_{\alpha\beta}{}^\mu{}_\nu X^\alpha Y^\beta Z^\nu$ [2, p. 7].

Often useful in relativity is the concept of associated vectors and associated covectors. The associated covector of a vector X is the covector $X^\flat$ such that $g(X, Y) = \langle X, Y \rangle = X^\flat(Y)$ for all vectors Y . The associated covector of X can be written as $X^\flat = X_\nu = g_{\mu\nu} X^\mu$. Similarly, we apply the inverse of this map and define the associated vector of a covector ω to be the vector $\omega^\sharp$ such that $g(\omega^\sharp, Y) = \omega(Y)$ for all vectors Y . In order to obtain a formula for the associated vectors, we need first the inverse metric g^{-1} , or $g^{\mu\nu}$ in the notations common among physicists, which is guaranteed to exist by the metric being positive-definite.¹ We can now write an equation for the associated vector as $\omega^\sharp = \omega^\nu = g^{\mu\nu} \omega_\mu$.

Such index lowering and raising can be generalized to arbitrary tensors. For example, the curvature can be rewritten as $R_{\alpha\beta\mu\nu} = g_{\mu\sigma} R_{\alpha\beta}{}^\sigma{}_\nu$. In fact, $R_{\alpha\beta\mu\nu} = g_{\sigma\mu} R_{\alpha\beta}{}^\sigma{}_\nu$ would have given the same result since the metrics we are dealing with are symmetric.

Following our introduction to the mathematics, we now rewrite the Levi-Civita connection as

$$(2.10) \quad \nabla_X Y = X^\mu \nabla_\mu Y^\nu,$$

where the *covariant derivative tensor* is defined as

$$(2.11) \quad \nabla Y = \nabla_\mu Y^\nu = \partial_\mu X^\nu + \Gamma_{\mu\alpha}^\nu X^\alpha,$$

with $\Gamma_{\mu\nu}^\alpha$ being the *Christoffel symbols* of the Levi-Civita connection and are given by [2, p. 7]

$$(2.12) \quad \Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}),$$

¹In the case of Lorentzian metric which allows negative distances, the metric is still non-degenerate: if $g(X, Y) = 0$ for all Y , then $X = 0$. The metric is therefore still invertible.

which can in turn be used to calculate the curvature tensor:

$$(2.13) \quad R_{\alpha\beta}{}^{\mu}{}_{\nu} = dx^{\mu}(R(\partial_{\alpha}, \partial_{\beta})\partial_{\nu}) = \partial_{\alpha}\Gamma_{\beta\nu}^{\mu} - \partial_{\beta}\Gamma_{\alpha\nu}^{\mu} + \Gamma_{\alpha\gamma}^{\mu}\Gamma_{\beta\nu}^{\gamma} - \Gamma_{\beta\gamma}^{\mu}\Gamma_{\alpha\nu}^{\gamma}.$$

Note, however, that Christoffel symbols are *not* tensors.

Following our calculation of the curvature, we now define the *Ricci Curvature Tensor* as

$$(2.14) \quad R_{\mu\nu} = R_{\alpha\mu}{}^{\alpha}{}_{\nu}.$$

Such an action of summing over an upper and lower index of a single tensor is called a *contraction*. The Ricci tensor is the only independent contraction of the curvature tensor: other contractions either vanish or are the same as the Ricci tensor.

We can take one step further and take the trace of the Ricci tensor. To do that, let us first raise the first index: $R^{\mu}{}_{\nu} = g^{\alpha\mu}R_{\alpha\nu}$. Then, we take the trace and obtain the *scalar curvature*

$$(2.15) \quad R = R^{\mu}{}_{\mu} = g^{\mu\nu}R_{\mu\nu}$$

Using the equation for connections and following (2.5), we can derive the connection on a covector field to be

$$\begin{aligned} (X^{\mu}\nabla_{\mu}\omega_{\nu})Y^{\nu} &= X^{\mu}\partial_{\mu}(\omega_{\nu}Y^{\nu}) - \omega_{\nu}(X^{\mu}\nabla_{\mu}Y^{\nu}) \\ &= X^{\mu}(\partial_{\mu}\omega_{\nu})Y^{\nu} + X^{\mu}\omega_{\nu}\partial_{\mu}Y^{\nu} - \omega_{\nu}(X^{\mu}\partial_{\mu}Y^{\nu} + X^{\mu}\Gamma_{\mu\alpha}^{\nu}Y^{\alpha}) \\ &= (\partial_{\mu}\omega_{\nu} - \Gamma_{\mu\nu}^{\alpha}\omega_{\alpha})X^{\mu}Y^{\nu} \end{aligned}$$

Hence,

$$\nabla_{\mu}\omega_{\nu} = \partial_{\mu}\omega_{\nu} - \Gamma_{\mu\nu}^{\alpha}\omega_{\alpha}$$

We can then extend this to all tensor fields. For example, for a $(2, 1)$ tensor $T_{\mu\nu}^{\beta}$, the connection is given by [2, p. 8]

$$(2.16) \quad \nabla_{\alpha}T_{\mu\nu}^{\beta} = \partial_{\alpha}T_{\mu\nu}^{\beta} + \Gamma_{\alpha\gamma}^{\beta}T_{\mu\nu}^{\gamma} - \Gamma_{\alpha\mu}^{\gamma}T_{\gamma\nu}^{\beta} - \Gamma_{\alpha\nu}^{\gamma}T_{\mu\gamma}^{\beta}$$

It is also worth noting that the equation for geodesics is

$$(2.17) \quad \ddot{x}^{\mu} + \Gamma_{\alpha\beta}^{\mu}\dot{x}^{\alpha}\dot{x}^{\beta} = 0.$$

3. EINSTEIN FIELD EQUATIONS

Now that we have ample background knowledge, let us show a brief derivation of the Einstein Field Equations. To begin with, let us write the curvature $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X, Y]}Z$ in the physics notations:

$$\begin{aligned} R_{\alpha\beta}{}^{\mu}{}_{\nu}X^{\alpha}Y^{\beta}Z^{\nu} &= X^{\alpha}\nabla_{\alpha}(Y^{\beta}\nabla_{\beta}Z^{\mu}) - Y^{\alpha}\nabla_{\alpha}(X^{\beta}\nabla_{\beta}Z^{\mu}) - \\ &\quad (X^{\alpha}\nabla_{\alpha}Y^{\beta} - Y^{\alpha}\nabla_{\alpha}X^{\beta})\nabla_{\beta}Z^{\mu} \\ &= (X^{\alpha}\nabla_{\alpha}Y^{\beta})(\nabla_{\beta}Z^{\mu}) + X^{\alpha}Y^{\beta}\nabla_{\alpha}\nabla_{\beta}Z^{\mu} \\ &\quad - (Y^{\alpha}\nabla_{\alpha}X^{\beta})(\nabla_{\beta}Z^{\mu}) - Y^{\alpha}X^{\beta}\nabla_{\alpha}\nabla_{\beta}Z^{\mu} \\ &\quad - (X^{\alpha}\nabla_{\alpha}Y^{\beta})\nabla_{\beta}Z^{\mu} + (Y^{\alpha}\nabla_{\alpha}X^{\beta})\nabla_{\beta}Z^{\mu} \\ &= X^{\alpha}Y^{\beta}(\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha})Z^{\mu}, \end{aligned}$$

where in the final step we rename $(Y^{\alpha}\nabla_{\alpha}X^{\beta})\nabla_{\beta}Z^{\mu} = (Y^{\beta}\nabla_{\beta}X^{\alpha})\nabla_{\alpha}Z^{\mu}$, etc. Thus we obtain:

$$(3.1) \quad R_{\alpha\beta}{}^{\mu}{}_{\nu}Z^{\nu} = (\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha})Z^{\mu}.$$

Lowering the index μ , we have

$$(3.2) \quad R_{\alpha\beta\mu\nu}Z^\nu = (\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)Z_\mu.$$

For simplicity, let us introduce the antisymmetrization $[\cdot, \cdot, \dots, \cdot]$ and the symmetrization $(\cdot, \cdot, \dots, \cdot)$ as

$$\begin{cases} T_{[x_1, \dots, x_n]} = \frac{1}{n!} \sum_{\sigma \in S_p} \text{sgn}(\sigma) T_{x_{\sigma(1)}, \dots, x_{\sigma(n)}}, \\ T_{(x_1, \dots, x_n)} = \frac{1}{n!} \sum_{\sigma \in S_p} T_{x_{\sigma(1)}, \dots, x_{\sigma(n)}}, \end{cases}$$

where S_p is the set of permutations of $\{1, 2, \dots, n\}$. For example, we can now rewrite (3.2) as $R_{\alpha\beta\mu\nu}Z^\nu = 2\nabla_{[\alpha}\nabla_{\beta]}Z_\mu$.

Now, let us choose $Z_\mu = \partial_\mu f$ for some function f . We then have

$$R_{[\alpha\beta\mu]\nu}Z^\nu = 2\nabla_{[[\alpha}\nabla_{\beta]}Z_\mu] = 2\nabla_{[\alpha}\nabla_{\beta]}Z_\mu = 0,$$

because

$$\nabla_{[\beta}Z_{\mu]} = \partial_{[\beta}Z_{\mu]} - \Gamma_{[\beta\mu]}^\alpha Z_\alpha = \partial_{[\beta}\partial_{\mu]}f = 0.$$

Since we can choose f arbitrarily, we have

$$(3.3) \quad R_{[\alpha\beta\mu]\nu} = 0,$$

which is the first Bianchi identity in the form of the physicist notations. Furthermore, we notice that the curvature is antisymmetric in the first two indices:

$$\begin{aligned} R_{\beta\alpha\mu\nu}Z^\nu &= (\nabla_\beta\nabla_\alpha - \nabla_\alpha\nabla_\beta)Z_\mu \\ &= -R_{\alpha\beta\mu\nu}Z^\nu. \\ \Rightarrow R_{\alpha\beta\mu\nu} &= -R_{\beta\alpha\mu\nu}. \end{aligned}$$

In fact, the full set of symmetries in the curvature tensor is given by [2, p. 9]:

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}.$$

Note that the Ricci curvature tensor $R_{\mu\nu}$ is then symmetric.

If we take the covariant derivative of equation (3.1), we have the *second Bianchi identity*:

$$(3.4) \quad \nabla_{[\alpha}R_{\beta\gamma]\mu\nu} = 0.$$

Now if we raise β and γ and contract them with μ and ν respectively, we have the *contracted Bianchi identity*:

$$\begin{aligned} \nabla_\alpha R - \nabla^\beta R_{\alpha\beta} - \nabla^\gamma R_{\alpha\gamma} &= 0 \\ \Rightarrow \nabla^\beta R_{\alpha\beta} - \frac{1}{2}\nabla_\alpha R &= 0 \\ \Rightarrow \nabla^\beta (R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}) &= 0. \end{aligned}$$

This inspired the definition of the *Einstein tensor*:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu},$$

which is divergenceless:

$$\nabla^\mu (G_{\mu\nu}) = 0.$$

Furthermore, in special relativity it is possible to define the *energy-momentum tensor*, so that the conservations of energy and momentum are equivalent to the vanishing of its divergence: [2, p. 11]

$$\nabla^\mu T_{\mu\nu} = 0.$$

Thus, Einstein proposed that the metric g should satisfy the *Einstein field equations*:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu},$$

where Λ is called the *cosmological constant*.

4. MINKOWSKI SPACETIME

Now that we have the equations at hand, let us take a look at a simple example of solutions to the equation. For simplicity, let us choose our units such that Newton's gravitational constant $G = 1$ and the speed of light $c = 1$. The simplest solution to the Einstein Field Equations is the Minkowski spacetime, which is $\mathbf{R}^4$ with the metric $g_{tt} = -1, g_{xx} = 1, g_{yy} = 1$, and $g_{zz} = 1$ with all other terms being zero. It is often written in the form $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, i.e. [2, p. 3]

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

With the metric defined, we can then divide vectors into three categories: a vector X^μ is said to be

- (1) Timelike if $g_{\mu\nu}X^\mu X^\nu < 0$,
- (2) Spacelike if $g_{\mu\nu}X^\mu X^\nu > 0$,
- (3) Null if $g_{\mu\nu}X^\mu X^\nu = 0$.

A spacelike surface is orthogonal to a timelike vector, a timelike surface a spacelike vector, and a null surface a null vector. Furthermore, a vector X is said to be future-pointing if $\langle X, \partial_t \rangle < 0$, and past-pointing if $-X$ is future-pointing.

Particles moving along timelike vectors, such as $(2, 1, 0, 0)$, have a speed less than the speed of light (in this case, the speed is $\frac{1}{2} < 1 = c$). Null vectors, on the other hand, are tangent to trajectories of light particles. It is impossible for particles to move along spacelike vectors. Therefore, if two events are connected only by spacelike curves, it is impossible for one event to affect the other event. These events are said to be causally unrelated.

Notice that in the Minkowski spacetime geodesics are straight lines with constant speed. Therefore, a particle moving without acceleration in the Minkowski spacetime moves in a straight line, similar to the case in Newtonian physics.

Furthermore, a key difference from Riemannian geometry is that geodesics in relativity no longer minimize the distance between two points, as negative distances are possible. Rather, they maximize the *proper time* τ defined by

$$d\tau^2 = -ds^2 = dt^2 - dx^2 - dy^2 - dz^2.$$

This quantity is invariant under coordinate changes, and therefore corresponds to the observer's own measure of time.

We now turn to a set of coordinates that better reflects the symmetries of problems we wish to study. In particular, the spherical coordinates (t, r, θ, φ) are important in describing central mass points such as the black holes. The metric in

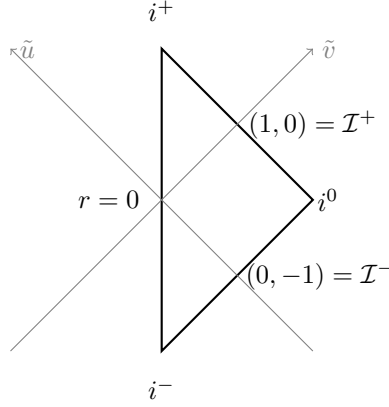


FIGURE 1. Penrose Diagram for Minkowski Spacetime

the spherical coordinate form is

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2).$$

If we further substitute with $u = t - r$ and $v = t + r$, we have

$$ds^2 = -dudv + r^2(d\theta^2 + \sin^2 \theta d\varphi^2).$$

Notice that u and v are null coordinates because both $\partial_u = \partial_t + \partial_r$ and $\partial_v = \partial_t - \partial_r$ are null vectors. Also, since $r = \frac{1}{2}(v - u) \geq 0$, we have $v \geq u$. If we further perform a coordinate change $\tilde{u} = \tanh u$ and $\tilde{v} = \tanh v$, we arrive at a new form of metric:

$$ds^2 = -\frac{1}{(1 - \tilde{u}^2)(1 - \tilde{v}^2)} d\tilde{u}d\tilde{v} + r^2(d\theta + \sin^2 \theta d\varphi^2).$$

This way we keep the coordinates as null coordinates but we are able to map the entire spacetime to a finite diagram called the Penrose diagram. [2, pp. 16-17]

Since $\tilde{u}$ and $\tilde{v}$ are null geodesics, we often tilt them by 45 degrees. Also, we have $-1 < \tilde{u} \leq \tilde{v} < 1$. Thus, we arrive at the Penrose diagram for the Minkowski spacetime as shown in Figure 1.

i^+ in the diagram represents the future timelike infinity and corresponds to $t \rightarrow \infty$. i^- , on the other hand, is the past timelike infinity with $t \rightarrow -\infty$. $\mathcal{I}^+$ represents the future null infinity and $\mathcal{I}^-$ is the past null infinity. i^0 , finally, is the spacelike infinity with $r \rightarrow \infty$.

The Penrose diagram is a useful tool for analyzing causalities. In the 1-dimensional spacetime with (t, x) , the null geodesics are represented by 45 degree straight lines. Since we keep $\tilde{u}$ and $\tilde{v}$ at 45 degrees, causalities are similar to the case in (t, x) : timelike vectors lie within the 45-degree lightcone and spacelike vectors remain outside of it.

5. STATIONARY BLACK HOLES

Another solution to the Einstein field equations is the Schwarzschild solution given by:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2).$$

The Schwarzschild solution describes the spacetime geometry outside a static, spherically symmetric mass. When the mass is sufficiently compact, the solution contains an event horizon at the Schwarzschild radius $r = 2M$. This horizon marks a one-way boundary: signals and particles can cross inward but can travel outwards.

The Birkhoff's theorem states that the Schwarzschild solution is the only form of spherically symmetric solution of the vacuum Einstein equations [3, p. 125].

Now, let us consider the proper time of a timelike curve parametrized by the coordinate time:

$$\tau = \int_{t_0}^{t_1} \left[\left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\theta}^2 - r^2 \sin^2 \theta \dot{\varphi}^2 \right]^{-\frac{1}{2}} dt.$$

If we make the approximation that the curve is far away from the central mass and that the speed is much smaller than the speed of light, i.e. $\frac{M}{r} \ll 1$ and $\dot{r} \ll 1$, we have for the integrand

$$\begin{aligned} L_S &= \left[\left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\theta}^2 - r^2 \sin^2 \theta \dot{\varphi}^2 \right]^{-\frac{1}{2}} \\ &\approx 1 - \frac{M}{r} - \frac{1}{2}(\dot{r}^2 + r^2 \dot{\theta}^2 + \sin^2(\theta) \dot{\varphi}^2). \end{aligned}$$

Notice that the Lagrangian for a particle moving under Newtonian gravity of a point mass M is precisely

$$L_N = T - V = \frac{1}{2}(\dot{r}^2 + r^2 \dot{\theta}^2 + \sin^2(\theta) \dot{\varphi}^2) + \frac{M}{r}.$$

Thus, the Schwarzschild solution is analogous to the classical point mass M . In fact, the gravitational field in Newtonian physics $\text{div} G = -4\pi\rho$ has inspired the definition of the Komar mass [2, pp. 119-121]. To define it properly, let us first introduce Killing vectors.

Definition 5.1. A *Killing vector* ξ^α is a vector with vanishing Lie derivative of the metric:

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0.$$

In fact, if the metric is independent of a coordinate α , then α is a Killing vector.

The Killing vectors generate an isometry of the spacetime, i.e. a symmetry of the metric. Thus, if ξ^α is timelike in a region, it represents time-translation symmetry and leads to a conserved quantity along the geodesics: $\xi_\alpha x^\alpha$ is constant along a geodesic. [3, p. 442]

Thus, we can define the Komar mass as:

$$(5.2) \quad M_{Komar} = -\frac{1}{8\pi} \int_{\Sigma} \star dK^\flat.$$

Here Σ is any surface enclosing the matter, K is the timelike Killing vector, and the Hodge star $\star$ is defined such that for a differential k form

$$\omega = \frac{1}{k!} \omega_{\mu_1 \dots \mu_k} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k},$$

where the volume form is defined in the space by orthogonal vectors $dx^{\mu_1}, \dots, dx^{\mu_n}$, then $\star\omega$ is defined as the $(n - k)$ form

$$\star\omega = \frac{\sqrt{|g|}}{k!(n-k)!} \omega_{\mu_1 \dots \mu_k} \epsilon^{\mu_1 \dots \mu_k}_{\nu_{k+1} \dots \nu_n} dx^{\nu_{k+1}} \wedge \dots \wedge dx^{\nu_n},$$

where $|g|$ is the determinant of the metric $g_{\mu\nu}$ and the Levi-civita symbol $\epsilon^{\mu_1 \dots \mu_k \nu_{k+1} \dots \nu_n}$ is the sign of the permutation $(\mu_1, \dots, \mu_k, \nu_{k+1}, \dots, \nu_n)$ and 0 if any index is repeated.

For example, since the Schwarzschild metric is independent of the time t , the timelike Killing vector in the Schwarzschild solution is $K = \partial_t$, and we have $K^\flat = -(1 - \frac{2M}{r})dt$, and therefore $dK^\flat = -\frac{2M}{r^2}dr \wedge dt = \frac{2M}{r^2}dt \wedge dr$. The determinant $|g|$ of the metric is $r^4 \sin^2 \theta$. Note, however, since the metric is negative in dt^2 , in the volume form there is a negative sign in the dt . We compensate that by adding a sign flip during the Hodge star operation. Thus, we obtain

$$\begin{aligned} \star dK^\flat &= -\frac{r^2 \sin \theta}{2} \frac{2M}{r^2} (d\theta \wedge d\varphi - d\varphi \wedge d\theta) \\ &= -2M \sin \theta \quad d\theta \wedge d\varphi. \end{aligned}$$

Therefore, using (5.2), the Komar mass enclosed in any sphere $r = R > 0$ is:

$$\begin{aligned} M_{Komar} &= -\frac{1}{8\pi} \int_0^\pi \int_0^{2\pi} 2M \sin \theta d\theta d\varphi \\ &= -\frac{1}{8\pi} \int_0^\pi 4\pi M \sin \theta d\theta \\ &= M. \end{aligned}$$

Returning to the metric we defined, we realize that it naturally splits the coordinates into two parts: $r \in (0, 2M)$ and $r \in (2M, \infty)$. These two regions behave quite differently. Noticeably, g_{tt} and g_{rr} switch signs in the two regions. This way, ∂_t , being a timelike vector on the outer region, becomes spacelike inside while ∂_r becomes timelike inside. This is often informally described as “time and space switching roles.”

To obtain a penrose diagram similar to the Minkowski case, let us begin with the first two terms of the metric:

$$\begin{aligned} ds^2 &= (1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 \\ &= -(1 - \frac{2M}{r}) \left[dt^2 - (1 - \frac{2M}{r})^{-2}dr^2 \right] \\ &= -(1 - \frac{2M}{r})(dt - (1 - \frac{2M}{r})^{-2}dr)(dt + (1 - \frac{2M}{r})^{-2}dr) \\ &= -(1 - \frac{2M}{r})dudv, \end{aligned}$$

if we define

$$\begin{cases} u = t - \int (1 - \frac{2M}{r})^{-1} dr &= t - r - 2M \log |r - 2M|, \\ v = t + \int (1 + \frac{2M}{r})^{-1} dr &= t + r + 2M \log |r - 2M|. \end{cases}$$

Often times, with a choice of constant in the integral, we write u and v in the following form:

$$\begin{cases} u = t - r - 2M \log \left| \frac{r}{2M} - 1 \right| &= t - r^*, \\ v = t + r + 2M \log \left| \frac{r}{2M} - 1 \right| &= t + r^*. \end{cases}$$

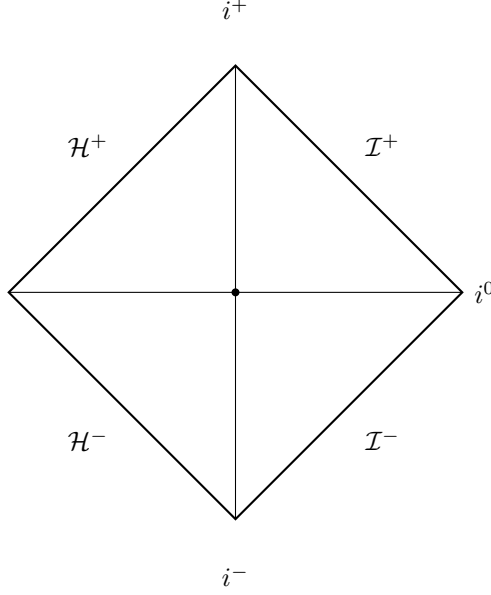


FIGURE 2. Penrose Diagram for the Schwarzschild Solution

Here, r^* is known as the tortoise coordinate. Again, with $M = 0$ it is the same as our treatment to the Minkowski spacetime. Now, however, the quantity $v - u$ can be any real number:

$$v - u = 2r + 4M \log \left| \frac{r}{2M} - 1 \right| \in (-\infty, \infty).$$

Thus, if we perform the coordinate change again by $\tilde{u} = \tanh u$ and $\tilde{v} = \tanh v$, we have $(\tilde{u}, \tilde{v}) \in (-1, 1) \times (-1, 1)$, a full square instead of the triangle in the Minkowski case. We then have the Penrose diagram as in Figure 2.

In the diagram, we obtain two new null infinities known as the past event horizon $\mathcal{H}^-$ and future event horizon $\mathcal{H}^+$. At these points, we have $r = 2M$. At first glance, it seems that this event horizon $r = 2M$ is a singularity in the spacetime. However, if we contract the curvature tensor:

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \frac{48M^2}{r^6},$$

we realize that it is well-behaved and finite at $r = 2M$, unlike $r = 0$ where it is a true singularity. In fact, since we have

$$r + 2M \log \left| \frac{r}{2M} - 1 \right| = \frac{1}{2}(v - u),$$

we can rewrite the metric as

$$ds^2 = -\frac{2Me^{-r/2M}}{r} e^{(v-u)/4M} du dv.$$

This is non-singular as $r \rightarrow 2M$. If we take one step further and take the new coordinates

$$\begin{cases} U = -e^{-u/4M} \\ V = e^{v/4M}, \end{cases}$$

we then have the metric as

$$ds^2 = -\frac{32M^3 e^{-r/2M}}{r} dU dV.$$

With the final transformation $T = \frac{U+V}{2}$ and $X = \frac{V-U}{2}$, we have the final form of the metric given by Kruskal:

$$ds^2 = -\frac{32M^3 e^{-r/2M}}{r} (-dT^2 + dX^2).$$

The original coordinates can be retrieved by

$$\begin{cases} \left(\frac{r}{2M} - 1\right)e^{r/2M} = X^2 - T^2 \\ \frac{t}{2M} = 2 \tanh^{-1}(T/X). \end{cases}$$

The only constraint on the coordinates now is that $r > 0$ and therefore we have $X^2 - T^2 > -1$. [3, p. 154]

The Kruskal coordinates predict four regions of spacetime: two exterior regions $r > 2M$, and two interior regions $r < 2M$ known as the black hole and white hole regions. For our purposes, we shall restrict ourselves to the black hole region and the exterior region in Figure 2.

To obtain the Penrose diagram for $r < 2M$ let us return to the tortoise coordinates (u, v) and consider the pair (v, r) . We have the metric as

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr.$$

This is a solution to $r < 2M$, $r > 2M$, and $r = 2M$ since its determinant $\det g = -1 \neq 0$. Thus, it is possible to glue an extra region $r < 2M$ along the future event horizon $\mathcal{H}^+$. To do that, we take the coordinate $u' = -u$. This way, we have $u' + v = 2r + 4M \log \left| \frac{r}{2M} - 1 \right| \in (-\infty, 0)$ in the region $r \in (0, 2M)$. If we take the finite version of the coordinates $\tilde{u}'$ and $\tilde{v}$ we obtain the extended Penrose diagram as shown in Figure 3.

In Region *II*, we have $r < 2M$. As discussed before, the vector ∂_r is now timelike and the vector ∂_t is now spacelike. Thus, the orientation of the light cones is such that all future-directed causal vectors point toward smaller r . Therefore, all future-directed timelike and null curves that start in Region *I* always enter region *II* and end up at $r = 0$. Physically, this represents the fact that nothing can escape the event horizon.

6. ROTATING BLACK HOLES

In general, however, black holes do not remain stationary and are expected to be rotating. The solution to the Einstein field equations that corresponds to a rotating black hole is known as the Kerr solution:

$$\begin{aligned} ds^2 = & -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar \sin^2 \theta}{\rho^2} dt d\varphi + \frac{\rho^2}{\Delta} dr^2 \\ & + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\rho^2}\right) \sin^2 \theta d\varphi^2 \end{aligned}$$

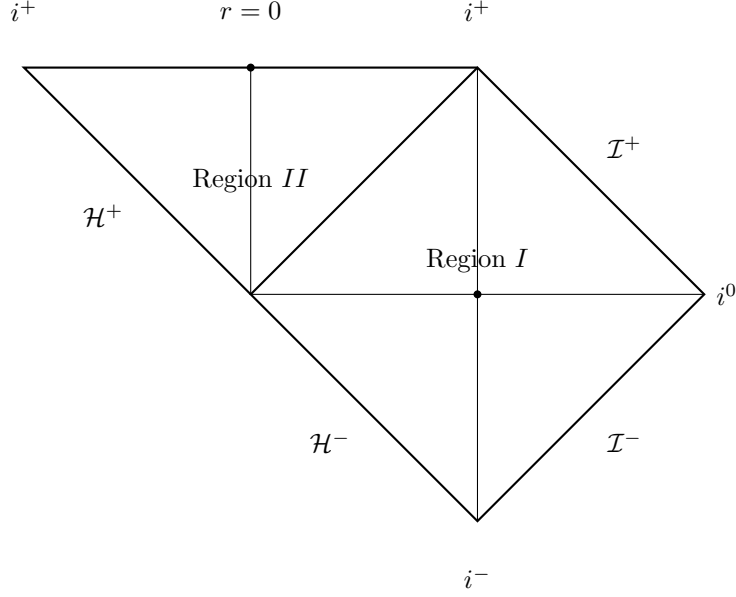


FIGURE 3. Penrose Diagram for the Half Extended Schwarzschild Solution

where

$$\begin{aligned}\rho^2 &= r^2 + a^2 \cos^2 \theta, \\ \Delta &= r^2 - 2Mr + a^2,\end{aligned}$$

where a and M are constants. This is in fact the same as the Schwarzschild solution if $a = 0$.

To demonstrate that this metric indeed demonstrates rotating black holes, let us begin by noticing that in the Kerr solution, apart from $X = \partial_t$ there is another timelike Killing vector $Y = \partial_\varphi$. This corresponds to the Komar angular momentum defined as:

$$(6.1) \quad J_{Komar} = \frac{1}{16\pi} \int_{\Sigma} \star dY^\flat.$$

Choosing Σ to be fixed t and $r \gg M, a$, we have

$$\begin{aligned}Y^\flat &= g_{\mu\nu} Y^\mu d^\nu \\ &= -\frac{2Mar \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} dt + (r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{r^2 + a^2 \cos^2 \theta}) \sin^2 \theta d\varphi \\ &\approx -\frac{2Ma \sin^2 \theta}{r} dt + r^2 \sin^2 \theta d\varphi.\end{aligned}$$

Since the metric is no longer diagonal, $dt, dr, d\theta, d\varphi$ is no longer an orthonormal coframe and their wedge product can no longer be used to define the volume in the 4 dimensional manifold. In this case, an orthonormal coframe is given by [2,

pp. 167]:

$$\begin{aligned}\omega^0 &\approx dt, \quad \omega^r \approx dr, \quad \omega^\theta \approx r d\theta, \\ \omega^\varphi &\approx r \sin \theta d\varphi - \frac{2Ma \sin^2 \theta}{r^2} dt.\end{aligned}$$

Thus,

$$\begin{aligned}dY^\flat &= \frac{2Ma \sin^2 \theta}{r^2} dr \wedge dt + 2r \sin^2 \theta dr \wedge d\varphi + \dots \\ &= -\frac{2Ma \sin^2 \theta}{r^2} \omega^0 \wedge \omega^r + 2\omega^r \wedge \omega^\varphi + \frac{4Ma \sin^2 \theta}{r^2} \omega^r \wedge \omega^0 + \dots \\ &= -\frac{6Ma \sin^2 \theta}{r^2} \omega^0 \wedge \omega^r + \dots\end{aligned}$$

We ignore terms that are not dr and dt because after the hodge star they would become dr and dt terms and vanish in the integral.

In the orthonormal coframe, the metric is given by $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, and therefore the determinant is -1 . Using our definition of the star, we have

$$\begin{aligned}\star dY^\flat &= -\frac{6Ma \sin^2 \theta}{r^2} \omega^\theta \wedge \omega^\varphi \\ &= -6Ma \sin^3 \theta d\theta \wedge d\varphi.\end{aligned}$$

Again, we discard the dt term. Finally, by (6.1), we have

$$\begin{aligned}J_{Komar} &= \frac{1}{16\pi} \int_{\Sigma} \star dY^\flat \\ &= \frac{1}{16\pi} \int_0^\pi \int_0^{2\pi} 6Ma \sin^3 \theta d\varphi d\theta \\ &= \frac{3Ma}{4} \int_0^\pi \sin^3 \theta d\theta \\ &= Ma.\end{aligned}$$

Using the other Killing vector $X = \partial_t$, it is possible to show that the Komar mass of the Kerr solution is also M as in the Schwarzschild case. Thus a can be interpreted as angular momentum per unit mass.

On the hypersurface $r = M + \sqrt{M^2 - a^2 \cos^2 \theta}$, the Killing vector $X = \partial_t$ becomes null as $1 - \frac{2Mr}{\rho^2} = 0$. Within this region, ∂_t becomes a spacelike vector. However, it can be shown that the metric induced on this hypersurface is non-degenerate and therefore Lorentzian, unlike the event horizon in the Schwarzschild case which is a null hypersurface with a null normal vector. Also, the ergosphere can be crossed both ways by a timelike curve. Therefore, it should not be considered the equivalent of the event horizon in the Schwarzschild case.

On the other hand, if we consider the hypersurface $r = r_+ = M + \sqrt{M^2 - a^2}$, we notice that the function Δ changes sign on both sides and therefore ∂_r becomes timelike within this region and therefore all future-directed timelike curves must have decreasing r within this region. Thus, this is the event horizon in the Kerr solution. This is strictly enclosed by the ergosphere and only touching on the poles $\theta = 0$ and $\theta = \pi$.

One of the most striking facts about black holes is that they closely resemble thermodynamic systems such as ideal gases. Just as an ideal gas can be described by a few parameters like temperature, pressure, and volume, a black hole can be

described entirely by a few parameters such as its mass and angular momentum. The geometry of black holes and the horizons actually obey a set of laws that mirror the fundamental laws of thermodynamics. To state these laws, let us first define some useful concepts.

Definition 6.2. A *Killing horizon* is a null hypersurface that is orthogonal to a nonvanishing Killing vector field.

Notice that the event horizon is a Killing horizon. In fact, its normal vector is

$$Z = X + \Omega Y = \partial_t + \Omega \partial_\varphi.$$

Definition 6.3. Ω is called the angular velocity of the event horizon.

Since the vector Z is null, using the metric we have:

$$-(1 - \frac{2Mr}{\rho^2}) - \frac{4Mar \sin^2 \theta}{\rho^2} \Omega + (r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\rho^2}) \sin^2 \theta \Omega^2 = 0.$$

Solving,

$$\Omega = \frac{a}{r_+^2 + a^2} = \frac{a}{2Mr_+}.$$

We can also define the surface gravity of the horizon:

Definition 6.4. The surface gravity of a Killing horizon $\mathcal{H}$ with vector field Z is the function $k : \mathcal{H} \rightarrow \mathbf{R}$ defined by:

$$\nabla_Z Z = kZ.$$

With these definitions, we are ready for the statements of the black hole thermodynamic laws:

Theorem 6.5. Zeroth Law of Black Hole Thermodynamics: *The surface gravity of a Killing horizon is a constant function.*

Theorem 6.6. First Law of Black Hole Thermodynamics: *The mass of the black hole $M = M(A, J)$ as a function of the area A of the event horizon with fixed time t and the angular momentum satisfies:*

$$dM = \frac{k}{8\pi} dA + \Omega dJ.$$

Often, A is denoted the spacelike cross-section of the event horizon.

Theorem 6.7. Second Law of Black Hole Thermodynamics: *For a collection of test fields propagating on a Kerr spacetime, the energy ΔM and the angular momentum ΔJ absorbed by the black hole satisfy:*

$$\Delta M \geq \Omega \Delta J.$$

This implies that the area A of any spacelike cross-section of the event horizon cannot decrease toward the future. [2, p. 166]

In fact, these three laws closely resemble the three laws of thermodynamics for other matter like ideal gases:

The **zeroth law** states that the temperature is throughout the gas in thermal equilibrium;

The **first law** states that $dU = TdS - pdV$;

The **second law** states that the entropy cannot decrease towards the future.

Comparing these laws to the three laws of black hole thermodynamics, we can establish the following correspondences: $U \Leftrightarrow M$, $TdS \Leftrightarrow \frac{k}{8\pi}dA$, and $pdV \Leftrightarrow -\Omega dJ$. In 1974, Hawking provided the last piece of evidence and showed that black holes emit particles as a form of radiation similar to the black body radiation by objects in thermodynamics. He then showed that such radiation corresponds to temperature $T = \frac{k}{2\pi}$ and entropy $S = \frac{A}{4}$ [2, pp. 166-167].

The proofs of these laws, however, require longer setup and is beyond the scope of this paper. Nataro has provided proofs of these theorems in his book [2]. Here, we shall only provide a proof of the first law with some of the nuances that the book left untouched.

Proof. To prove the first law, let us begin by deriving a formula for the area A of the spacelike cross-section S of the event horizon $\mathcal{H}$. Notice that on this surface, the induced metric is given by

$$\begin{aligned} ds^2 &= \rho^2 d\theta^2 + (r_+^2 + a^2 + \frac{2Ma^2 r_+ \sin^2 \theta}{\rho^2}) \sin^2 \theta d\varphi^2 \\ &= \rho^2 d\theta^2 + 2Mr_+ \sin^2 \theta (1 + \frac{a^2 \sin^2 \theta}{\rho^2}) d\varphi^2 \\ &= \rho^2 d\theta^2 + 2Mr_+ \sin^2 \theta \frac{r_+^2 + a^2}{\rho^2} d\varphi^2 \\ &= \rho^2 d\theta^2 + (2Mr_+)^2 \frac{\sin^2 \theta}{\rho^2} d\varphi^2, \end{aligned}$$

since $r_+^2 + a^2 = 2Mr_+$.

Therefore the area A of the surface is given by

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^\pi \sqrt{\det(g_{AB}|_S)} d\theta d\varphi \\ &= \int_0^{2\pi} \int_0^\pi \sqrt{g_{\theta\theta} g_{\varphi\varphi}} d\theta d\varphi \\ &= \int_0^{2\pi} \int_0^\pi 2Mr_+ \sin \theta d\theta d\varphi \\ &= \int_0^{2\pi} \int_0^\pi 2Mr_+ \sin \theta d\theta d\varphi \\ &= 8\pi Mr_+ = 4\pi(r_+^2 + a^2) \\ &= 8\pi(M^2 + M\sqrt{M^2 - a^2}). \end{aligned}$$

Thus,

$$\begin{aligned} 64\pi^2(M^4 - M^2 a^2) &= A^2 - 16\pi M^2 A + 64\pi^2 M^4, \\ 16\pi M^2 A &= A^2 + 64\pi^2 J^2, \\ M &= \sqrt{\frac{A}{16\pi} + \frac{4\pi J^2}{A}}. \end{aligned}$$

We notice that M as a function of A and J is homogeneous of degree $\frac{1}{2}$, i.e. we have $M(\lambda A, \lambda J) = \lambda^{\frac{1}{2}} M(A, J)$ for all λ . If we differentiate with respect to λ at

$\lambda = 1$, we have

$$(6.8) \quad A \frac{\partial M}{\partial A} + J \frac{\partial M}{\partial J} = \frac{1}{2} M,$$

which is more commonly known as the Euler's homogeneous function theorem.

In fact, using the definitions of Komar mass and Komar angular momentum, we have

$$\begin{aligned} M - 2\Omega J &= -\frac{1}{8\pi} \int_{\Sigma} \star dX^b + \Omega \star dY^b \\ &= -\frac{1}{8\pi} \int_{\Sigma} \star dZ^b. \end{aligned}$$

Now, let us choose the surface Σ to be a spacelike cross-section of the event horizon $\mathcal{H}$, i.e. the event horizon with fixed time t . Furthermore, we choose a future-directed timelike unit vector N and a spacelike unit vector n such that $Z = N + n$ and they are both orthogonal to Σ . Since Z is a Killing vector, we have

$$\nabla_{\mu} Z_{\nu} + \nabla_{\nu} Z_{\mu} = 0.$$

Thus,

$$(dZ^b)_{\mu\nu} = \nabla_{\mu} Z_{\nu} - \nabla_{\nu} Z_{\mu} = 2\nabla_{[\mu} Z_{\nu]}.$$

In particular, it is a 2-form since it is an antisymmetric $(0, 2)$ tensor.

Now, we can extend N and n to an orthonormal frame $\{N, n, E_1, E_2\}$ where E_1 and E_2 are tangent to Σ . By a choice of order between E_1 and E_2 , we can choose the frame to be positive, i.e. $\{-N^b, n^b, E_1^b, E_2^b\}$ is a positive coframe, so that the volume form is defined by $-N^b \wedge n^b \wedge E_1^b \wedge E_2^b$. Thus,

$$\nabla Z^b = -\nabla Z^b(N, n) N^b \wedge n^b + \dots$$

Therefore,

$$\begin{aligned} M - 2\Omega J &= -\frac{1}{4\pi} \int_{\Sigma} \star \nabla Z^b \\ &= \frac{1}{4\pi} \int_{\Sigma} \nabla Z^b(N, n) E_1^b \wedge E_2^b. \end{aligned}$$

Notice that by definition of the surface gravity,

$$\nabla Z^b(N, n) = \nabla Z^b(Z, n) = \langle \nabla_Z Z, n \rangle = \langle kZ, n \rangle = k,$$

since $Z = N + n$. By the zeroth law we know that k is constant over Σ . Notice also that the wedge product $E_1^b \wedge E_2^b$ is in fact the area element on Σ . Thus, integrating over Σ , we arrive at the Smarr formula:

$$M = \frac{kA}{4\pi} + 2\Omega J.$$

Comparing to (6.8), we have:

$$\frac{\partial M}{\partial A} = \frac{k}{8\pi}, \quad \frac{\partial M}{\partial J} = \Omega.$$

Thus, we finally arrive at the first law:

$$dM = \frac{k}{8\pi} dA + \Omega dJ.$$

□

7. CONCLUSION

This paper has traced a path from the mathematical foundations of general relativity to the physical interpretation of black holes. Beginning with the geometry of manifolds and curvature, we motivated Einstein's field equations and illustrated their simplest solution in Minkowski spacetime. From there, we examined stationary black holes, first through the Schwarzschild metric and its causal structure, then through the Kerr metric, which introduced angular momentum, the ergosphere, and richer horizon dynamics. Building on this analysis, we concluded with an exploration of black hole thermodynamics, providing a proof of the first law and considering its connection to classical thermodynamic principles.

Taken together, these results underscore the central role of black holes in uniting geometry and physics. They illustrate not only the predictive power of general relativity but also the surprising depth of black hole behavior, with concepts parallel to those in thermodynamics and statistical mechanics. Although much remains to be understood - particularly regarding quantum aspects of black holes - the study of their classical geometry already reveals profound insights into the structure of spacetime.

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REFERENCES

- [1] Manfredo P. do Carmo. Riemannian Geometry. Translated by Francis Flaherty. Birkhäuser. 1992.
- [2] Jose Natario. An Introduction to Mathematical Relativity Springer Nature. 2021.
- [3] Robert M. Wald. General Relativity. The University of Chicago Press. 1984.